Fundamentals Of

Fluid Mechanics

Theory and Applications

(Class Notes for CE / ME 370)

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Preface

Fluid mechanics is one of the most fundamental and ubiquitous of all the sciences. It has intrigued mankind since ancient times when Archimedes established the basic principles of buoyancy, and its study continues to shed light on important pure phenomena, as well as scientific and engineering applications. There are many excellent fluid mechanics texts and I do not consider this assembly of notes to be any sort of replacement. This merely represents a collection of concepts I consider to be the most valuable in a one–semester introductory course. The material is therefore intended to amplify and augment that which appears in your primary course textbook. In large part, we follow Munson et al. (2006) in terms of the way material is arranged. A sufficient familiarity with mathematics is presumed: calculus and its theorems (e.g. the Chain Rule), linear differential equations, and complex analysis. Problems and examples are minimal, since this volume serves as a collection of notes rather than a complete text. Supplementary margin notes appear at appropriate places suggesting example problems from Munson et al. (M, Y & O) that could be discussed in class.

The style of presentation leans toward the theoretical and mathematical side of the spectrum rather than the empirical side. My experience is that instilling good mathematical habits early in an undergraduate career is of significant benefit. Moreover, this allows us not to ask readers to take too much on faith. For the most part, we shall prove everything from first principles so that the reader can derive a truly fundamental understanding of the material. Where formal proofs are beyond our present scope, we will cite appropriate sources. Phenomena which cannot readily be presented in terms of first principles, e.g. turbulence, will be described in terms of a few empiricisms, but will largely be left for in–class discussion by the instructor.

There is minimal mention of English units of measurement. Where units are necessary, we work primarily in the International System (SI) of units. The reader will also find no treatment of compressible flows. This reflects my feeling that "starting incompressibly" provides the most intuitive entry into the field. Finally, I am grateful to those who have called attention to any mistakes appearing here and to those who will do so in the future.

Michael C. Wendl
2006

CHAPTER 1

Introduction

The science of fluid mechanics is concerned with the behavior of *fluids* at rest or in motion, i.e. flowing. While we have yet to formally define the term “fluid”, let us assume for the moment that we can speak of ordinary liquids and gases as fluids. This then is an extremely diverse subject! Many people initially equate fluid mechanics with the phenomena of aerodynamics, e.g. flow around lifting bodies. Yet the subject is much broader. It also encompasses essentially all of the traditional engineering applications, such as flow in machinery, HVAC[1.1], structural aerodynamics, weather behavior, lubrication, convection heat transfer, etc. Moreover, many phenomena not typically thought of as traditional engineering fare are fundamentally based on fluid phenomena, including biomedical flows[1.2], electrophoresis of DNA molecules, etc. Add to all of this that mathematics has often developed symbiotically with the study of fluid mechanics, and one is truly looking at a remarkable field of study. Here, we will mostly confine ourselves to developing a firm understanding of the fundamentals of fluid mechanics. Naturally, we must first define the entity of a “fluid”.

1.1. Definition of a Fluid

An introductory physics class will typically divide matter into three categories: solids, liquids, and gases and describe their various behaviors when placed in a container. In fluid mechanics, however, there are only two categories: fluids and non–fluids, i.e. solids (White, 1974). The actual definition is framed in the context of an applied shear force (Figure 1.1). You probably already have a general idea of the difference between fluid and solid behavior. A solid tends to resist the shear load. It will deform to some finite degree, but when the resistive force equals the applied load, deformation will halt. Conversely, fluids experiencing a shear load will continue to deform. Perhaps the clearest example of this is to imagine a dammed reservoir holding water at rest. If the dam is suddenly removed, the water will flow continuously. This leads us to a working definition:

definition of a fluid

Definition 1.1. *A fluid is a substance that deforms continuously upon application of a shear stress.*

[1.1] This is the standard acronym for heating, ventilating, and air conditioning.

[1.2] Most living organisms are fluid based, for example the cardiovascular system of blood flow and the pulmonary system of oxygenation.

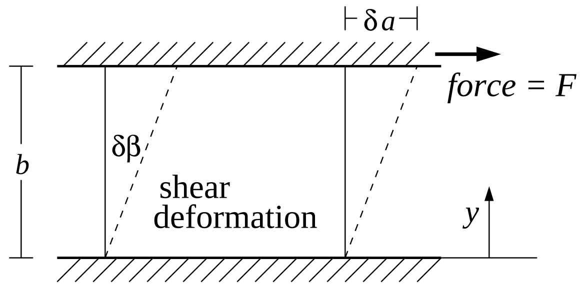

FIGURE 1.1. *Shear deformation realized by applying a shearing load to the upper surface, while the lower surface is held fixed.*

Note that the magnitude of the shear is irrelevant.

Our definition applies to "true fluids", such as all gases, and the common liquids like water, oil, alcohol, etc. That being said, we quickly point out that some substances cannot be classified exactly according to this definition. For example, tar and toothpaste act as solids for sufficiently small shear loads, then behave as fluids once this load is exceeded. We will omit such fluids from our present treatment and stick to the aforementioned true fluids.

1.2. The Continuum Assumption

If we were to look at a mass of fluid at the microscopic level, what we would see are individual molecules interacting with each other. We are not actually interested in the behavior of individual molecules for our engineering applications. Rather, we want to understand the overall (or macroscopic) behavior of the system as a whole. That is, it is the macroscopic properties such as density, temperature, or pressure drop that are of engineering interest. What we are doing from the mathematical perspective is taking averages over small elemental volumes. These volumes must be large enough such that they contain enough molecules at any instant in time to yield a statistically significant average. Yet they must also be small enough so that the statistical average does not vary over the volume — it should be a constant. If these conditions are met, the fluid properties will have definite point values. In other words, they will be continuous functions of space and time. This is the so–called *continuum assumption.*

the continuum assumption

To illustrate this concept, consider the density ρ of a fluid. A region of fluid is shown in Fig. 1.2(a). We desire to find the density at a location C having coordinates (x_0, y_0, z_0), where density is defined simply as mass per unit volume. For the larger region, the average density within its volume V would simply be $\rho = m/V$. Since this is an average, we would expect ρ to vary within V since V is large. Specifically, the average ρ would not be expected to be equal to the density at point C. To determine the density at C, we compute $\delta m/\delta V$. The question governing the continuum assumption is how small or large is δV?

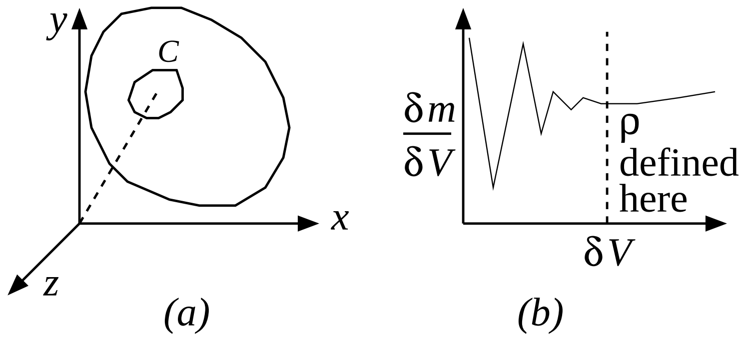

FIGURE 1.2. *(a) Defining density as a point function according to the continuum assumption. Large region has volume V and mass m, while the point region C has volume δV and mass δm. (b) Value of density according to size of δV.*

Let us plot $\delta m/\delta V$ as a function of δV as in Fig. 1.2(b) starting from large values of δV. The average density tends to approach an asymptotic value as the volume is lowered to the point such as to enclose only homogeneous fluid in the immediate neighborhood of point C. Below this point, the volume is small enough such that the number of molecules contained at a given instant of time is not constant in the average sense. Rather, the number varies erratically, causing a corresponding variation in mass δm, which in turn leads to an erratic value for density $\delta m/\delta V$. There is clearly a lower limit for δV which restricts the continuum assumption. Luckily, this limit is very small compared to the scales we are interested in for engineering calculations[1.3]. Therefore, we are permitted mathematically to approximate $\delta V \to 0$ relative to the size of our engineering length scales. By this, we mean that the volume approaches but does not reach 0. Therefore, the continuum assumption will be valid for our problems of interest. Panton (1984) contains a more detailed discussion.

Since the (x_0, y_0, z_0) location of point C is arbitrary, ρ at any point in the region could likewise be determined. If ρ were computed simultaneously for all δV in the fluid, we could formulate an expression as a function of the location and time of the measurements, i.e. $\rho = \rho(x, y, z, t)$. Thus, the continuum assumption leads naturally to functional definitions for the properties of interest.

1.3. Fundamental Fluid Properties

Let us now discuss the fundamental fluid properties of interest. First and perhaps most difficult is temperature T. Everyone is familiar with the

[1.3]For example, a cubic meter of air at standard temperature and pressure ($15C$, $101.3\,kPa$) contains about 2.5×10^{25} molecules. Therefore, the number of molecules in a volume about the size of a grain of sand, about 10^{-12} cubic meters, would be 2.5×10^{13}, which is large enough to ensure that the average mass would be constant.

qualitative connotation of temperature as "hot" and "cold", yet an exact quantitative definition is difficult[1.4]. For our purposes, we will define an *equality of temperature*: two bodies have equal temperature T if there is no change in observable properties when the bodies are in thermal contact. By the transitive property, we can extend this to any number of bodies. We can then construct an arbitrary scale of temperature with respect to a standard body.

definition of T

Pressure is another entity that we are familiar with from experience. Scuba divers in particular have experienced hydrostatic pressure when under the surface. Microscopically, the molecules of a fluid are moving about randomly due to their thermal energy. A surface within this fluid is struck by these molecules, even if it is at rest. By Newton's Second Law, $F = m\,a$, we can describe the force exerted by a molecule in terms of its time rate of change of momentum, where $m\,v$ is momentum and $a = \dot{v}$ is acceleration. *Pressure* is simply the magnitude of this force per unit area of surface where the area approaches zero within the confines of the continuum assumption

definition of P

$$P = \lim_{\delta A \to 0} \frac{F}{\delta A}\,. \tag{1.1}$$

Pressure acts normal to the surface and carries units of force per unit area.

We have already introduced density above. Formally, we define *density* as mass per unit volume where volume approaches zero within the confines of the continuum assumption

definition of ρ

$$\rho = \lim_{\delta V \to 0} \frac{\delta m}{\delta V}\,. \tag{1.2}$$

The dimensions of this quantity are mass per length to the third power. Gases are highly compressible relative to fluids and in many cases there is a special relationship among density, pressure, and temperature known as the *Ideal Gas Law*

$$P = \rho\,R\,T\,, \tag{1.3}$$

where R is a constant that depends upon the gas.

Specific weight γ is also sometimes important and it is defined simply as weight per unit volume of a fluid. Water at $4\,C$ has a specific weight of roughly $980.7\,N/m^3$. It follows that density and specific weight are related by

definition of γ

$$\gamma = \rho\,g\,, \tag{1.4}$$

where g is the gravitational constant $9.81\,m/sec^2$. A related quantity is the *specific gravity* SG, which is defined as the ratio of two densities. Almost

definition of SG

[1.4] Temperature is a surprisingly difficult concept to define in terms of first principles. This is usually deferred to graduate courses in thermodynamics where it arises from considerations of energy and entropy. Here, we will take the simplistic, but typical approach of understanding temperature merely as a quantity that indicates thermal energy.

always, the reference density in the denominator is ρ of water at $4\,C$, i.e. $\rho = 1000\,kg/m^3$.

When fluids are in motion, the shear deformation discussed above is accompanied by shear stress. It is helpful to refer back to Fig. 1.1. Let b be the distance between the two plates, $\delta\beta$ be the angle, and δa be the distance swept out between the tips of the solid to the tips of the dashed lines, F be the applied force, and u be the velocity at which the top plate will slide. Examining the system over a small increment of time δt, we see that $\delta a = u\,\delta t$, and by trigonometry we get[1.5]

$$\tan\delta\beta \;\approx\; \delta\beta \;=\; \frac{\delta a}{b} \;=\; \frac{u\,\delta t}{b}\,.$$

For solids, we would attempt to relate shear stress τ to $\delta\beta$, but of course for fluids this is not meaningful since $\delta\beta$ varies as a function of time. That is, it increases without bound. We solve this equation to obtain an approximation for the strain rate

$$\frac{\delta\beta}{\delta t} \;=\; \frac{u}{b}\,,$$

which can be made exact by taking the limit. That is, δ symbols become differentials. The quantities u and b are also differentials and can be written du and dy, respectively. Thus,

$$\frac{du}{dy} \;=\; \frac{d\beta}{dt}\,.$$

The resistive force realized by the plate is caused by shear stress imparted at the plate surface by the fluid. This force is equal to the stress multiplied by the area of the plate over which it acts: $F = \tau A$. After taking many measurements for various shear rates and forces, we would find that τ is directly proportional to the shear rate, $\tau \propto du/dy$, for the true fluids described above. The constant of proportionality is the *viscosity* definition of μ

$$\tau \;=\; \mu\,\frac{du}{dy}\,. \tag{1.5}$$

Note the similarity to Hooke's Law for an elastic solid where τ is proportional to strain itself rather than the rate of strain (Popov, 1976). This is the fundamental difference between fluids and solids. Eq. (1.5) shows that the resistance to flow is related to μ. Specifically, a fluid with higher μ would require higher applied force to reach the same strain rate as a fluid with lower μ. We note that the full name for μ is the *dynamic viscosity*. One obtains the *kinematic viscosity* according to $\nu = \mu/\rho$. We will see that this quantity arises naturally in the fluid equations of motion. definition of ν

[1.5] The approximation is valid for small values of $\delta\beta$ because "$\tan\delta\beta$" expands as (Beyer, 1984)

$$\tan\delta\beta \;=\; \delta\beta + \frac{(\delta\beta)^3}{3} + \frac{2\,(\delta\beta)^5}{15} + \frac{17\,(\delta\beta)^7}{315} + \cdots$$

meaning that terms of order 3 and higher are very small compared to the first term. Thus, the tangent of a small angle is approximately equal to the angle itself.

There are some additional quantities that textbooks normally introduce at this point such as those related to compressibility and evaporation. Instead, we defer mentioning these quantities since we omit discussion of flows requiring them.

1.4. Anatomy of a Fluid Mechanics Problem

What is it that we would like to know regarding the arbitrary fluid mechanics problem? A problem is completely characterized if we know the *velocity distribution* and the *pressure distribution*. Having these expressions enables us to compute quantities of engineering interest such as lift, drag, friction coefficient, etc. We will see that velocity and pressure are related through a set of conservation equations.

We have already defined pressure according to the continuum assumption, but we have not yet done so for velocity. The entity of velocity should be familiar from particle dynamics. It is a vector quantity that gives the speed of a particle in three orthogonal coordinate directions: $\mathbf{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$, where (u, v, w) represent the component magnitudes along the unit vectors $(\hat{i}, \hat{j}, \hat{k})$ in the coordinate directions (x, y, z). This is also the form we will use to denote the velocity distribution in fluids. However, we must realize that, like our other quantities, velocity is actually defined according to the continuum assumption. That is, *velocity* is the collective momentum of all the particles in δV divided by the total mass of these particles (Panton, 1984). Mathematically, we write this as definition of $\mathbf{V}$

$$\mathbf{V} = \lim_{\delta V \to 0} \frac{\Sigma\, m_i\, \mathbf{V}_i}{\Sigma\, m_i}\,, \tag{1.6}$$

i.e. velocity is nothing more than momentum per unit mass. This definition permits us to use velocity as a continuum function defined at all points in the fluid.

CHAPTER 2

Fluid Statics

The first class of problems we will consider are those where the fluid is at rest, i.e. $\mathbf{V} = 0$. Here, there is no relative motion between fluid particles, so there are no shear stresses[2.1]. Thus, pressure is the primary quantity of interest. Fluid static problems, also known as *hydrostatic* problems, are mathematically more simple than dynamic problems. Beginning our study with hydrostatics will allow us to work up through some of mathematical difficulties from a manageable level.

2.1. Pressure at a Point: Pascal's Law

As we discussed in Chapter 1, pressure refers to the force per unit area acting in a normal direction on a surface. The first question we address is whether the magnitude of pressure at a point varies according to the direction of the normal vector. In other words, we ask whether pressure is uniform at a point. To answer this question, we start with a somewhat arbitrary control volume of fluid shaped like a wedge (Figure 2.1). Imagine that this control volume wedge lies submerged. Pressure acts on the five faces, which all have surface normal vectors in different directions. Moreover, the weight of the fluid acts in the $-z$ direction. For now, let consider only pressure forces in the (y, z) plane as shown in the Fig. 2.1 and neglect forces in the x direction. Also, we will assume for generality that the element is being uniformly accelerated with components in the y and z directions. Note that because of the wedge shape, the volume is $\delta V = \delta x\, \delta y\, \delta z/2$.

First, let us write Newton's Second Law $\mathbf{F} = m\, \mathbf{a}$ in the y and z directions. In the y direction, we have pressure acting positively on the left face and a horizontal component $(\sin\theta)$ of pressure on the top face acting negatively. If a_y denotes the acceleration in the y direction, we obtain

$$P_y\, \delta x\, \delta z \;-\; P_s\, \delta x\, \delta s\, \sin\theta \;=\; \rho\, \frac{\delta x\, \delta y\, \delta z}{2}\, a_y\, . \tag{2.1}$$

In the z direction, we have pressure acting upwardly on the bottom face, a vertical component $(\cos\theta)$ of pressure on the top face acting downwardly,

[2.1] In actuality, we could also fulfill the $\tau = 0$ condition by imagining other scenarios where no relative motion exists between fluid particles. A tank of liquid that has been spinning for a long time and a mass of fluid that is accelerated uniformly would be two such examples. Here, the flow domains behave as rigid bodies. We shall consider these mathematical generalities, however, most problems will deal with the practical case of fluids at rest.

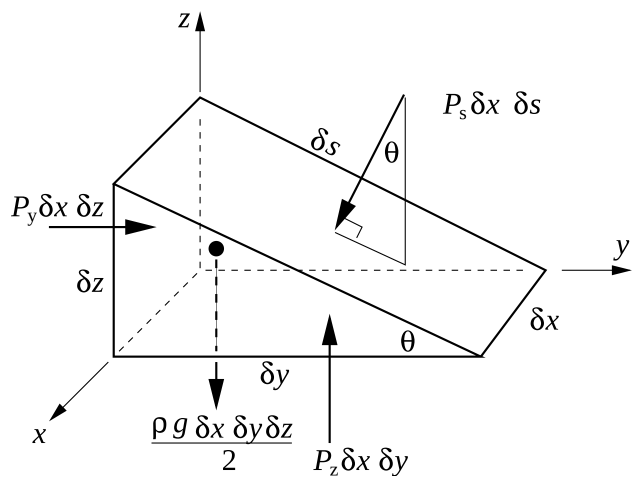

FIGURE 2.1. *Pressure forces on an arbitrary wedge–shaped element of fluid.*

and the weight also acting downwardly. If a_z denotes the acceleration in the z direction, we obtain

$$P_z\, \delta x\, \delta y \;-\; P_s\, \delta x\, \delta s\, \cos\theta \;-\; \rho\, g\, \frac{\delta x\, \delta y\, \delta z}{2} \;=\; \rho\, \frac{\delta x\, \delta y\, \delta z}{2}\, a_z\,. \tag{2.2}$$

Furthermore, from geometry, we have $\delta y = \delta s\, \cos\theta$ and $\delta z = \delta s\, \sin\theta$. Substituting these expressions, we see that Eqs. (2.1) and (2.2) simplify to

$$P_y \;-\; P_s \;=\; \rho\, a_y\, \frac{\delta y}{2} \tag{2.3}$$

and

$$P_z \;-\; P_s \;=\; \rho(a_z + g)\, \frac{\delta z}{2}\,. \tag{2.4}$$

Because we are interested in the behavior of pressure at a point, we must allow our elemental volume to become small, i.e. we allow it to approach "zero" within the continuum assumption. In other words, we take the limits $(\delta x, \delta y, \delta z) \to 0$, from which we immediately deduce $P_y = P_z = P_s$ since the right hand sides of both equations vanish. Since we did not assume any particular value for θ, we can deduce the following theorem

Pascal's Law

THEOREM 2.1 (Pascal's Law). *The pressure at any point within a fluid at rest is independent of direction.*

We could extend this same treatment to the x direction, in which case we would find that P_x is equivalent to the other pressures. There is an interesting analogy to Mohr's Circle of stress which can also be used to demonstrate this result as well (see Appendix A).

2.2. The Equation of Hydrostatics

In the previous section we determined that pressure is constant with respect to a point. Let us now generalize the problem to ask how pressure varies from point to point. In other words, how do we obtain the pressure distribution?

To address this question, we must first derive an equation which governs the pressure in hydrostatic situations[2.2]. This will derive once again from Newton's Second Law. We will approach this task once again by analyzing the pressure forces acting on a differential fluid element of dimensions $\delta x \times \delta y \times \delta z$ (Figure 2.2). If we let the pressure at the geometric center of the

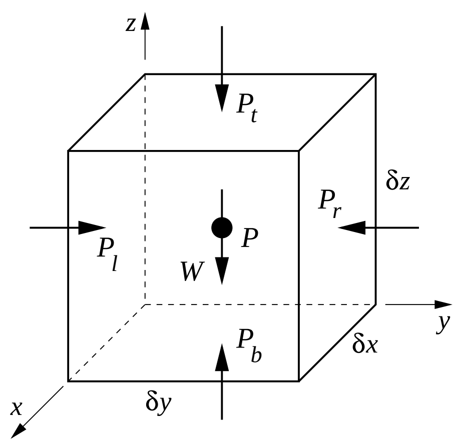

FIGURE 2.2. *Surface and body forces on an element of fluid.*

element be P, we can expand P in terms of Taylor series to obtain pressure at the faces of the element. In actuality, we only use 1–term expansions[2.3], which can be written as

$$P_l = P - \frac{\partial P}{\partial y}\frac{\delta y}{2}, \tag{2.5}$$

$$P_r = P + \frac{\partial P}{\partial y}\frac{\delta y}{2}, \tag{2.6}$$

$$P_b = P - \frac{\partial P}{\partial z}\frac{\delta z}{2}, \tag{2.7}$$

$$P_t = P + \frac{\partial P}{\partial z}\frac{\delta z}{2}, \tag{2.8}$$

[2.2] As mentioned at the beginning of the chapter, we will actually treat the more general case of uniform acceleration. The resulting no–shear situation is identical to the true hydrostatic case.

[2.3] All terms of second order and above can be neglected since they will involve products of δx, δy, and δz.

where ∂ represents the *partial derivative*. Although not shown in Fig. 2.2, there are the corresponding contributions in the x direction, which can similarly be deduced. Also, the weight of the element in the $-z$ direction is $W = \rho\, g\, \delta x\, \delta y\, \delta z$.

Forces on the faces of the element corresponding to these pressures can be calculated by simply multiplying pressure by area, e.g.

$$F_l \;=\; P_l\, \delta x\, \delta z \;=\; \left(P \;-\; \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\, \delta x\, \delta z\;, \tag{2.9}$$

$$F_r \;=\; P_r\, \delta x\, \delta z \;=\; \left(P \;+\; \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\, \delta x\, \delta z\;, \tag{2.10}$$

$$F_b \;=\; P_b\, \delta x\, \delta y \;=\; \left(P \;-\; \frac{\partial P}{\partial z}\frac{\delta z}{2}\right)\, \delta x\, \delta y\;, \tag{2.11}$$

$$F_t \;=\; P_t\, \delta x\, \delta y \;=\; \left(P \;+\; \frac{\partial P}{\partial z}\frac{\delta z}{2}\right)\, \delta x\, \delta y\;. \tag{2.12}$$

We now sum the surface forces in each of the coordinate directions. For example, in the y direction, we do $\delta F_y = F_l - F_r$, which gives

$$\delta F_y \;=\; \left(P \;-\; \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\, \delta x\, \delta z \;-\; \left(P \;+\; \frac{\partial P}{\partial y}\frac{\delta y}{2}\right)\, \delta x\, \delta z\;. \tag{2.13}$$

This expression simplifies to

$$\delta F_y \;=\; -\,\frac{\partial P}{\partial y}\, \delta x\, \delta y\, \delta z\;. \tag{2.14}$$

Performing the same operations in the x and z directions, we find

$$\delta F_x \;=\; -\,\frac{\partial P}{\partial x}\, \delta x\, \delta y\, \delta z\;. \tag{2.15}$$

and

$$\delta F_z \;=\; -\,\frac{\partial P}{\partial z}\, \delta x\, \delta y\, \delta z\;. \tag{2.16}$$

Let us consolidate the surface force expressions into vector form. Recall from Chapter 1 that we have the unit vector basis $(\hat{\imath}, \hat{\jmath}, \hat{k})$ along the coordinate directions (x, y, z). Therefore, attaching the components of surface force to their corresponding directions, we get

$$\delta \mathbf{F} \;=\; \delta F_x\, \hat{\imath} \;+\; \delta F_y\, \hat{\jmath} \;+\; \delta F_z\, \hat{k}\;, \tag{2.17}$$

or by substituting our expressions in Eqs. (2.14), (2.15), and (2.16), we obtain

$$\delta \mathbf{F} \;=\; -\,\left(\frac{\partial P}{\partial x}\,\hat{\imath} \;+\; \frac{\partial P}{\partial y}\,\hat{\jmath} \;+\; \frac{\partial P}{\partial z}\,\hat{k}\right)\, \delta x\, \delta y\, \delta z\;. \tag{2.18}$$

We see that the expression in parenthesis is simply the gradient of pressure, ∇P, which is a vector quantity.

∇P is a vector quantity

Before we write down Newton's Second Law, we must also account for the weight of the fluid element due to gravity g acting in the $-z$ direction. We write this as a force directed negatively along the $\hat{k}$ unit vector as $-\rho\, g\, \delta x\, \delta y\, \delta z\, \hat{k}$. In vector notation, we can now write Newton's Second Law for each of the 3 directions as a compact vector expression $\Sigma\, \delta \mathbf{F} = \delta m\, \mathbf{a}$.

$$-\nabla P\, \delta x\, \delta y\, \delta z \;-\; \rho\, g\, \delta x\, \delta y\, \delta z\, \hat{k} \;=\; \rho\, \delta x\, \delta y\, \delta z\, \mathbf{a}\,. \tag{2.19}$$

The first term represents the pressure gradient in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ directions while the second term represents the weight of the fluid *which only acts in the* $-\hat{k}$ *direction*. The right hand side is simply the mass (density $\times$ volume of the element) times the acceleration. Of course, the volume of the element can be dropped to reveal the final generalized equation

$$-\nabla P \;-\; \rho\, g\, \hat{k} \;=\; \rho\, \mathbf{a}\,. \tag{2.20}$$

In a strictly hydrostatic situation, $\mathbf{a} = 0$, which yields the final *equation of hydrostatics*

equation of hydrostatics

$$\nabla P \;+\; \rho\, g\, \hat{k} \;=\; 0\,, \tag{2.21}$$

where we have moved the non–vanishing terms to the other side of the equation.

Of course, we can expand Eq. (2.21) into its 3 orthogonal components as

$$\frac{\partial P}{\partial x} \;=\; 0\,, \qquad \frac{\partial P}{\partial y} \;=\; 0\,, \qquad \frac{\partial P}{\partial z} \;=\; -\,\rho\, g\,. \tag{2.22}$$

We can readily deduce from these equations that pressure along lines in x and y directions is constant and that P only varies in the z, or vertical direction. This is of course the basis for the layman's hydrostatic law "water always seeks its own level". We will delve further into the meaning of this later in the chapter.

Since we have established that P is only a function of the single variable z, the partial differential equation becomes an ordinary differential equation, i.e. we replace ∂ with the differential operator d. We now obtain

$$\frac{dP}{dz} \;=\; -\,\rho\, g\,. \tag{2.23}$$

Eq. (2.23) is applicable to fluids at rest for determining how pressure changes with elevation. It is a simple first–order ordinary differential equation.

2.3. Hydrostatics for Incompressible Fluids

Solving Eq. (2.23) is simply a matter of direct integration. That is, to determine the pressure behavior between 2 elevations z_1 and z_2, where $z_2 > z_1$, we integrate as

$$\int_{P_1}^{P_2} dP \;=\; -\int_{z_1}^{z_2} \rho\, g\, dz\,. \tag{2.24}$$

Gravity is constant, so it can be moved out of the integral sign, however, what about ρ? If the fluid is compressible ρ will vary as a function of the elevation. We would then require some sort of state equation relating density of the fluid to its local elevation. Here is where we introduce the concept of *incompressibility*, i.e. the idea that a fluid cannot be compressed. Thus, incompressible fluid

DEFINITION 2.1. *An incompressible fluid is one in which density is essentially constant.*

Intuitively, we feel that this is probably a pretty good assumption for liquids, since molecules are already very closely packed. It is also a reasonable assumption for gases at rest if the changes in elevation are not too overwhelmingly large. Gases can also behave incompressibly in a dynamic situation, depending upon the flow. For example, in the low–speed aerodynamics of cars and buildings, air, which is compressible, does not experience any significant changes in density.

Since it is constant, ρ also comes outside of the integral sign, and Eq. (2.24) is easily evaluated as. *MY&O Ex. 2.1 pp 44*

$$P_2 - P_1 = -\rho\, g\, (z_2 - z_1)\,. \tag{2.25}$$

If $h = z_2 - z_1$ is the elevation, we have the simplified form $P_2 = P_1 - \rho\, g\, h$. Note that pressure decreases as elevation increases and vice versa.

2.4. The Nomenclature of Pressure

Like every physical quantity, "pressure" must be measured relative to something. This context is readily illustrated by temperature: it can be measured relative to either an absolute thermodynamic state (the Kelvin scale) or to a physical manifestation, e.g. freezing and boiling of water (the Celsius scale). Pressure measurement is treated similarly. In the absolute sense, we can measure pressure relative to a perfect vacuum. This is *absolute pressure*. Conversely, pressure can be measured relative to a local atmospheric pressure, i.e. as a *gage pressure*.

Absolute pressure is always positive because a vacuum is "zero pressure", while gage pressure can be positive (higher than local atmospheric pressure) or negative (lower than atmospheric pressure). Fig. 2.3 shows these examples as positions 1 and 2, respectively.

2.5. Hydrostatic Pressure Measurement: The Manometer

As we have seen, the hydrostatic equation relates pressure changes to corresponding changes in vertical elevation through a fluid. This is of course a phenomenon which can be exploited to measure pressure, i.e. P can be determined as a function of the height of a liquid column of known density which can be supported. A device implementing this principle is known as a *manometer* (Figure 2.4). Previously in the chapter, we determined that horizontal lines are also lines of constant pressure. We formalize this concept in terms of a simple theorem:

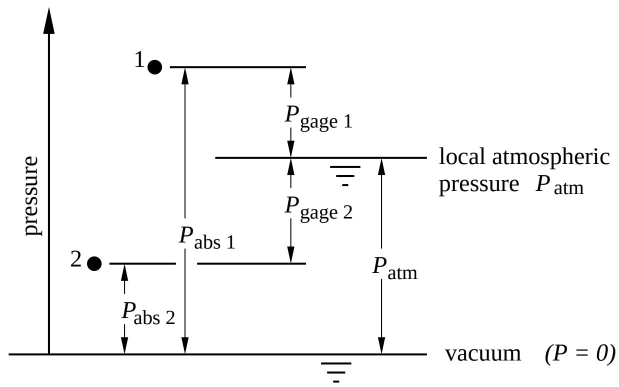

FIGURE 2.3. *Representative pressures measured with respect to absolute and local references.*

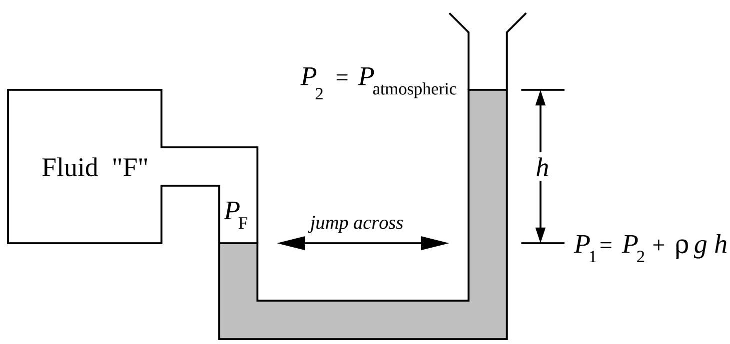

FIGURE 2.4. *Simple open–ended U–tube manometer.*

THEOREM 2.2. *Any two points at the same elevation connected by the same fluid are at equal pressure.*

This in turn leads to a simple procedure for solving manometer problems, sometimes referred to as the "jump across" method: *MY&O Ex. 2.4 pp 52*

- Identify point where an unknown pressure must be determined
- Identify point where a pressure is known
- Beginning with the known pressure, work your way toward the unknown pressure by applying the hydrostatic equation for any vertical changes. Ignore all horizontal translations since they trace lines of constant pressure.
- Whenever you arrive at a point which has a companion location at the same elevation connected by the same fluid closer to your destination, "jump across" at constant pressure (Fig. 2.4).
- Iterate until you arrive at the location of desired pressure, which can now be computed from cumulative pressure changes you have kept track of.

Note that it is common to see slanted tubes in some cases to increase the scale resolution for reading pressure. For example, if the scale consists simply of graduated lines marked off on the tube, it can be stretched by a factor of $(\sin\,\theta)^{-1}$ to obtain more accurate readings (Figure 2.5). That is, a height

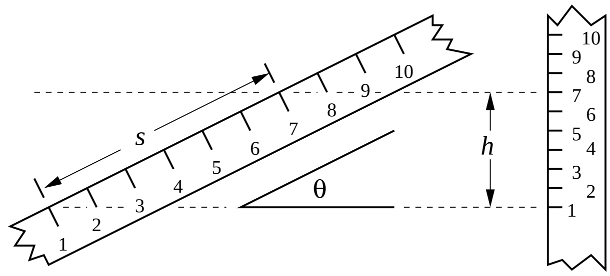

FIGURE 2.5. *Slanting tube for increasing scale resolution.*

h, which registers a specific pressure reading on the vertical tube, can be read on an extended scale $s = h/\sin\,\theta$. This design is particularly useful for measuring small changes in pressure.

One remaining approximation often employed in engineering problems is to neglect elevation changes for any gases when computing pressure. Since liquid density is much greater than density of gases, the pressure changes associated with gases are small.

2.6. Hydrostatic Forces on Planar Submerged Surfaces

Now that we have determined how pressure varies in static fluids, we can extend our treatment to study the resulting *hydrostatic forces* acting on submerged objects. Such calculations are important in the design of large hydraulic structures such as dams, ships, petroleum tanks, etc. To completely characterize the force acting on a submerged surface we must know the magnitude, direction, and line of action of the *resultant* force. Since pressure is a distributed load, we can treat it in the same fashion as in statics and strength of materials. That is, the hydrostatic loading problem reduces to formulas related to centroids and cross–sectional moments of inertia.

Consider the submerged planar surface shown in Fig. 2.6. It is tilted at an angle θ relative to the liquid free surface. Assume zero pressure at the free surface. Therefore, the pressure at any given depth is simply $\gamma\,h$, where $\gamma = \rho\,g$ is the specific weight and h is the measured depth, which is related to the coordinate system by $h = y\,\sin\,\theta$. It follows that the differential force dF acting on differential area dA must be $dF = \gamma\,h\,dA$. Furthermore, it must be $\perp$ to the surface since pressure is defined to act in the normal direction when fluid is at rest[2.4].

[2.4]Because there are no shearing forces in a static fluid.

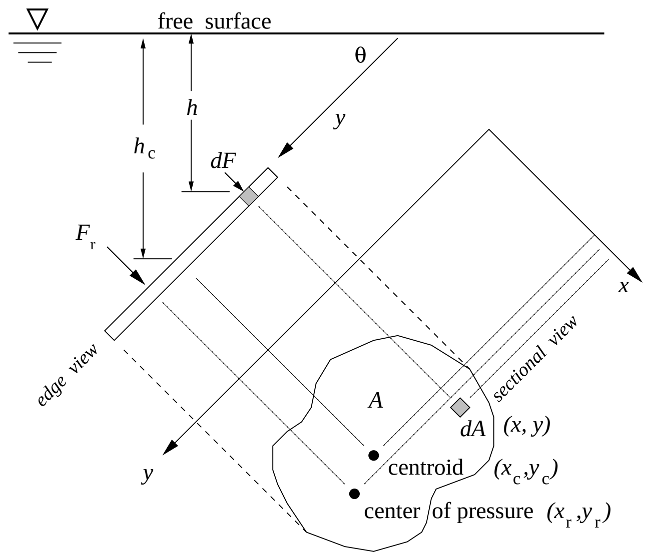

FIGURE 2.6. *Submerged planar surface showing centroid and center of pressure.*

The magnitude of the resulting force can be found simply by integrating dF over the entire area A, that is, we simply sum the entire contribution of the distributed force

$$F_r = \int_A dF = \int \gamma\, h\, dA = \int \gamma\, y\, \sin\theta\, dA = \gamma\, \sin\theta \int y\, dA\,. \tag{2.26}$$

The last step is permitted for γ once again because of the incompressible assumption. The last integral quantity in Eq. (2.26) is the definition of the *first moment of area* familiar from statics (Beer and Johnston, 1984)

$$y_c A \equiv \int y\, dA\,, \tag{2.27}$$

where y_c is the coordinate location of the *centroid*. Therefore, the magnitude of the applied pressure load is $F_r = \gamma\, y_c\, A\, \sin\theta$, which can be written independent of θ as

$$F_r = \gamma\, h_c\, A\,, \tag{2.28}$$

where h_c is the vertical distance of the centroid below the surface. This result can be stated as

THEOREM 2.3. *The magnitude of a hydrostatic force applied to a planar surface is equal to the pressure at its centroid (i.e. the average pressure) multiplied by its area.*

Intuitively we might suspect that the line of action of the resultant force should also pass through the centroid[2.5]. However, this is not the case here because the pressure distribution is not uniform over the surface (Fig. 2.7). To compute the location of the line of action y_r, we can sum the contribution

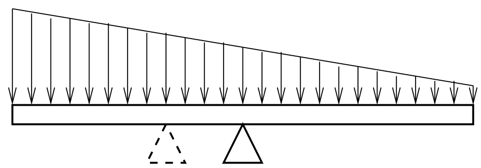

FIGURE 2.7. *The geometric centroid, represented by the solid fulcrum in the middle, does not coincide with the line of force of the resulting pressure load. That is, the sum of the moments (torques) of this system is not zero. Moving the fulcrum to the left, i.e. to a lower fluid depth, would support the load since the sum of the moments vanishes.*

of all moments about the x axis and set this equal to $y_r F_r$ to obtain the static equilibrium given by

$$y_r\, F_r \;=\; \int_A y\, dF \;=\; \int \gamma\, y^2\, \sin\theta\, dA \;=\; \gamma\, \sin\theta \int\, y^2\, dA\,. \tag{2.29}$$

Since we have already established $F_r = \gamma\, y_c\, A\, \sin\theta$, we divide both sides by this expression to obtain

$$y_r \;=\; \frac{\int\, y^2\, dA}{y_c\, A}\,. \tag{2.30}$$

The integral quantity in Eq. (2.30) is the definition of the *second moment of area*, also called the *moment of inertia* I_x (Beer and Johnston, 1984). Eq. (2.30) can then be written as

$$y_r \;=\; \frac{I_x}{y_c\, A}\,, \tag{2.31}$$

where I_x is computed with respect to the x axis.

Note that Eq. (2.31) is not in a very convenient form for calculation, because we would explicitly have to compute I_x depending upon the problem. For example I_x varies as a function of depth even for the same surface! It would be better to utilize moments of inertia that are constant for the surface in question, i.e. I_{xc}, which is based upon the centroid of the surface. We can relate I_x and I_{xc} via the *Parallel Axis Theorem* (Beer and Johnston, 1984)

Parallel Axis Theorem

$$I_x \;=\; I_{xc} \;+\; A y_c^2\,, \tag{2.32}$$

[2.5] This is according to our recollection that the centroid is the location where a support could be placed to hold the surface in perfect balance.

which then can be substituted into Eq. (2.31) to yield the final solution

$$y_r = \frac{I_{xc}}{y_c\,A} + y_c\,. \tag{2.33}$$

Eq. (2.33) shows that the line of action of the applied force is *always* below the centroid of the surface[2.6]. Referring back to Fig. 2.7, this is what we would expect given that the distribution of higher pressures is biased toward lower depths.

To find the location of the applied force relative to the x coordinate, i.e. x_r, we can perform a similar derivation:

$$x_r\,F_r = \int_A x\,dF = \int \gamma\,x\,y\,\sin\theta\,dA = \gamma\,\sin\theta \int x\,y\,dA\,, \tag{2.34}$$

which yields

$$x_r = \frac{I_{xy}}{y_c\,A}\,, \tag{2.35}$$

where $I_{xy} = \int x\,y\,dA$ is the inertia with respect to the x and y axes[2.7] Once again, applying the Parallel Axis Theorem, we obtain: *MY&O Ex. 2.6 pp 61*

$$x_r = \frac{I_{xyc} + Ax_cy_c}{y_c\,A} = \frac{I_{xyc}}{y_c\,A} + x_c\,. \tag{2.36}$$

Eqs. (2.36) and (2.33) allow us to find the point of application of the resultant for the general case.

2.7. A Review of Centroids and Moments of Inertia

Let us review in greater detail the concepts of centroids, moments of inertia, and the Parallel Axis Theorem. As an example relevant to hydrostatics, consider the rectangular plate as e.g. a dam gate or channel wall (Fig. 2.8). First, let us calculate the centroid location relative to the y'' coordinate, i.e. $y''_c = d + y'_c$, where y'_c is computed from the definition of the centroid as measured from $y'' = d$, or equivalently $y' = 0$. We obtain

$$y'_c = \frac{\int y dA}{A} = \frac{\int_0^h \int_{-b/2}^{b/2} y\,dx\,dy}{bh} = \frac{\int_{-b/2}^{b/2} dx \int_0^h ydy}{bh} = \frac{bh^2}{2bh} = \frac{h}{2}\,, \tag{2.37}$$

which, when measured from the fluid free surface yields $y''_c = d + h/2$. This is a fairly intuitive result.

[2.6] Since I_{xc}, y_c, and A are all positive quantities, it must be the case that $y_r > y_c$, meaning the line of action lies deeper than the centroid.

[2.7] This is sometimes referred to as the product of inertia (Beer and Johnston, 1984).

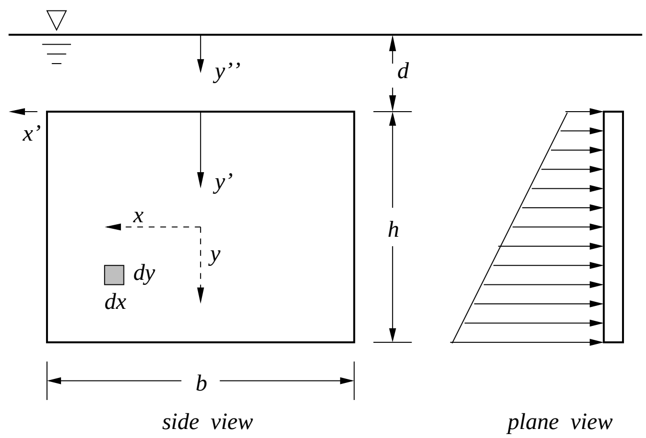

FIGURE 2.8. *A rectangular surface is commonly used for e.g. dam gates.*

The calculation is similar for I_x. Let us compute both I_x with respect to $y'' = 0$ and I_{xc} then double–check the results via the Parallel Axis Theorem[2.8]. We obtain

$$I_x = \int y''^2 \, dA = \int_d^{d+h} \int_{-b/2}^{b/2} y''^2 \, dx \, dy'' = \int_{-b/2}^{b/2} dx \int_d^{d+h} y''^2 \, dy'' , \tag{2.38}$$

which evaluates to

$$I_x = b \left. \frac{y''^3}{3} \right|_d^{d+h} = b \left(\frac{h^3}{3} + hd^2 + dh^2 \right) . \tag{2.39}$$

Alternatively, we can calculate the moment of inertia with respect to the centroid as

$$I_{xc} = \int y^2 \, dA = \int_{-h/2}^{h/2} \int_{-b/2}^{b/2} y^2 \, dx \, dy = \int_{-b/2}^{b/2} dx \int_{-h/2}^{h/2} y^2 \, dy , \tag{2.40}$$

which evaluates to

$$I_{xc} = b \left. \frac{y^3}{3} \right|_{-h/2}^{h/2} = \frac{bh^3}{12} . \tag{2.41}$$

[2.8] This theorem, introduced in Eq. (2.32), is straightforward to prove. With respect to Fig. 2.8, let $r = d + h/2$ represent the fixed distance to the centroid. Then $I_x = \int y''^2 \, dA$ can be written in the frame of reference of the centroid as $I_x = \int (r+y)^2 dA$ since $y'' = r+y$ according to the figure. This gives $I_x = r^2 \int dA + 2r \int y dA + \int y^2 dA$. The first integral is simply the area while the second must vanish since y is in the frame of reference of the centroid, i.e. evaluating this integral yields a centroid of $y = 0$. The last integral is simply the moment of inertia about the centroid I_{xc}. We thus obtain the required theorem: $I_x = I_{xc} + Ar^2 = I_{xc} + Ay_c^2$.

If we were to re–compute I_x via the Parallel Axis Theorem[2.9], we would then obtain the identical result as in Eq. (2.39).

Usually we have a less intuitive feel for the product of inertia, e.g. I_{xyc} which is taken with respect to the centroid, however, it is computed once again according to the same sort of differential calculation. According to its definition

$$I_{xyc} = \int x\, y\, dA\,, \tag{2.42}$$

we see that there will be a cancellation effect if either lines parallel to the x or the y axes (or both) are lines of symmetry. In these cases $I_{xyc} = 0$. This leads to

THEOREM 2.4. *If the submerged surface is symmetric with respect to an axis passing through the centroid and parallel to either the x or y axis (or both), then $x_r = x_c$.*

Once again referring Fig. 2.8, we can show for example

$$I_{xyc} = \int x\, y\, dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} x\, y\, dx\, dy = \int_{-b/2}^{b/2} x dx \int_{-h/2}^{h/2} y dy\,. \tag{2.43}$$

These terms will have the forms x^2 and y^2 which will clearly cancel over the symmetric limits. Note that this is a special case where contributions in *both* coordinate directions cancel. We hope it is clear, however, that it only requires symmetry in a single direction to realize this effect.

The rectangular plate bears additional mention because it arises so frequently in problems. We generically call this the "dam problem" since it involves hydrostatic forces acting on a vertical surface (Fig. 2.9). In Fig. 2.9(a), we find for an arbitrary width b from Theorem 2.3 that the magnitude of the resultant force (depicted by the large arrow) will be $\gamma(h/2)(bh)$, where $h/2$ is the centroid of the surface. From Eq. (2.33), we find that the point of application would be at $h/3$ above the bottom of the dam[2.10]. This point also represents the centroid of the triangle swept out by the loci of the arrows. We note in Fig. 2.9(b) that distributions are additive. In particular,

---

[2.9] Again, the Parallel Axis Theorem says $I_x = I_{xc} + A(d + h/2)^2$, so that we would find

$$\begin{aligned} I_x &= \frac{bh^3}{12} + bh\left(d + \frac{h}{2}\right)\left(d + \frac{h}{2}\right) \\ &= \frac{bh^3}{12} + \left(\frac{bh^3}{4} + bdh^2 + bhd^2\right) \\ &= \frac{bh^3}{3} + bdh^2 + bhd^2 \end{aligned}$$

[2.10] We have as the location measured from the surface

$$y_r = \frac{I_{xc}}{y_c\, A} + y_c = \frac{bh^3/12}{h/2 \times bh} + \frac{h}{2} = \frac{bh^3}{6bh^2} + \frac{h}{2} = \frac{h}{6} + \frac{h}{2} = \frac{4h}{6} = \frac{2h}{3}\,,$$

which is the same as $h/3$, as measured from the bottom.

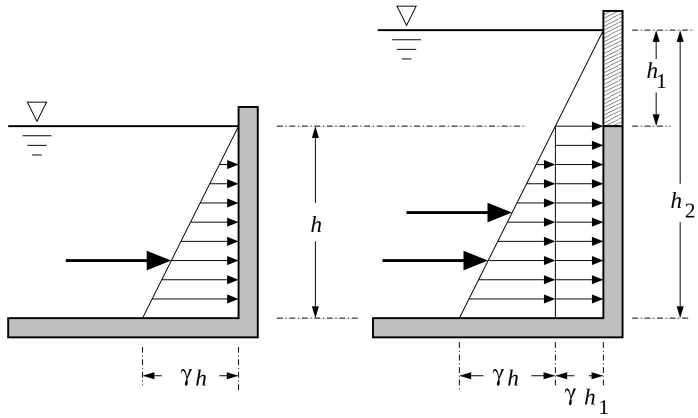

FIGURE 2.9. *Graphical representation of hydrostatic forces on a vertical rectangular surface.*

for the case where we require the loading on a surface that does not extend to the top of the fluid pool, we can break the pressure distribution into 2 contributions, the "triangular part", which is identical to that in Fig. 2.9(a) and a "square" part whose magnitude is γh_1 and point of application is at the centroid[2.11]

2.8. Archimedes' Principles of Buoyancy and Flotation

We are all familiar with the facts that boats "float" and that we feel lighter in a swimming pool. These phenomena arise because of hydrostatic pressures acting on the object in question. That is, a net upward force, called *buoyancy*, is generated because pressure forces acting from below are larger than those acting from above. These observations are embodied in Archimedes' Principles, in particular

THEOREM 2.5 (Buoyancy). *A body that is fully immersed in a fluid realizes a net upward buoyancy force equal to the weight of the fluid that it displaces.*

Archimedes' Principles

THEOREM 2.6 (Flotation). *A body that is partially immersed in a fluid (a floating body) displaces its own weight of the fluid in which it is floating.*

These theorems are straightforward to prove. For example, we assume a submerged body of arbitrary shape (Figure 2.10) for Thm. 2.5. For a vertical differential section through the body, the hydrostatic pressure on the top will be $\rho\ g\ y_1$, while the on the bottom it will be $\rho\ g\ y_2$. Here, ρ is

[2.11]This derives from the fact that the centroid of the rectangle swept out by the loci of the arrows is $h/2$ above the bottom.

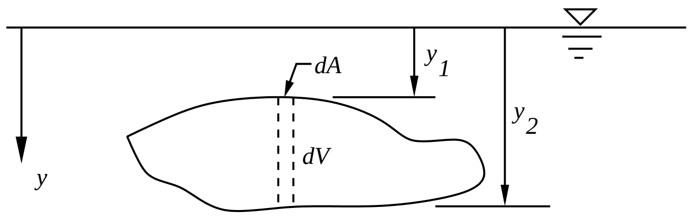

FIGURE 2.10. *Archimedes' Principle of buoyancy.*

the fluid density. The net buoyant upward force on this element will be

$$dF_B \;=\; \rho\, g\, y_2\, dA \;-\; \rho\, g\, y_1\, dA \;=\; \rho\, g\, dA\,(y_2 \;-\; y_1) \;=\; \rho\, g\, dV\ .$$

This expression can be integrated over the entire body as

$$F_B \;=\; \int dF_B \;=\; \int \rho\, g\, dV \;=\; \rho\, g \int dV \;=\; \rho\, g\, V \;=\; \gamma\, V\ .$$

Note that integration relied on the assumption that the fluid was incompressible, i.e. its density is constant. Had density been dependent upon depth, we could not have moved it directly outside of the integral. The specific weight of the fluid is γ and V is the volume of the body, which is equal to the volume of the fluid that the body displaces by its presence. Thm. 2.5 is thus proved. Thm. 2.6 is addressed via similar arguments.

<u>EXAMPLE 2.1:</u>

Legend has it that Archimedes' was asked in 220 B. C. to deduce the gold content in a crown made for King Hiero II of Syracuse. The crown was supposed to be solid gold, but there was a suspicion that it was only gold plating over a lighter, cheaper metal. How could Archimedes' answer the question without damaging the crown?

Archimedes could calculate the density of the crown: if this density is less than the density of gold then the crown contains impurities. First, determine the crown's weight W in still air. Denoting its density and volume as ρ_c and V, respectively, we have the relationship

$$W \;=\; \rho_c\, g\, V\ ,$$

however, ρ_c cannot be determined directly because V is not known. The second step is to weigh the crown when submerged in a liquid, say water having a known density ρ_w. The second weight would be less than the first by an amount equal to the net upward buoyancy force of $F_B = \rho_w\ g\ V$. Archimedes' would then know the volume of the crown as $V = F_B/\,(\rho_w\ g)$, from which he could calculate the density of the crown as

$$\rho_c \;=\; \frac{W}{g\, V} \;=\; \frac{W}{g}\,\frac{\rho_w\, g}{F_B} \;=\; \frac{\rho_w\, W}{F_B}\ .$$

This was surely one of the first techniques of non–destructive testing. According to legend, he found $\rho_c < \rho_{gold}$ so that the crown was a fraud. $\diamondsuit\diamondsuit\diamondsuit$

CHAPTER 3

Elementary Fluid Dynamics

Fluid phenomena are governed by the same physical laws you've already studied in statics, particle dynamics, etc., although they appear in slightly different form. In particular, Newton's Second Law, which relates applied forces to rates of change of momentum, is one of the basic principles of fluid dynamics. Here, we will focus on a special idealized Second Law description of fluid motion and use it to describe some elementary flow situations.

3.1. Newton's Second Law — The Bernoulli Equation

We introduced the Second Law in Chapter 2 as $\mathbf{F} = m\,\mathbf{a}$. The full translation of $\mathbf{F} = m\,\mathbf{a}$ into an equation valid for fluid flows is a fairly complicated matter which we'll discuss later. For now, we focus on an idealized case in which it is assumed that viscous forces[3.1] vanish. Conceptually, we can refer back to Eq. (1.5) on pp. 5, the definition of viscosity, and say that $\mu = 0$. Yet there are practical instances in which the assumption is also applicable, for example when the *product* of a small viscosity and a small rate of strain leads to very small values of τ. Flows in which viscous forces can be neglected are *inviscid flows*. Even in the presence of viscous forces, we may be interested in obtaining an approximate solution in order to estimate quantities of interest. Consequently, the inviscid case is an important phenomenon in its own right.

inviscid flow

Before we can derive this equation, let us introduce the concept of a *streamline*, which will form the basis of a coordinate system for our idealized Second Law.

definition of a streamline

Definition 3.1. *A streamline is a locus of points (curve) to which the local fluid velocity is tangent.*

Mathematically, let us define a radius vector $\mathbf{r}$ that specifies the local position of a particle in the fluid relative to an inertial coordinate system (Figure 3.1). An instant later, the particle has moved to $\mathbf{r} + d\mathbf{r}$ such that the vector tracing its path is simply the difference of these two vectors, i.e. $d\mathbf{r}$. If the velocity of the particle is $\mathbf{V}$, then the mathematical condition for a streamline is that the vector cross–product between $\mathbf{V}$ and $d\mathbf{r}$ must

[3.1] Viscous forces are synonymous with the frictional forces that arises via the action of viscosity. Our idealized case is therefore frictionless flow.

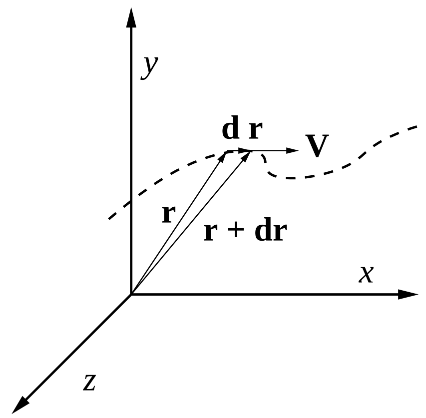

FIGURE 3.1. *Streamline in a flow field.*

vanish[3.2]

$$\mathbf{V} \times d\mathbf{r} = 0 \,. \tag{3.1}$$

We now use the concept of a streamline as an orthogonal curvilinear coordinate system. Since we have framed the definition of a streamline in terms of tangency, we immediately have the remaining coordinate of our system, the normal to the streamline. These observations suggest that at an instant of time, our coordinate system $(\hat{s}, \hat{n})$ is anchored to the point from which emanates the local radius of curvature r_c of the streamline. Since the velocity is tangent, we have $\mathbf{V} = v\hat{s} + 0\hat{n}$.

We can now write Newton's Second Law in terms of our streamline coordinate system. First, we determine the acceleration vector, which has a component along the streamline a_s and another component normal to the streamline a_n. The streamwise component arises because the speed of the particle along the streamline is not constant. If we denote a particle's position as a function of time along the streamline $s = s(t)$, we can compute its speed by simple differentiation as $v = ds/dt$. We can utilize this result in computing a_s by the Chain Rule of Calculus

$$a_s = \frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{ds}{dt} = v\,\frac{\partial v}{\partial s} \,. \tag{3.2}$$

From the idea of the instant center, we know that the normal component is equal to the centrifugal acceleration

$$a_n = \frac{v^2}{r_c} \,. \tag{3.3}$$

[3.2]Recall that the magnitude of the vector cross–product is $|\mathbf{V}|\ |d\mathbf{r}| \sin\alpha$, where α is the angle between the two vectors (Kreyszig, 1988). Their directions are coincident if $\alpha = 0$, which implies that the velocity vector is tangent to the streamline.

Next, let us determine the forces acting on a fluid particle. Absent viscous forces, fluid motion is governed by pressure forces and gravity forces. That is, the Second Law has the form:

$$\Sigma \mathbf{F_p} + \Sigma \mathbf{F_g} = m\,\mathbf{a}\,, \tag{3.4}$$

where $\mathbf{F_p}$ and $\mathbf{F_g}$ are the net pressure and gravity forces, respectively and the components of $\mathbf{a}$ are computed as in Eqs. (3.2) and (3.3). Forces can be computed according to a differential analysis similar to the one we conducted in deriving the hydrostatic equation in Chapter 2. Here, we analyze a differential element in streamline coordinates (Figure 3.2). Similar to Eqs. (2.5)

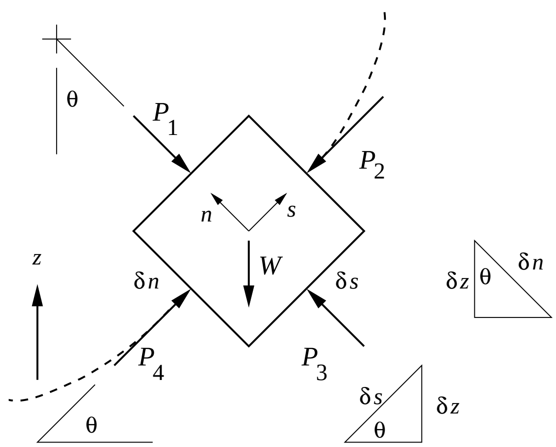

FIGURE 3.2. *Differential analysis of pressure and gravity forces in streamline coordinates.*

to (2.8) on pp. 9, if we let the pressure at the geometric center of the element be P, we can expand P in terms of 1–term truncated Taylor series to obtain pressure at the faces of the element as

$$P_1 = P + \frac{\partial P}{\partial n}\frac{\delta n}{2}\,, \tag{3.5}$$

$$P_2 = P + \frac{\partial P}{\partial s}\frac{\delta s}{2}\,, \tag{3.6}$$

$$P_3 = P - \frac{\partial P}{\partial n}\frac{\delta n}{2}\,, \tag{3.7}$$

$$P_4 = P - \frac{\partial P}{\partial s}\frac{\delta s}{2}\,. \tag{3.8}$$

Forces are then obtained by multiplying pressures by the corresponding areas over which they act

$$F_1 = \left(P + \frac{\partial P}{\partial n}\frac{\delta n}{2}\right)\delta s\,\delta y\,, \tag{3.9}$$

$$F_2 \;=\; \left(P \;+\; \frac{\partial P}{\partial s}\,\frac{\delta s}{2}\right)\;\delta n\,\delta y\;, \tag{3.10}$$

$$F_3 \;=\; \left(P \;-\; \frac{\partial P}{\partial n}\,\frac{\delta n}{2}\right)\;\delta s\,\delta y\;, \tag{3.11}$$

$$F_4 \;=\; \left(P \;-\; \frac{\partial P}{\partial s}\,\frac{\delta s}{2}\right)\;\delta n\,\delta y\;. \tag{3.12}$$

Therefore, we can write the net force due to pressure in the streamline direction as $F_4 - F_2$ to obtain

$$F_{ps} \;=\; -\,\frac{\partial P}{\partial s}\,\delta s\,\delta n\,\delta y\;. \tag{3.13}$$

Also, note that the net gravity force is simply due to the component of the fluid weight along the streamline

$$F_{gs} \;=\; -\,\gamma\,\sin\theta\,\delta s\,\delta n\,\delta y\;. \tag{3.14}$$

Now, noting that the mass of the element is simply ρ times the volume $\delta s\,\delta n\,\delta y$, we substitute all quantities to get the Second Law in the streamline direction

$$\rho\,v\,\frac{\partial v}{\partial s} \;=\; -\,\gamma\,\sin\theta \;-\; \frac{\partial P}{\partial s}\;. \tag{3.15}$$

The question now becomes how to integrate this differential equation to obtain practical results. First, we see that $\sin\theta = dz/ds$. Also, we re–cast

$$v\,\frac{dv}{ds} \rightarrow \frac{1}{2}\,\frac{dv^2}{ds}\;.$$

Next, we convert the partial derivative of pressure to an ordinary derivative via a clever observation which relies on the fact that along a streamline the normal coordinate is constant. In other words, $P = P(s,n)$ in general, but in a direction strictly along the streamline the derivative in the normal direction vanishes $dn = 0$ so that the Chain Rule yields

$$dP \;=\; \frac{\partial P}{\partial s}\,ds \;+\; \frac{\partial P}{\partial n}\,dn\;. \tag{3.16}$$

Since $dn = 0$ we obtain $dP/ds = \partial P/\partial s$. We can then cast Eq. (3.15) as

$$-\,\gamma\,\frac{dz}{ds} \;-\; \frac{dP}{ds} \;=\; \frac{1}{2}\,\rho\,\frac{d\left(v^2\right)}{ds}\;, \tag{3.17}$$

which simplifies to

$$dP \;+\; \frac{1}{2}\,\rho\,d\left(v^2\right) \;+\; \gamma\,dz \;=\; 0\;. \tag{3.18}$$

It is important to remember that Eq. (3.18) is valid only along a streamline. We can now integrate each term. Note that this is possible under the assumption of incompressible flow since density is constant. If this were not the case, we would need to specify additional information. Integrating each term, we obtain

$$P \,+\, \frac{1}{2}\,\rho\, v^2 \,+\, \gamma\, z \,=\, C_1\,, \tag{3.19}$$

where C_1 is a constant. Eq. (3.19) is the *Bernoulli Equation*. Let us recall one more time the assumptions that this equation is based upon

Bernoulli Equation

- viscous effects are negligible
- flow is steady
- flow is incompressible
- applicable along a streamline

The constant C_1 can be evaluated in certain cases where there is enough information available at a single location.

Likewise, we can write Newton's Second Law in the normal direction. Recall from Eq. (3.3) that the acceleration in the normal direction, directed along the vector pointing toward the instant center, is $a_n = v^2/r_c$. Moreover, recall that the mass of the element is simply ρ times the volume $\delta s\,\delta n\,\delta y$. The product of these quantities will give us the "right hand side" of the Second Law equation. The sum of the forces acting on the fluid element in Fig. 3.2 can be calculated in the normal direction in same manner as was performed in the streamline direction. The net pressure force in the normal direction is simply $P_3 - P_1$, which yields

$$F_{pn} \,=\, -\,\frac{\partial P}{\partial n}\,\delta s\,\delta n\,\delta y \tag{3.20}$$

and the net gravity force is simply due to the component of the fluid weight in the normal direction

$$F_{gn} \,=\, -\,\gamma\,\cos\theta\,\delta s\,\delta n\,\delta y\,. \tag{3.21}$$

These quantities give us the "left hand side" of the equation. Also, from Fig. 3.2, we see by trigonometry that $\cos\theta = dz/dn$. Combining all of this information, we find

$$-\,\gamma\,\frac{dz}{dn} \,-\, \frac{\partial P}{\partial n} \,=\, \rho\,\frac{v^2}{r_c}\,, \tag{3.22}$$

If the fluid is a gas, the first term would be neglected according to the fact that γ is small. We can take this one step further by using the same argument from Eq. (3.16) to show that $\partial P/\partial n \to dP/dn$ at a constant s, i.e. $ds = 0$. The equation can be arranged as

$$dP \,+\, \rho\,\frac{v^2}{r_c}\,dn \,+\, \gamma\,dz \,=\, 0\,. \tag{3.23}$$

In this case, when we attempt to integrate the equation, i.e.

$$\int dP \,+\, \int \rho\,\frac{v^2}{r_c}\,dn \,+\, \int \gamma\,dz \,=\, 0\,, \tag{3.24}$$

we run into a wrinkle. While the first (pressure) and last (gravity) terms are straightforward, the second term (acceleration) cannot be integrated since we do not know how v and r_c vary as functions of n. Recall in Eq. (3.18) that the acceleration term appeared as the differential itself, so this was easily integrated. Here, v and r_c are separate arguments in an integral of n. Therefore, the most we can say for incompressible flow is

$$P + \rho \int \frac{v^2}{r_c}\, dn + \gamma z = C_2 \,, \tag{3.25}$$

where C_2 is a constant[3.3].

3.2. Physical Interpretation of the Bernoulli Equation

We have derived the two components of the Bernoulli Equation[3.4] in Eqs. (3.19) and (3.25). Both these equations are actually statements of the conservation of energy. Recall that they were derived under the concept of conservation of forces, however, once we integrated them, they became energy statements. In fact, Eqs. (3.19) and (3.25) are merely statements of the work–energy principle (Beer and Johnston, 1984) familiar from the dynamics of a particle

Theorem 3.1 (Work–Energy Principle). *The net work exerted on a particle by applied forces is equal to the change in kinetic energy of the particle.*

In normal engineering practice, Eq. (3.25) is not as useful as Eq. (3.19), although it does account for some interesting physical phenomena. Essentially, Eq. (3.25) says that a change of flow direction, i.e. flow along a curved path $r_c < \infty$, is accomplished by a combination of pressure gradient and weight contribution in a direction normal to the flow. Larger speed and/or density and smaller radius of curvature requires larger force to produce the motion. Consider the special case of a gas where γ can be neglected. The normal component of the Bernoulli Equation becomes

MY&O Ex. 3.4 pp 105

$$\frac{\partial P}{\partial n} = -\rho \frac{v^2}{r_c} \,, \tag{3.26}$$

which shows that the pressure increases with distance away from the center[3.5]. This explains, for example, the fact that the pressure outside of a tornado (atmospheric pressure) is much higher than at its center.

[3.3] This equation appears incorrectly in Munson et al. (2006), i.e. their Equation 3.12 on page 111. They used y in place of γ.

[3.4] Technically, both Eqs. (3.19) and (3.25) make up the full Bernoulli Equation, however, when speaking generically of the "Bernoulli Equation", engineers are typically referring to Eq. (3.19) alone.

[3.5] The individual variables on the right hand side of Eq. (3.26) must all be positive, therefore, the right hand side as a whole is negative. This makes the pressure gradient $\partial P/\partial n$ negative, i.e. the rate of change of pressure along n is negative, meaning that the further along n we travel, the lower the pressure is. However, since we defined the normal direction to point toward the center, the pressure actually *increases* as we move away from the center.

Notice that in Eq. (3.19), every term carries units of force per unit area

- P : pressure is defined in terms of force per unit area
- ρv^2 : density is mass per length dimension to the third power and v^2 is the square of length dimension over time dimension. Their product gives force times length per unit volume, which is equivalent to force per unit area.
- γz : specific weight is force per unit volume. Multiplying by z gives force times length per unit volume, which is again equivalent to force per unit area.

Thus, we can speak of these terms as various kinds of pressures. For instance, P is simply the *static pressure* we would experience as an observer moving along (static to) the flow. In the simple fluid device shown in Fig. 3.3, we could measure P by placing a manometer at port 3. We can apply the

static pressure

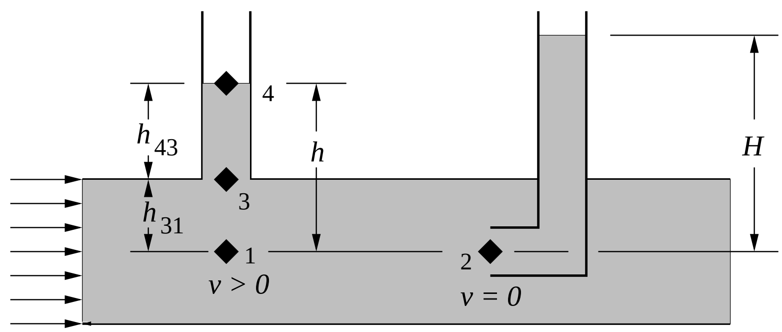

FIGURE 3.3. *Measurement of static and stagnation pressures.*

normal form of the Bernoulli Equation (3.25) *across* the streamlines, which run horizontally along the device centerline. Since the device is straight, the radius of curvature is $r_c \to \infty$, so Eq. (3.25) simplifies to

$$P + \gamma z = C_2 . \tag{3.27}$$

Thus, we see that, even though the fluid is in motion, in this case pressure changes in the vertical direction are equivalent to those in hydrostatics[3.6]. Therefore, this becomes a manometer problem, and we measure $P_3 = \gamma h_{43}$ and $P_1 = \gamma h$.

The term γz in Eq. (3.19) is the *hydrostatic pressure* in obvious regard to the principles introduced in Chapter 2. It represents changes in pressure due to potential energy variations as a result of elevation changes. The term $\rho v^2/2$ is the *dynamic pressure* and its meaning is interpreted as follows. A manometer tube is inserted into the flow stream with one end pointing upstream and the other one open to the atmosphere. Fluid will come to a stop at point 2 and the manometer level will rise to a height H above this

hydrostatic pressure

dynamic pressure

[3.6]If the device were to be curved, pressure changes *would not be equivalent* to the hydrostatic case!

point. Point 2 is termed a *stagnation point*. Now, assuming once again that streamlines follow the horizontal contour of the device, we can apply the streamwise Bernoulli Equation between points 1 and 2 to obtain

$$P_1 + \frac{1}{2}\,\rho\,v_1^2 + \gamma\,z_1 = P_2 + \frac{1}{2}\,\rho\,v_2^2 + \gamma\,z_2\,. \tag{3.28}$$

We can simplify Eq. (3.28) by noticing that $v_2 = 0$ and $z_1 = z_2$. This gives

$$P_2 = P_1 + \frac{1}{2}\,\rho\,v_1^2\,. \tag{3.29}$$

Since point 2 is a stagnation point, P_2 is a *stagnation pressure* and it is greater than the static pressure P_1 by an amount $\rho\,v^2/2$. In other words, static pressure plus dynamic pressure yields stagnation pressure. If we conceptualize any flow stream in which a body is placed, it is clear that there will be at least 1 stagnation point on the body. If elevation effects are neglected, then the stagnation pressure is the largest pressure that can be obtained along a given streamline. It represents conversion of the all the available kinetic energy into pressure head. stagnation pressure

The sum of all three pressures in Eq. (3.19) is termed the *total pressure*. Clearly, this equation states that the total pressure remains constant along a streamline, i.e. there is no pressure loss[3.7].

3.3. Example Applications of the Bernoulli Equation

The Bernoulli equation is useful for many practical flow calculations. For example, Eq. (3.29) is the basis for measuring flow speeds using a *Pitot–static tube* (Figure 3.4). Specifically, port 2 is once again a stagnation point Pitot–static tube

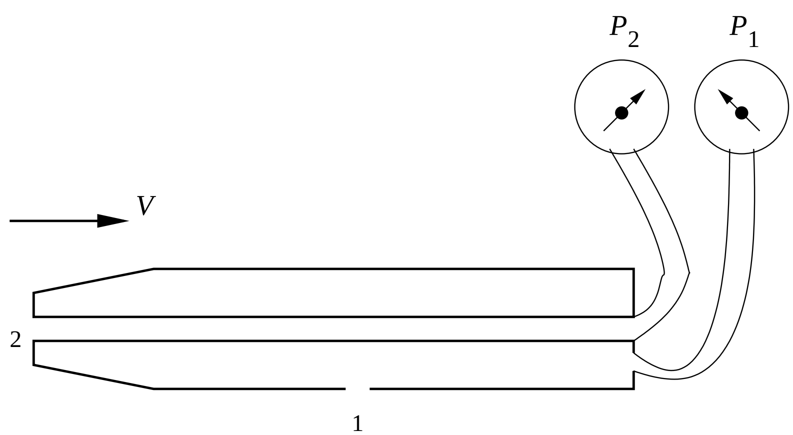

FIGURE 3.4. *Pressure taps on a Pitot–Static tube.*

in Fig. 3.4, so it measures the sum of the static plus dynamic pressures,

[3.7] We will see quite the opposite when we arrive at the point of studying viscous (frictional) flows.

i.e. the stagnation pressure P_2. Conversely, port 1 only measure the static pressure P_1. Their difference is the dynamic pressure, which can be solved to yield flow speed.

$$v = \sqrt{2\,(P_2 - P_1)/\rho}\,. \tag{3.30}$$

Many other other cases can be derived from appropriate simplifications of Eq. (3.28). Perhaps the most fundamental configuration is the draining a tank of liquid (Figure 3.5). Approximate streamlines are shown for a flow

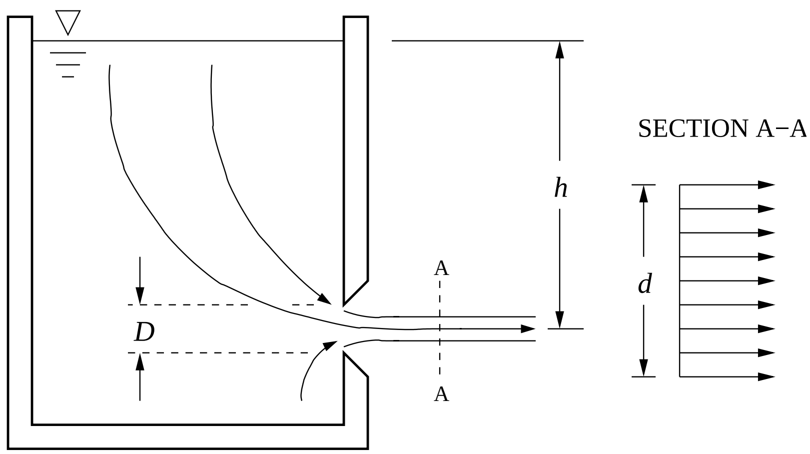

FIGURE 3.5. *Free jet of liquid from draining a tank.*

that develops due hydrostatic pressure of the elevated fluid. We can write Eq. (3.28) connecting any two points along any of these streamlines. Let us choose a point on the free surface and a point at the edge of the flow stream (section A—A in Fig. 3.5). Assuming the diameter of the tank is large with respect to the hole, then the velocity of any point on the liquid free surface will be small, so its square will be even smaller. Also, with respect to a gage reading, the pressure at the surface is zero. This leaves only the "potential energy" contribution, which can be expressed via γh. In the stream, the gage reading of pressure is zero[3.8] and the potential energy contribution is also zero, however, the stream speed is v. Thus, we see that Eq. (3.28) simplifies to

$$\gamma\, h = \frac{1}{2}\,\rho\, v^2\,. \tag{3.31}$$

[3.8]This is a reasonable approximation for a straight "free jet". In particular, since the streamlines are straight (infinite radius of curvature), we can apply Eq. (3.27) which states that the vertical distribution of pressure obeys the hydrostatic law. However, we have assumed that the diameter of the stream is small, therefore, vertical pressure changes are negligible, i.e. the pressure is uniform. Since the pressure at the top edge of the stream is atmospheric, it follows that the pressure throughout the stream must also be atmospheric.

Substituting the definition $\gamma = \rho g$, we can solve Eq. (3.31) to obtain

$$v = \sqrt{2\,g\,h}\,, \tag{3.32}$$

which is sometimes referred to as *Torricelli's Law*. Essentially, it states that the potential energy of the fluid is converted entirely to kinetic energy in the stream, which is consistent with neglecting friction and the fact that no pressure work is performed. We note that this result is identical to the motion[3.9] of a uniformly accelerated particle, e.g. a particle falling freely from height $z = h$ to the ground $z = 0$ (Beer and Johnston, 1984). Torricelli's Law

We can extend these results to measure the *mass flow rate* of the jet. That is, the amount of liquid that drains per unit time is simply mass flow rate

$$\dot{m} = \rho\,v\,A_j\,, \tag{3.33}$$

where A_j is the cross–sectional area of the jet stream[3.10]. We must mention a subtle point here having to do with dynamic effects at the orifice exit. We applied the Bernoulli equation at section A—A in Fig. 3.5; Velocity was presumed essentially uniform because we invoked the argument of constant pressure. Assuming the jet is round, the mass flow rate is calculated as $\dot{m} = \rho v \pi d^2/4$. However, we do not know d and, in general, d is less than the orifice exit size D. This is called the *vena contracta* phenomenon and arises because the stream cannot negotiate the sharp corner with an exact right–angle turn. The problem is that we cannot effectively apply the Bernoulli equation at the exit orifice because we do not know how the pressure varies in the stream. Here, we would have to account for streamline curvature, which implies a non–uniform pressure. Specifically, since the radius of curvature of a streamline is not zero, there will be a pressure change in the normal direction through the stream, so it would be impossible to obtain a simple relationship between h and v. Fortunately, we can wrap all these dynamic effects into an empirically determined coefficient which depends on the orifice geometry[3.11]. This parameter is called the *contraction coefficient* and is defined simply as the ratio of the area of the vena contracta (jet) A_j to the area of the orifice exit A_e *vena contracta* contraction coefficient

$$C_c = A_j/A_e\,, \tag{3.34}$$

which implies the mass flow rate given in Eq. (3.33) can be expressed as

$$\dot{m} = C_c\,\rho\,v\,A_e\,, \tag{3.35}$$

[3.9]This can be shown quite simply — Using Eq. (3.2), we can write particle's vertical acceleration due to gravity as $g = v\,dv/dz$, which can be integrated between $z = 0$ and $z = h$ as $\int_0^v v\,dv = g\int_0^h dz$. This yields $v^2 = 2\,g\,h$.

[3.10]This equation is valid only for the special case of a uniform velocity profile. Later we will examine a more general form that is valid for non–uniform profiles.

[3.11]The exact geometry affects the curvature of the streamlines and thus the pressure distribution through the stream.

The exit geometry in Fig. 3.5 has a contraction coefficient of approximately $C_c \approx 0.61$. More values can be found in standard texts and handbooks, e.g. Munson et al. (2006).

The Bernoulli equation can also be utilized in cases of confined flow, e.g. pipes and nozzles, where the pressure cannot be determined *a priori* as it was for free jets. Here we must also make use of a simple *conservation of mass* principle[3.12] which says the rate of inflow into a control volume must equal the rate of outflow. In other words

$$\dot{m}_{in} = \dot{m}_{out} \, , \tag{3.36}$$

which simplifies further to

$$Q = A_{in}\, v_{in} = A_{out}\, v_{out} \tag{3.37}$$

in light of the fact that density is constant for incompressible flow. Here, Q is the volume rate of flow, e.g. m^3/sec.

This extra equation allows us to examine internal flows, for example other types of flow metering devices which depend upon flow constrictions. The basic idea for all such devices is that constrictions cause a pressure difference, which can be measured, and the converted into a velocity or flow rate. Here the conversion occurs via the Bernoulli equation, but later we will also discuss corrections in cases where conditions for applying this equation are not identically satisfied. Consider a typical Venturi–type flow meter (Figure 3.6). Neglecting changes in elevation, the Bernoulli equation (3.28)

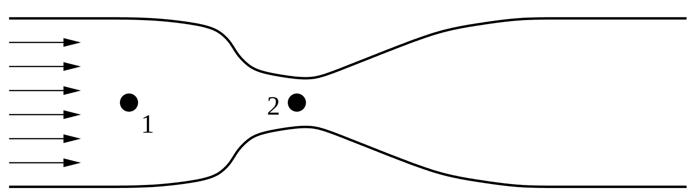

FIGURE 3.6. *A typical Venturi–type flow meter device.*

can be written between the upstream section and the throat section as

$$P_1 + \frac{1}{2}\,\rho\, v_1^2 = P_2 + \frac{1}{2}\,\rho\, v_2^2 \tag{3.38}$$

Assuming the velocity profiles are uniform in both regions, we can invoke mass conservation in the form of Eq. (3.37) to write another relationship for the velocities to obtain

$$Q = A_1\, v_1 = A_2\, v_2 \, . \tag{3.39}$$

[3.12] Like the mass flow rate, we presently will use a simplified form that will be generalized in a later discussion.

Eqs. (3.38) and (3.39) can be combined to obtain an idealized flow meter law[3.13]

$$Q = A_2 \sqrt{\frac{2(P_1 - P_2)}{\rho(1 - A_2^2/A_1^2)}}\,, \tag{3.40}$$

where $Q = A_1 v_1 = A_2 v_2$ is the volume flow rate. Like free jet orifices discussed above, "real world" performance of these flow meters is both flow and geometry dependent, e.g. there can be separation upstream and/or downstream of the constriction. *MY&O Ex. 3.7 pp 115* *MY&O Ex. 3.8 pp 116*

Open channel flows can also be metered using *sluice gates*. A sluice gate is a dam–like device which backs up the flow such that there is a reservoir height that can be measured, z_1. Flow is allowed to escape at the bottom, for which the downstream flow height can also be measured, z_2 (Figure 3.7). Upstream and downstream velocity profiles are assumed *MY&O Ex. 3.9 pp 118*

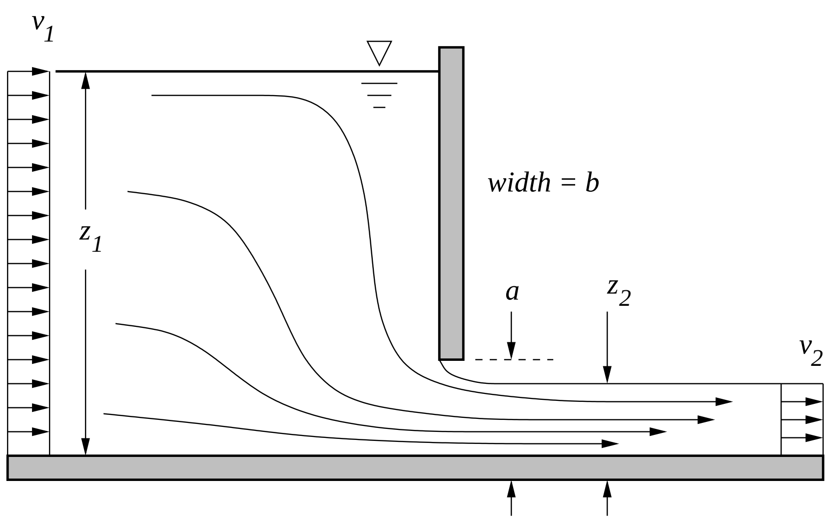

FIGURE 3.7. *Open channel flow metering using a sluice gate.*

uniform. We can apply Bernoulli's equation along a stream line emanating from the upstream free surface, where velocity is v_1, to the downstream free surface (i.e. downstream of the gate after the flow has straightened out), where velocity is v_2 to obtain an expression identical to Eq. (3.28). Assuming steady flow has been achieved, z_1 and z_2 are constant with respect to time. Therefore, conservation of mass dictates that the volume rate of flow upstream must be equal to the volume rate of flow downstream. If the channel has a width b, then we obtain

$$Q = A_1\, v_1 = b\, z_1\, v_1 = A_2\, v_2 = b\, z_2\, v_2\,. \tag{3.41}$$

[3.13]See Appendix B for derivation.

It is convenient to write the equation with respect to the two free surfaces because each is at atmospheric pressure, meaning that $P_1 = P_2$. Solving Eqs. (3.28) and (3.41), we find the flow rate to be governed by[3.14]

$$Q = z_2\, b \sqrt{\frac{2\, g(z_1 - z_2)}{1 - z_2^2/z_1^2}}\,. \tag{3.42}$$

Derivation is shown in Appendix B. If the outlet is small with respect to the reservoir height, $z_1 \gg z_2$, then Eq. (3.42) simplifies further to

$$Q = z_2\, b \sqrt{2\, g\, z_1}\,. \tag{3.43}$$

Eq. (3.43) implies that most of the energy upstream is in the form of potential energy, which is completely converted to kinetic energy in the downstream flow. Notice the similarity between Eqs. (3.43) and (3.32).

Like the draining problem in Fig. 3.5, we have the complication of the *vena contracta*, i.e. we can compute v_2 downstream but do not *a priori* know z_2. Conversely, we know the height at the mouth of the gate, but cannot effectively apply the Bernoulli equation to obtain the velocity distribution at that location. Once again, we employ the concept of an empirically–determined contraction coefficient introduced in Eq. (3.34), *MY&O Ex. 3.12 pp 124*

$$C_c = \frac{z_2\, b}{a\, b} = \frac{z_2}{a}\,, \tag{3.44}$$

where $C_c \approx 0.61$ for the upstream conditions $0 < a/z_1 < 0.2$ (Munson et al., 2006).

[3.14] Eq. (3.42) could alternatively be derived by writing a Bernoulli equation along a stream line connecting an upstream point and a downstream point both at $z = 0$. Since streamlines are straight, the pressure at these locations is simply equal to the hydrostatic pressure contribution, but the potential energies are zero. The net effect is that, while the potential energy terms vanish, they are replaced by pressure terms of equivalent magnitude. Eq. (3.42) then follows directly.

CHAPTER 4

Kinematics

Thus far, we have defined the basic properties of incompressible fluids and examined situations where they are either at rest or moving in an elementary fashion. For more general cases, it is useful to first study the *kinematics* of fluid motion, i.e. the velocity and acceleration components. We'll then be in a better position to study the dynamic portion having to do with the associated forces.

kinematics

4.1. The Velocity Distribution

We discussed the continuum assumption in Chapter 1 and, under this assumption, we defined the velocity at a point in Eq. (1.6). This implies that we can formulate the velocity of an entire flow field as a function of location. If the flow is unsteady, time would also be included. This leads to the definition of a velocity distribution[4.1]

$$\mathbf{V} \;=\; u\,(x,y,z,t)\,\hat{\imath} \;+\; v\,(x,y,z,t)\,\hat{\jmath} \;+\; w\,(x,y,z,t)\,\hat{k}\,, \tag{4.1}$$

where u, v, and w are the components of the velocity vector in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ coordinate directions. In other words, we can describe the velocity at any time and at any point in a flow field by a triplet of scalar functions. Velocity as described by Eq. (4.1) is a vector quantity, so it has both magnitude and direction. The magnitude of $\mathbf{V}$ is $|\mathbf{V}| = \sqrt{u^2 + v^2 + w^2}$ and this is properly called the *speed* of the fluid. One must be careful not to casually confuse these quantities. For example, acceleration arises via changes in velocity, which derive from changes in speed, or direction, or both.

Our formulation of Eq. (4.1) implies that we have identified an inertial coordinate system[4.2] as a frame of reference for values of (x, y, z). We therefore focus on points in (x, y, z) space, through which particles will flow. However, any particle at a particular (x, y, z) location will have properties characteristic of that particular location. For example, a fluid particle passing through a specific point (x_1, y_1, z_1) has a velocity determined by that particular location, but when it passes through another neighboring point (x_2, y_2, z_2) it will likely have a different velocity which is characteristic of that point. In other words, properties (in this case velocity) are given with

[4.1] This is also called a velocity field.

[4.2] Also known as an inertial frame of reference, Newtonian frame of reference, etc. This simply means that coordinate system is not accelerating.

respect to coordinates rather than particular fluid particles. This is called an *Eulerian* description of motion.

Eulerian description

Alternatively, one could follow individual particles through a flow stream, determining flow properties on a per particle basis simply as a function of time. This is called a *Lagrangian* description of motion. Eulerian and Lagrangian descriptions of motion have direct analogs to the following example. Suppose that a researcher wants to gather data on the yearly north–south migration patterns of a certain species of bird. The Lagrangian approach would involve tagging a sufficient number of birds with transmitters and setting up an automated monitoring system to track each individual bird's location as a function of time. Conversely, the Eulerian approach would require hiring a sufficient number of observers in various northern and southern cities to report how many birds move through. So, Eulerian descriptions are associated with fixed frames of reference while Lagrangian descriptions are associated with frames of reference attached to the moving body.

Lagrangian description

In fluid mechanics, we usually use the Eulerian description since the resulting equations are easier to apply (but not always straightforward to solve). This is the description we will apply from here on. Yet in some special cases the Lagrangian basis is more applicable, for example using a set of weather balloons to gather atmospheric data or radioactive tracers to analyze blood movement.

Eq. (4.1) gives the most general 3–dimensional representation of the velocity distribution, however, for many problems, only 2 of the 3 dimensions are important. In most of these instances, one of the components is much smaller than the remaining two and it can be neglected to a reasonable engineering approximation. The form is then

$$\mathbf{V} \;=\; u\,(x,y,t)\,\hat{\imath} \;+\; v\,(x,y,t)\,\hat{\jmath}\;, \tag{4.2}$$

where we have arbitrarily assumed that w is small relative to u and v and thus can be neglected. Recall in Chapter 3 we introduced the concept of a streamline (Definition 3.1) and used it as the basis for developing the Bernoulli equation. Now that we have formally defined velocity, we can derive a simple mathematical representation of streamlines. Later we will derive a more formal definition which is related directly to the principle of conservation of mass. In a two dimensional flow, the slope of a streamline, dy/dx, must be equal to the angle of the velocity vector, v/u, if the two are tangent, i.e.

$$\frac{dy}{dx} \;=\; \frac{v}{u}\;, \tag{4.3}$$

which can be integrated to a form $f(x, y) = C$, where C is a constant, if u and v are known as functions of x and y. The resulting equation is a mathematical description of a family of streamlines ψ, which depend upon C. For example, if $u = x$ and $v = -y$, we obtain $dy/dx = -y/x$, which can be integrated to the form $\ln y = -\,\ln x + C$. This can be developed further[4.3] to yield $x\,y = C_1 = \psi$, where C_1 is a new constant. Various values of $\psi = C_1$ then yield a family of streamlines. As we have already seen, streamlines are quite valuable from a theoretical standpoint. Two other entities are *streaklines*[4.4] and *pathlines*[4.5], however, these are more useful for experimental applications. For unsteady flow, these three entities are not necessarily the same, but for steady flow they are identical.

4.2. Acceleration

Eq. (4.1) allows us to describe fluid motion in a systematic fashion, however, in order to be able to apply Newton's Second Law, we must also develop a general mathematical description of acceleration. Like velocity, we must derive this in the context of an Eulerian frame of reference. Acceleration is simply the time rate of change of velocity, which can be due to changes in speed, direction, or both. Consider again a particle moving along a path, where its position is given by a radius vector $\mathbf{r} = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$. Because the particle is moving, its (x, y, z) coordinates are functions of time. In general, the velocity itself can also change with respect to time, so we can write velocity in a functional form as

$$\mathbf{V} \;=\; \mathbf{V}(\mathbf{r}, t) \;=\; \mathbf{V}\big(x(t), y(t), z(t), t\big)\;, \tag{4.4}$$

i.e. velocity is a function of time and position, and, furthermore, position is also a function of time. This is consistent with our formulation in Eq. (4.1), although it is shown in a slightly different form. A simple application of the Chain Rule then allows us to derive acceleration as the time rate of change of velocity in Eq. (4.4)

$$\mathbf{a} \;=\; \frac{d\mathbf{V}}{dt} \;=\; \frac{\partial \mathbf{V}}{\partial t} \;+\; \frac{\partial \mathbf{V}}{\partial x}\frac{dx}{dt} \;+\; \frac{\partial \mathbf{V}}{\partial y}\frac{dy}{dt} \;+\; \frac{\partial \mathbf{V}}{\partial z}\frac{dz}{dt}\;. \tag{4.5}$$

Upon inspection, we notice that the velocity components

$$u \;=\; \frac{dx}{dt} \qquad v \;=\; \frac{dy}{dt} \qquad w \;=\; \frac{dz}{dt}\;, \tag{4.6}$$

[4.3] Exponentiating both sides yields

$$e^{\ln y} \;=\; e^{-\;\ln x + C} \;=\; e^{-\;\ln x}\, e^{C} \;=\; C_1\, e^{-\;\ln x} \;=\; C_1\, \frac{1}{e^{\ln x}}\;.$$

Evaluating the exponentials, we find $y = C_1\, x^{-1}$, which shows that $x\,y = C_1 = \psi$ is the equation of a family of streamlines.

[4.4] A streakline consists of all particles in a flow which have previously passed through a specific common (x, y, z) point. In the lab, for example, a streakline can be produced by continuously injecting dye into a flow from a fixed point then taking a single exposure of the flow.

[4.5] A pathline is traced by a given particle as it flows from one point to another. In the lab, this could be produced by tagging a single particle and taking a series of closely–spaced exposures. Example 4.3 in Munson et al. (2006) illustrates these concepts.

which are the time rates of change of the positional coordinates, appear explicitly. Substituting we find

$$\mathbf{a} = \frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\,\frac{\partial \mathbf{V}}{\partial x} + v\,\frac{\partial \mathbf{V}}{\partial y} + w\,\frac{\partial \mathbf{V}}{\partial z}\,. \tag{4.7}$$

Eq. (4.7) is the Eulerian description of the acceleration, which is a vector quantity.

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Let us compare the vector properties of velocity and acceleration. Velocity has 3 components (u, v, w), defined in Eq. (4.6), oriented in the $(\hat{\imath}, \hat{\jmath}, \hat{k})$ directions, respectively. Acceleration also has 3 components, which can be written by substituting the appropriate components into Eq. (4.7). For example, in the $\hat{\imath}$ direction, the velocity component is u. We then substitute $\mathbf{V} \to u$ in Eq. (4.7) to obtain the $\hat{\imath}$ component of acceleration as

$$a_x = \frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\,. \tag{4.8}$$

Similarly, we can obtain the other 2 components as

$$a_y = \frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} \tag{4.9}$$

and

$$a_z = \frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\,. \tag{4.10}$$

Eqs. (4.8) through (4.10) are the (x, y, z) components of the acceleration vector given in Eq. (4.7).

Eq. (4.7) is often written in a conventional shorthand form

$$\mathbf{a} = \frac{D\mathbf{V}}{Dt}\,, \tag{4.11}$$

where the operator

$$\frac{D(\ \ )}{Dt} \equiv \frac{\partial(\ \ )}{\partial t} + u\,\frac{\partial(\ \ )}{\partial x} + v\,\frac{\partial(\ \ )}{\partial y} + w\,\frac{\partial(\ \ )}{\partial z} \tag{4.12}$$

is called the *total derivative*, *material derivative*, or *substantial derivative*. Mathematically, Eq. (4.7) can also be cast as[4.6]

$$\frac{D(\ \ )}{Dt} \equiv \frac{\partial(\ \ )}{\partial t} + \mathbf{V}\cdot\nabla(\ \ )\,, \tag{4.13}$$

where ∇ is the gradient operator

$$\nabla(\ \ ) \equiv \frac{\partial(\ \ )}{\partial x}\,\hat{\imath} + \frac{\partial(\ \ )}{\partial y}\,\hat{\jmath} + \frac{\partial(\ \ )}{\partial z}\,\hat{k}\,. \tag{4.14}$$

Eq. (4.13) prescribes the general rate–of–change operator for an Eulerian basis. It is important to note that Eq. (4.13) by itself is a scalar operator,

[4.6]Throughout this monograph we employ vector notation, as in e.g. Eq. (4.13). Another commonly found representation is the so–called Einstein summation notation (Einstein, 1916), which provides rules for summing components. This system is slightly more confusing for new students, but provides significant economy in writing long equations.

not a vector operator — The dot product between the vector quantities $\mathbf{V}$ and ∇ is a scalar. The nature of the result depends upon the *argument*: a vector argument, such as the velocity $\mathbf{V}$ will yield a vector result, as in Eq. (4.7). Conversely, a scalar argument yields a scalar result. For example, an important quantity in convection heat transfer is the rate of change of temperature (a scalar) in the Eulerian system. This is obtained as

$$\frac{DT}{Dt} \equiv \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} = \frac{\partial T}{\partial t} + \mathbf{V}\cdot\nabla T \tag{4.15}$$

by direct substitution into Eqs. (4.12) and (4.13).

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As shown in Eqs. (4.5) through (4.15), the total derivative involves two types of terms: those involving the *local derivative* with respect to time, $\partial/\partial t$, and the remainder involving the *convective derivative*, which encompasses velocities and local spatial derivatives. If a flow is steady, then the local acceleration vanishes, $\partial/\partial t \to 0$, so there are no changes in a flow parameter with respect to a fixed point in space. However, there *are* point–to–point changes in the parameter for a fluid particle as it moves about.

4.3. Reynolds' Transport Theorem

We have already discussed the Lagrangian versus Eulerian description of motion for infinitesimal fluid particles. If we amplify the scale, we can speak of a finite–size collection of fluid particles. Here, the analog of the Lagrangian description is the *system description*, whereby we would hope to analyze an identifiable mass of fluid as it moves about. Conversely, we could use a *control volume description*, which is analogous to the Eulerian system, whereby we prescribe a finite control volume in space and formulate analyses based on the fact that fluid moves into and out of this space. As before, we will find, in most cases, that it is more convenient to handle flow problems based on describing fixed points and volumes in space. It is quite difficult to identify and follow a specific mass of fluid as would be required with a *system* formulation. Moreover, we are often interested on the effect of a flow on a specific device fixed in space, e.g. the friction on the inside wall of a pipe. These problems lend themselves naturally to a control volume formulation.

Physical laws framed on the basis of a *system* description are often quite straightforward. For example, the law of conservation of mass can be stated as simply as "the mass of a system remains constant", or, equivalently, "the time rate of change of mass in a system is zero". Note the use of the word *system* rather than *control volume*. To utilize physical laws in terms of a control volume formulation, we need an analytical tool that permits to convert between the two bases. Reynolds' Transport Theorem is such a tool. We will employ this theorem to derive the equations of motion in the next chapter.

Reynolds Transport Theorem

Let us define B as any extensive[4.7] fluid parameter and b as its corresponding intensive[4.8] parameter, e.g. mass, kinetic energy, etc. The amount of this extensive property that a system possesses at any given instant, B_{sys}, can be determined simply by summing the amount associated with each fluid particle in the system. Specifically, if b is the parameter per unit mass then $b\,\rho$ specifies the parameter on a per unit volume basis. Then $b\,\rho\,dv$ is the amount of the parameter in a differential volume dv. This can be integrated over the entire volume of the system to obtain the total amount of the parameter as

$$B_{sys} = \int_v b\,\rho\,dv\,. \tag{4.16}$$

Eq. (4.16) is fairly straightforward, but what we are really interested in are the time rates of change of extensive properties. For example, the rate of change of momentum, $m\mathbf{V}$, is what we require to formulate Newton's Second Law for a fluid[4.9]. In the *system* formulation, we can simply take the time derivative of Eq. (4.16), however, it is not so simple in the *control volume* formulation because mass crosses control volume boundaries. We must therefore also consider the associated fluxes.

Consider the conceptual flow field in Figure 4.1. Frame (a) represents a

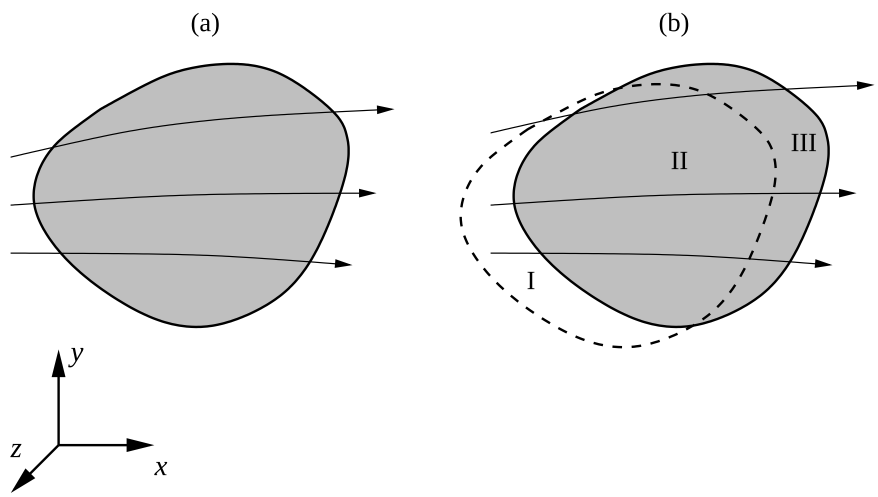

FIGURE 4.1. *System and control volume configuration: (a) at time t_0 the system and control volume coincide, (b) at time $t_0 + \Delta t$ the system has moved away from the fixed control volume.*

[4.7] *Extensive* implies that the value of the parameter being considered depends upon the amount of mass m associated with it.

[4.8] *Intensive* implies that the value of the parameter is independent of mass. That is, the parameter is given on a per unit mass basis. The relationship between extensive and intensive representations of a parameter is then $B = m\,b$.

[4.9] Note here that momentum per unit mass, or simply $\mathbf{V}$, is the intensive property.

control volume fixed in space, which happens to coincide with a "system" of fluid at a particular instant of time, t_0. The system must consist at all times of the same collection of fluid particles. Frame (b) shows the flow field an instant later at time $t_0 + \Delta t$, where the control volume remains fixed, but the system has moved. At this later time, the system encompasses regions II and III. During this time, more fluid, represented by region I, has entered the control volume, while the fluid in region III has left the control volume. First, let us simply write down the rate of change of B_{sys} with respect to time in terms of Fig. 4.1 as

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{B_{sys}|_{t_0+\Delta t} - B_{sys}|_{t_0}}{\Delta t} \,. \tag{4.17}$$

According to the fluxes crossing the control volume boundary we mentioned above, we can account for the total amount in the system at both t_0 and $t_0 + \Delta t$ as

$$B_{sys}|_{t_0} = B_{cv}|_{t_0} \tag{4.18}$$

and

$$B_{sys}|_{t_0+\Delta t} = (B_{II} + B_{III})|_{t_0+\Delta t} = (B_{cv} - B_I + B_{III})|_{t_0+\Delta t} \,. \tag{4.19}$$

Substituting into Eq. (4.17), we find

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{(B_{cv} - B_I + B_{III})|_{t_0+\Delta t} - B_{cv}|_{t_0}}{\Delta t} \,. \tag{4.20}$$

We can take the limit over each of these terms individually, i.e. Eq. (4.20) can be written

$$\left.\frac{dB}{dt}\right|_{sys} = \lim_{\Delta t \to 0} \frac{B_{cv}|_{t_0+\Delta t} - B_{cv}|_{t_0}}{\Delta t} + \lim_{\Delta t \to 0} \frac{B_{III}|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{B_I|_{t_0+\Delta t}}{\Delta t} \,. \tag{4.21}$$

We must now evaluate each of these terms individually. The first term is fairly straightforward: It is simply the derivative with respect to time of the amount in the control volume, i.e.

$$\lim_{\Delta t \to 0} \frac{B_{cv}|_{t_0+\Delta t} - B_{cv}|_{t_0}}{\Delta t} = \frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t} \int_{cv} b\, \rho\, dv \,. \tag{4.22}$$

The last component derives from the fact that the amount of extensive property in the control volume B_{cv} can be computed by integrating the intensive property over the volume, similar to Eq. (4.16).

For the second term in Eq. (4.21), we examine the flux B_{III} across the boundary at a differential area dA which has an outward–pointing unit normal vector $d\hat{n}$ (Figure 4.2). In general, the local velocity vector is not aligned with the unit normal vector. Therefore, assume that they differ by an angle θ, which must be in the range $-\pi \leq 2\theta \leq \pi$ under the assumption that mass is flowing *out* of the control volume. We can write the differential amount of extensive property that is in the differential volume of region III

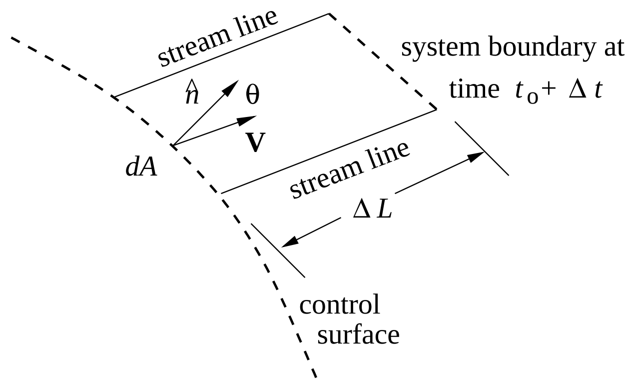

FIGURE 4.2. *Small differential area of the outflow region.*

simply as the product of intensive property, density, and differential volume, much like we did to obtain the the integrand of Eq. (4.16). Thus

$$dB_{III}|_{t_0+\Delta t} = (b\,\rho\,dv)|_{t_0+\Delta t} = (b\,\rho\,\Delta L\,\cos\theta\,dA)|_{t_0+\Delta t}\ . \tag{4.23}$$

The last term derives from the dot product of $\hat{n}dA$ and a vector of length ΔL directed along the streamline[4.10]. Now, to get the total amount of extensive property in region III, we can simply integrate Eq. (4.23) over the control volume bounding region III:

$$B_{III}|_{t_0+\Delta t} = \left[\int_{III} b\,\rho\,\Delta L\,\cos\theta\,dA\right]_{t_0+\Delta t}\ . \tag{4.24}$$

In turn, we can now substitute Eq. (4.24) into the corresponding quantity in Eq. (4.21) to obtain

$$\lim_{\Delta t\to 0}\frac{B_{III}|_{t_0+\Delta t}}{\Delta t} = \frac{\int_{III} b\,\rho\,\Delta L\,\cos\theta\,dA}{\Delta t}\ . \tag{4.25}$$

We are free to take the Δt under the integral sign, which gives

$$\lim_{\Delta t\to 0}\frac{B_{III}|_{t_0+\Delta t}}{\Delta t} = \int_{III} b\,\rho\,\frac{\Delta L}{\Delta t}\,\cos\theta\,dA\ . \tag{4.26}$$

By definition, the idea that a particle beginning on the control surface at t_0 travels along a streamline a distance ΔL is our representation of how the system volume moves. Because this takes place over an elapsed time of Δt, it represents the speed of the particle, i.e. the speed of the system. In the

[4.10]Let $\mathbf{a} = \hat{n}\,dA$ denote the vector normal to surface dA having length dA and let $\mathbf{b} = \Delta L\,\hat{s}$ be a vector directed along the streamline having length ΔL. Then, the volume of the parallelpiped swept out by surface dA and altitude ΔL is simply the dot product of these two vectors (Kreyszig, 1988), i.e. $\hat{n}\,dA\cdot\Delta L\,\hat{s} = |\hat{n}\,dA|\ |\hat{s}\,\Delta L|\ \cos\theta = dA\,\Delta L\,\cos\theta$, where θ is the angle between the two vectors.

limit $\Delta t \to 0$, we then obtain the magnitude of the velocity

$$\lim_{\Delta t \to 0} \frac{\Delta L}{\Delta t} = |\mathbf{V}| \; . \tag{4.27}$$

Moreover, we can express dA as the magnitude of the vector $|\hat{n}\, dA|$, where $\hat{n}$ is simply the unit normal vector. Thus, we obtain

$$\lim_{\Delta t \to 0} \frac{B_{III}|_{t_0+\Delta t}}{\Delta t} = \int_{III} b\, \rho\, |\mathbf{V}|\, \cos\theta\, |\hat{n}\, dA| \; . \tag{4.28}$$

We can go through a very similar process to find the remaining term characterizing region I in Eq. (4.21) as

$$\lim_{\Delta t \to 0} \frac{B_{I}|_{t_0+\Delta t}}{\Delta t} = - \int_{I} b\, \rho\, |\mathbf{V}|\, \cos\theta\, |\hat{n}\, dA| \; . \tag{4.29}$$

Eqs. (4.28) and (4.29) correspond to the surfaces of regions I and III, which together make up the total control surface of our control volume in Fig. 4.2. We can sum these terms by simply changing the integral to reflect the entire control surface. Let us at the same time resolve $|\mathbf{V}|\, \cos\theta\, |\hat{n}\, dA|$ back to the dot product $\mathbf{V} \cdot \hat{n}\, dA$. We then obtain

$$\lim_{\Delta t \to 0} \frac{B_{III}|_{t_0+\Delta t}}{\Delta t} - \lim_{\Delta t \to 0} \frac{B_{I}|_{t_0+\Delta t}}{\Delta t} = \int_{cs} b\, \rho\, \mathbf{V} \cdot \hat{n}\, dA \, . \tag{4.30}$$

Finally, we can substitute Eqs. (4.22) and (4.30) into (4.21) to obtain *Reynolds' Transport Theorem*

$$\left.\frac{dB}{dt}\right|_{sys} = \frac{\partial}{\partial t} \int_{cv} b\, \rho\, dv + \int_{cs} b\, \rho\, \mathbf{V} \cdot \hat{n}\, dA \, , \tag{4.31}$$

which gives the relationship between system and control volume descriptions of any extensive property.

To review, the individual parts of Eq. (4.31) represent:

- The first term is the rate of change with respect to time of any extensive system property
- The second term is the rate of change with respect to time of the extensive property within the control volume. The integral gives the amount of property within the control volume as introduced in Eq. (4.16) and $\partial/\partial t$ specifies its rate of change.
- The third term is the net rate of flux of the extensive property through the boundary (control surface). The product $\rho\, \mathbf{V} \cdot \hat{n}\, dA$ is the rate of mass flux through a differential portion of the boundary. For some areas, mass is being carried out, $\mathbf{V} \cdot \hat{n} > 0$, while for others it is being carried in, $\mathbf{V} \cdot \hat{n} < 0$. (Recall that $\hat{n}$ always points *outward*.) When multiplied by the intensive property b, this yields the differential amount of flux of the extensive property. When integrated over the whole boundary, this gives the net amount of extensive property flux.

Eq. (4.31) is a general result which will allow us to derive the integral form of the equations of motion for fluid flow in a straightforward, almost trivial fashion.

4.4. Generalization for Translating Control Volumes

The above discussion implicitly assumed that the control volume was fixed in space. Although valid for many problems, there will be some cases where the control volume must be considered in a more general capacity. For example, the configuration in Fig. 4.3 shows a control volume fixed to a cart that is translating to the right at a constant velocity, V_{cv}. The cart's

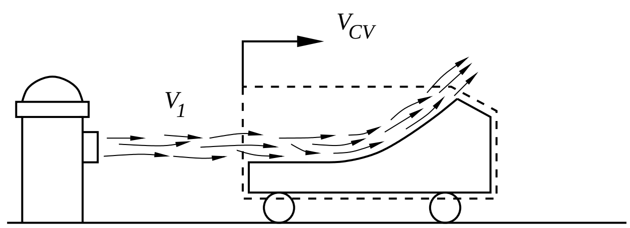

FIGURE 4.3. *Control volume translating at constant velocity.*

motion is driven by a stream of water moving at $V_1 > V_{cv}$. How do we modify Reynold's Transport Theorem for this more general case?

The primary difference between the fixed control volume and one that is translating is that we must now consider the *relative* nature of the fluid velocity at the control volume surfaces. For example, if the cart were to be fixed to the ground, fluid would cross the left control surface at velocity V_1. In the fixed frame of reference of the ground, V_1 is an absolute velocity. With the cart in motion (toward the right), the fluid is moving faster than the cart only by an amount $V_1 - V_{cv}$. In other words, fluid now crosses the left control surface at a relative velocity $W = V_1 - V_{cv}$. This relative velocity is the value that would be seen by an observer anchored to the cart itself, i.e. anchored to the moving control volume.

Velocity is, of course, a vector quantity. A simple graphical proof (Fig. 4.4) shows that the above conclusion is valid in general, i.e.

$$\mathbf{V} \;=\; \mathbf{W} \,+\, \mathbf{V}_{cv}\,. \tag{4.32}$$

If we were to go through the same derivation of Reynolds' Transport Theorem for this case, we would find simply that the absolute velocity $\mathbf{V}$ in Eq. (4.31) would simply be replaced by the relative velocity $\mathbf{W}$

$$\left.\frac{dB}{dt}\right|_{sys} \;=\; \frac{\partial}{\partial t}\int_{cv} b\,\rho\,dv \;+\; \int_{cs} b\,\rho\,\mathbf{W}\cdot\hat{n}\,dA\,. \tag{4.33}$$

This intuitively makes sense in light of the cart example. In particular, it is the relative velocity across the control surface that determines the flux of our parameter of interest.

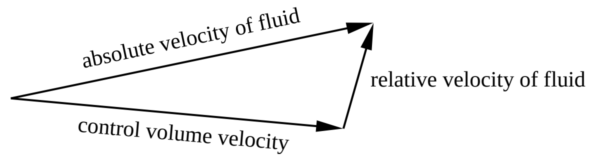

FIGURE 4.4. *Relationship among relative and absolute velocities and the control volume velocity.*

CHAPTER 5

Integral Analysis

Armed with a better understanding of the kinematics of fluid motion and Reynolds' Transport Theorem, we are now in a position to derive the integral form of the fluid equations of motion, i.e. the conservation equations for mass, momentum, and energy. These equations are applicable to a wide variety of practical engineering and science problems. All are based upon the concept of the control volume introduced in Chapter 4. We illustrate the principles with only the basic forms of the equations. Advanced concepts, such as deforming control volumes and accelerating frames of reference are not discussed. See e.g. Fox and McDonald (1998).

5.1. Conservation of Mass: The Continuity Equation

In Chapter 4, we stated the conceptual definition for the conservation of mass with respect to the system formulation as "the mass of a system remains constant". We can state this equivalently via the following theorem:

THEOREM 5.1 (Conservation of Mass of a System). *The time rate of change of mass in a system is zero.*

This concept is obvious in light of the fact that we defined a system to be a fixed collection of fluid particles which are not permitted to cross the boundary of the system. Unfortunately, we desire to construct problems in terms of the control volume formulation rather than the system formulation. Here is where we apply Reynolds' Transport Theorem, Eq. (4.31) on pp. 44, to derive a conservation of mass equation in the control volume formulation with practically no effort. Converting our conceptual definition of conservation of mass for a system into an equation, we obtain

$$\left.\frac{dm}{dt}\right|_{sys} = 0\,, \tag{5.1}$$

where m is the total mass of the system

$$m = \int_{v} \rho\, dv\,. \tag{5.2}$$

That is, m is the extensive property B in Eq. (4.31). If we recall the relationship between extensive and intensive representations of a parameter, $B = m\, b$, we see that, in this case, $b = 1$. In other words "mass per unit mass" is simply unity[5.1]. Using Eq. (5.1) and $b = 1$, we can substitute into Eq. (4.31) to obtain the conservation of mass equation, also called the continuity equation, for a control volume

$$0 = \frac{\partial}{\partial t} \int_{cv} \rho \, dv + \int_{cs} \rho \, \mathbf{V} \cdot \hat{n} \, dA \,. \tag{5.3}$$

Eq. (5.3) states that the time rate of change of mass within a control volume plus the net rate of mass flow (flux) across the boundary of the control volume must equal zero. It is completely general in the sense that we have not made any assumption about the fluid density, ρ. Specifically, we have not yet applied that fact the ρ is constant for incompressible flow.

If we now invoke the assumption of incompressible fluid, we can extract ρ, which is independent of both space and time, from under the integral signs

$$\rho \frac{\partial}{\partial t} \int_{cv} dv + \rho \int_{cs} \mathbf{V} \cdot \hat{n} \, dA = 0 \,. \tag{5.4}$$

The density is therefore simply a multiplicative factor which can be eliminated. Also, the control volume term can now be evaluated directly — it is simply the volume itself. This yields

$$\frac{\partial v}{\partial t} + \int_{cs} \mathbf{V} \cdot \hat{n} \, dA = 0 \,. \tag{5.5}$$

Let us invoke the additional restriction that the control volume is fixed, i.e. its volume is constant. This yields the basic continuity equation for incompressible flow

continuity equation for $\rho =$ constant

$$\int_{cs} \mathbf{V} \cdot \hat{n} \, dA = 0 \,. \tag{5.6}$$

The interesting aspect of Eq. (5.6) is that it is independent of time, however, *we have not assumed a steady flow!* We have only invoked the incompressible flow assumption and restricted our control volume to be fixed.

The dimensions of the integrand in Eq. (5.6) are $m/s \times m^2 = m^3/s$, which gives volume flow rate[5.2]. Therefore, Eq. (5.6) can be interpreted as saying that the volume rate of flow into a control volume must equal the volume rate of flow out of the control volume. This is perhaps intuitively obvious for incompressible fluids[5.3]. It follows from these observations and

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[5.1] Notice that $b = 1$ has already been incorporated into Eq. 5.2 in light of Eq. 4.16.

[5.2] Velocity has dimensions m/s and differential area has dimensions m^2. The outward unit normal vector $\hat{n}$ is dimensionless.

[5.3] For compressible fluids, ρ is variable and thereby can change within the control volume. For example, if the net rate of inflow of mass is *more* than the net rate of outflow, the fluid in the control volume will be compressed, i.e. ρ will increase, and the mass itself within the control volume will also increase. These factors are accounted for by the volumetric term in Eq. (5.3). The fact that density does not change for incompressible flow forces the rates of inflow and outflow to be equal.

Eq. (5.6) that we can compute the volume rate of flow across any bounding area or component of any control surface as

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$$Q = \int_A \mathbf{V} \cdot \hat{n} \, dA \,, \tag{5.7}$$

where Q is the volume flow rate and A is the total area of the boundary. Let us also define the magnitude of the average velocity $\overline{V}$ as the volume flow rate divided by the area: $\overline{V} = Q/A$, which gives

$$\overline{V} = \frac{Q}{A} = \frac{1}{A} \int_A \mathbf{V} \cdot \hat{n} \, dA \,, \tag{5.8}$$

Eqs. (5.7) and (5.8) are often convenient for incompressible flow calculations. For example, if we can frame a problem such that we can compute terms in Eqs. (5.7) and (5.8) at all locations where mass crosses control volume boundaries, we can compute the continuity equation as $\Sigma \, Q_{inflow} = \Sigma \, Q_{outflow}$ or, equivalently, $\Sigma \, (\overline{V} A)_{inflow} = \Sigma \, (\overline{V} A)_{outflow}$. In other words, the sum of all the volume flow rates of all incoming streams equals the sum of all the volume flow rates of all outgoing streams.

In §4.4, we derived the extension to Reynolds' Transport Theorem for a control volume moving at constant velocity. We found that we merely had to use relative velocity rather than absolute velocity to treat such configurations. Under this condition, Eq. (5.3) becomes

$$0 = \frac{\partial}{\partial t} \int_{cv} \rho \, dv + \int_{cs} \rho \, \mathbf{W} \cdot \hat{n} \, dA \tag{5.9}$$

and Eq. (5.6) becomes.

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$$\int_{cs} \mathbf{W} \cdot \hat{n} \, dA = 0 \,. \tag{5.10}$$

5.2. Newton's Second Law: The Momentum Equation

As we did for conservation of mass, we wish to develop Newton's Second Law for a control volume formulation. The procedure is once again the same: frame the conservation law in terms of a system formulation, then apply Eq. (4.31) on pp. 44 to transform it to a control volume basis. For a system, we obtain

$$\left. \frac{d}{dt} \, m\mathbf{V} \right|_{sys} = \Sigma \, \mathbf{F}_{sys} \,, \tag{5.11}$$

where $(m\mathbf{V})_{sys}$ is the total momentum of the system

$$(m\mathbf{V})_{sys} = \int_v \rho \, \mathbf{V} \, dv \,. \tag{5.12}$$

Here, $m\mathbf{V}$ is the extensive property B in Eq. (4.31). Once again, recalling the relationship between extensive and intensive representations of a parameter, $B = m \, b$, we see that, in this case, $b = \mathbf{V}$. In other words "momentum per unit mass" is simply velocity[5.4]. Eq. 5.11 is simply a restatement of Newton's Second Law for a system, i.e.

THEOREM 5.2 (Conservation of Momentum of a System). *The time rate of change of momentum of a system equals the sum of the forces acting upon it.*

We must now make a subtle point regarding forces applied to the system, $\Sigma\,\mathbf{F}_{sys}$ versus forces applied to the control volume, $\Sigma\,\mathbf{F}_{cv}$. Because we take the limit $\Delta t \to 0$ in deriving Reynolds' Transport Theorem, Eq. (4.31), the concept is valid at the instantaneous moment when the system and control volume coincide. Therefore the forces applied to the system are identical to the forces applied to the control volume, i.e. $\Sigma\,\mathbf{F}_{sys} = \Sigma\,\mathbf{F}_{cv}$. Using this observation, along with Eq. (5.11) and $b = \mathbf{V}$, we can substitute into Eq. (4.31) to obtain Newton's Second Law, also called the momentum equation, for a control volume

$$\Sigma\,\mathbf{F}_{cv} = \frac{\partial}{\partial t}\int_{cv} \mathbf{V}\,\rho\,dv + \int_{cs} \mathbf{V}\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,. \tag{5.13}$$

Eq. (5.13) states that the sum of the forces (both body and surface forces) acting on the control volume equals the rate of change of momentum of the material inside the control volume plus the net rate of momentum flux across the boundary of the control volume. Note that this equation is actually a vector equation having three components,

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$$\Sigma\,F_{cv,x} = \frac{\partial}{\partial t}\int_{cv} u\,\rho\,dv + \int_{cs} u\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,, \tag{5.14}$$

$$\Sigma\,F_{cv,y} = \frac{\partial}{\partial t}\int_{cv} v\,\rho\,dv + \int_{cs} v\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,, \tag{5.15}$$

and

$$\Sigma\,F_{cv,z} = \frac{\partial}{\partial t}\int_{cv} w\,\rho\,dv + \int_{cs} w\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,, \tag{5.16}$$

where $\mathbf{V} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$ contains the 3 velocity components (u, v, w) in the (x, y, z) directions from Eq. (4.1).

As we discussed in the previous section and in §4.4, a coordinate system moving at constant velocity is a useful generalization. Making the required substitution of $\mathbf{V}$ to $\mathbf{W}$, we find

$$\Sigma\,\mathbf{F}_{cv} = \frac{\partial}{\partial t}\int_{cv} \mathbf{V}\,\rho\,dv + \int_{cs} \mathbf{V}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA\,. \tag{5.17}$$

Notice that only the dot–product term was affected; This represents the contribution crossing the control volume boundary, which must be specified relative to the control volume motion itself. The variables that remain $\mathbf{V}$ represent our intensive property "momentum per unit mass", i.e. $b = \mathbf{V}$,

[5.4] As with the continuity equation, the intensive property $b = \mathbf{V}$ has already been incorporated into Eq. 5.12 in light of Eq. 4.16.

so these remain unchanged. However, further simplification can be made. First, let us assume a steady–state case, so that the $\partial/\partial t$ term vanishes. If we substitute Eq. (4.32) on pp. 45 for the absolute velocity, i.e. $\mathbf{V} = \mathbf{W} + \mathbf{V}_{cv}$, Eq. (5.17) can be written as

$$\begin{aligned}
\Sigma\,\mathbf{F}_{cv} &= \int_{cs} \mathbf{V}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \\
&= \int_{cs} (\mathbf{W} + \mathbf{V}_{cv})\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \\
&= \int_{cs} \mathbf{W}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \;+\; \int_{cs} \mathbf{V}_{cv}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \\
&= \int_{cs} \mathbf{W}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \;+\; \mathbf{V}_{cv}\,\rho \int_{cs} \mathbf{W}\cdot\hat{n}\,dA \\
\Sigma\,\mathbf{F}_{cv} &= \int_{cs} \mathbf{W}\,\rho\,\mathbf{W}\cdot\hat{n}\,dA \qquad (5.18)
\end{aligned}$$

The fourth line results from the fact that both $\mathbf{V}_{cv}$ and ρ are constant, while the fifth line, Eq. (5.18), results from invoking the continuity equation in Eq. (5.10). The last term vanishes. *MY&O Ex. 5.17 pp 219*

5.3. The Equation of Angular Momentum

Newton's Second Law *per se* governs conservation of linear momentum. In many engineering problems the moment of a force with respect to an axis (a torque) is important. While the linear momentum equation can be used to treat such problems, it is usually more convenient to formulate a special *angular momentum equation*[5.5], which relates rates of change of angular momentum to the applied torques. This, too, can be derived directly by formulating the principle from a system standpoint, then converting to a control volume basis.

Let us begin with a basic review of angular dynamics of a particle (Beer and Johnston, 1984). The relationship between force and torque is

$$\mathbf{T} \;=\; \mathbf{r} \times \mathbf{F}\,, \qquad (5.19)$$

where $\mathbf{r}$ is the radius vector from the origin of an inertial coordinate system to the point of application of the force $\mathbf{F}$ and the angular momentum is defined as

$$\mathbf{H} \;=\; \mathbf{r} \times m\,\mathbf{V}\,, \qquad (5.20)$$

where m is the mass of a particle[5.6]. Here, $\mathbf{H}$ is the extensive property in Eq. (4.31), since it depends upon mass. Once again, recalling the extensive–intensive relationship $B = m\,b$, we see that, in this case, $b = \mathbf{r} \times \mathbf{V}$ is

[5.5] This is also called the *moment of momentum equation*.

[5.6] The notation in Eq. (5.20) may at first appear to be ambiguous, however, recall that the cross product can only be taken between 2 vectors, therefore the $\times$ applies to $\mathbf{V}$, not m.

the intensive parameter. Thus, $\mathbf{r} \times \mathbf{V}$ is the "angular momentum per unit mass". To write down the total angular momentum of a system, we employ Eq. (4.16) on pp. 41

$$H_{sys} \;=\; (\mathbf{r} \times m\,\mathbf{V})_{sys} \;=\; \int_{\upsilon} \rho\,\mathbf{r} \times \mathbf{V}\,d\upsilon\,, \tag{5.21}$$

where υ again represents the volume of our system.

The principle of angular momentum in an inertial frame of reference is given by the following theorem (Beer and Johnston, 1984):

THEOREM 5.3 (Conservation of Momentum of a System). *The time rate of change of angular momentum of a system equals the sum of the torques acting upon it.*

Clearly, this is a form, albeit a special one, of Newton's Second Law. From this, we conclude that the "system" formulation of conservation of angular momentum is

$$\frac{d}{dt}\,\mathbf{H}\bigg|_{sys} \;=\; \frac{d}{dt}\,\mathbf{r} \times m\,\mathbf{V}\bigg|_{sys} \;=\; \Sigma\,\mathbf{T}_{sys} \;=\; \Sigma\,(\mathbf{r} \times \mathbf{F})_{sys}\,. \tag{5.22}$$

Now it is a simple process of applying Reynolds' Transport Theorem, again given by Eq. (4.31), to derive a conservation of angular momentum equation for a control volume formulation

$$\Sigma\,(\mathbf{r} \times \mathbf{F})_{cv} \;=\; \frac{\partial}{\partial t}\int_{cv} \mathbf{r} \times \mathbf{V}\,\rho\,d\upsilon \;+\; \int_{cs} \mathbf{r} \times \mathbf{V}\,\rho\,\mathbf{V} \cdot \hat{n}\,dA\,. \tag{5.23}$$

Note that we have applied a similar argument as with the linear momentum equation whereby system and control volume converge as $\Delta t \to 0$ such that $\Sigma\,(\mathbf{r} \times \mathbf{F})_{sys}$ is identical to $\Sigma\,(\mathbf{r} \times \mathbf{F})_{cv}$. The left hand side of Eq. (5.23) is the sum of all torques acting on the control volume. This can include surface forces, shaft torque, etc. The first term on the right hand side is simply the time rate of change of angular momentum within the control volume, while the second term is the net rate of flux of angular momentum across the boundary of the control volume. It is emphasized that the assumption of an inertial frame of reference is embedded in Eq. (5.23). It is sometimes natural to use a rotating coordinate system for such problems, however, *a rotating system is not inertial!* Extra terms compensate for this fact in a rotating frame (Fox and McDonald, 1998).

MY&O Ex. 5.18 pp 225

5.4. First Law of Thermodynamics: The Energy Equation

So far we have examined two independent conservation laws: conservation of mass and conservation of momentum (both linear and angular). A third basic law is the conservation of energy, one of the main concepts introduced in thermodynamics[5.7]. It too can be described by a fundamental theorem:

[5.7] The conservation of energy equation is also commonly known as the First Law of Thermodynamics.

THEOREM 5.4 (Conservation of Energy of a System). *The time rate of increase of energy of a system equals the net time rate of energy added by heat transfer plus the net time rate of work done on the system.*

Thus, we obtain the system representation as

$$\left.\frac{dE}{dt}\right|_{sys} = \dot{Q} + \dot{W}\,, \tag{5.24}$$

where $\dot{Q}$ is the rate of heat energy added, $\dot{W}$ is the rate of work, and E is the total energy of the system given by

$$E = \int_{v} e\,\rho\,dv\,. \tag{5.25}$$

Referring again to Eq. (4.31) on pp. 44, E is the extensive property B, and the extensive–intensive relationship, $B = m\,b$, shows that $b = e$, where e is energy per unit mass. Here we must consider the various types energy that belong to e: internal energy, kinetic energy, and potential energy.

$$e = u_{int} + \frac{V^2}{2} + gz\,, \tag{5.26}$$

where u_{int} is internal energy and $V^2/2$ and gz are kinetic and potential energy, respectively.

With the usual observation that

$$\left(\dot{Q} + \dot{W}\right)_{sys} = \left(\dot{Q} + \dot{W}\right)_{cv} \tag{5.27}$$

as a result of taking the limit $\Delta t \to 0$ such that the system and control volume entities converge, we can immediately write a conservation of energy equation for a control volume as

$$\left(\dot{Q} + \dot{W}\right)_{cv} = \frac{\partial}{\partial t}\int_{cv} e\,\rho\,dv + \int_{cs} e\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,. \tag{5.28}$$

Eq. (5.28) can be interpreted in the usual fashion according to Reynolds' Transport Theorem.

The heat transfer term, $\dot{Q}$, represents all the modes of transferring energy that arise because of a temperature gradient: radiation, conduction, and convection. However, we will treat $\dot{Q}$ generically here without delving into the specifics of these modes. Heat transfer directed into the control volume is considered positive, while heat transfer directed outward is negative.

Likewise, the work transfer term, $\dot{W}$, is positive when work is done *on* the control volume, otherwise it is considered negative. It represents all the varieties of work associated with the control volume. We will consider two main types of work: mechanical shaft work and work exerted by normal (pressure) stresses.

- Mechanical work can be transferred across the boundary of a control surface by one or more rotating shafts. The rate of work, or *power*, is simply the torque exerted by the shaft multiplied by the rate of rotation: $\dot{W}_{shaft} = T_{shaft}\,\omega$.

- Work is also exerted when a force acts over some distance. In fact, the amount of work is simply the dot product of the force and the differential distance, or, equivalently, the rate of work is the dot product of the force and velocity (Beer and Johnston, 1984), which can be expressed differentially as $\delta\dot{W}_{normal} = \delta\mathbf{F}\cdot\mathbf{V}$. The main contribution of interest in incompressible flow is from mechanical pressure. Therefore, the "force" is actually pressure P acting over a differential area dA oriented in a specific direction according to a unit normal vector $\hat{n}$. Since this work is not performed *on* the control volume by some external agent, it contribution is negative. That is $\delta\dot{W}_{normal} = -(P\,dA\,\hat{n})\cdot\mathbf{V}$, which can be written $\delta\dot{W}_{normal} = -P\,\mathbf{V}\cdot\hat{n}\,dA$. The total rate of work is then obtained by integrating this quantity over the entire boundary of the control volume: $\int_{cs} -P\,\mathbf{V}\cdot\hat{n}\,dA$.

Using these two more specific forms of work, we can write Eq. (5.28) as

$$\dot{Q} + \dot{W}_{shaft} - \int_{cs} P\,\mathbf{V}\cdot\hat{n}\,dA = \frac{\partial}{\partial t}\int_{cv} e\,\rho\,dv + \int_{cs} e\,\rho\,\mathbf{V}\cdot\hat{n}\,dA\,. \tag{5.29}$$

Of course, we can simplify Eq. (5.29) by multiplying P by ρ/ρ on the left hand side, moving this entire term to the right hand side, and incorporating it into the flux term. We then obtain the conservation of energy equation as

$$\dot{Q} + \dot{W}_{shaft} = \frac{\partial}{\partial t}\int_{cv} e\,\rho\,dv + \int_{cs}\left(e + \frac{P}{\rho}\right)\rho\,\mathbf{V}\cdot\hat{n}\,dA\,, \tag{5.30}$$

where e is once again given by Eq. (5.26). This equation can also be cast in terms of the *enthalpy* h_{int}, i.e. $h_{int} = u_{int} + P/\rho$, which gives enthalpy

$$\dot{Q} + \dot{W}_{shaft} = \frac{\partial}{\partial t}\int_{cv} e\,\rho\,dv + \int_{cs}\left(h_{int} + \frac{V^2}{2} + gz\right)\rho\,\mathbf{V}\cdot\hat{n}\,dA\,, \tag{5.31}$$

5.5. Relationship of the Energy and Bernoulli Equations

Let us assume a one–dimensional steady flow where the flow properties, i.e. velocity, pressure, internal energy, etc., are uniform over individual cross–sectional areas of the flow and there is no shaft work. This could be, for example, flow in a pipe or a streamtube where there's a cross–sectional area coming *in* to the control volume and another going *out* of the control volume (Figure 5.1). The unsteady term (the one having $\partial/\partial t$) vanishes and the boundary integral simplifies: $\rho\,\mathbf{V}\cdot\hat{n}\,dA \rightarrow \dot{m}$ and the integral vanishes. Expanding e into all of its corresponding terms then gives

$$\dot{Q} = \dot{m}\left[u_{int}|_2 - u_{int}|_1 + \frac{P_2 - P_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1)\right] \tag{5.32}$$

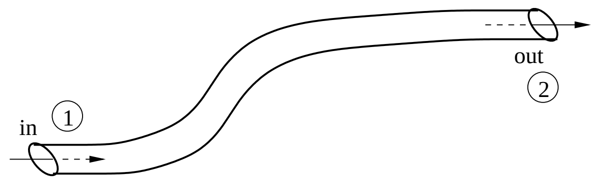

FIGURE 5.1. *One–dimensional steady flow in a stream tube. The only boundary locations where $\mathbf{V} \cdot \hat{n}\, dA \neq 0$ are at the "in" (1) and "out" (2) cross sections.*

Let us divide by the mass flow rate $\dot{m}$ and re–arrange terms so that inlet terms are on the left hand side and outlet terms are on the right. This gives

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 - \left(\Delta u_{int} - \frac{\dot{Q}}{\dot{m}}\right) = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2 \tag{5.33}$$

where $\Delta u_{int} = u_{int}|_2 - u_{int}|_1$ is the change in internal energy. Now, we notice, in light of Eq. (3.19), that Eq. (5.33) closely resembles the Bernoulli equation[5.8], except that there are the 2 extra terms $\Delta u_{int} - \dot{Q}/\dot{m}$ on the left hand side. In other words, if $\Delta u_{int} - \dot{Q}/\dot{m} = 0$, then Eq. (5.33) is identical to the Bernoulli Equation.

We said at the beginning of this section that we assumed a steady incompressible flow along a streamline in order to obtain Eq. (5.32). Of course, the true Bernoulli Equation also requires the flow to be *frictionless*[5.9]. We can therefore deduce that frictionless flow implies $\Delta u_{int} - \dot{Q}/\dot{m} = 0$. However, a study of the Second Law of Thermodynamics (Munson et al., 2006) indicates $\Delta u_{int} - \dot{Q}/\dot{m} \geq 0$, where the equality applies to frictionless flow and the $>$ applies to real "viscous" (frictional) flows. Therefore, we can classify this term as representing the frictional losses of a real flow, i.e.

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 - h_L = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2\,, \tag{5.34}$$

where h_L is a head loss.

In this discussion, we neglected shaft work $\dot{W}_{shaft}$, which we shall now re–introduce in a form commensurate with Eq. (5.34). That is, we divide by mass flow rate, so that *MY&O Ex. 5.23 pp 237*

$$h_S = \frac{\dot{W}_{shaft}}{\dot{m}}\,.$$

Checking signs carefully, we find that Eq. (5.34) becomes *MY&O Ex. 5.24 pp 239*

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 - h_L + h_S = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2\,, \tag{5.35}$$

[5.8] Actually, it resembles the Bernoulli equation where both sides of Eq. (3.19) have been divided by the density, ρ.

[5.9] The terms "frictionless" and "inviscid" are synonymous.

Eq. (5.35) is sometimes called the *extended Bernoulli equation*. It takes into account both head losses, h_L, and boosts, h_S, which result from shaft work.

MY&O Ex. 5.25 pp 241

You may have noticed something of a paradox in this development. Specifically, we derived Bernoulli's equation in Chapter 3 from Newton's Second Law, which is a statement of force. However, we now see that this same concept equates to an equation of energy, i.e. Eq. (5.33), etc. Clearly force and energy do not represent the same entities. How can this be?

If we go back to Chapter 3 and carefully examine how the Bernoulli equation was derived, we will recall that the force component was formulated with respect to a differential element along a streamline, i.e. Eq. (3.15) on pp. 26. This equation was then integrated over a streamline, i.e. a "distance", to obtain Eq. (3.19). The act of integrating effectively introduces the product of a force and a distance, thus making the Bernoulli equation a statement of energy.

CHAPTER 6

Differential Analysis

In Chapter 5 we discussed integral formulations of the governing equations. The integral form only enforces conservation around the boundary of a finite control volume, rather than throughout the control volume. In this sense, integral formulations are approximate, i.e. we cannot know the velocity and pressure distributions inside the control volume. In fact, we cannot even really know whether the conservation laws are satisfied within the control volume. It is essentially a "black box".

We will now extend our treatment of conservation laws using the classic differential approach. The resulting equations will be valid for every differential point in a flow domain, i.e. conservation laws must be satisfied simultaneously for all (x, y, z, t) in the flow. In this sense, the differential formulation is *exact*, however, it typically presents a more challenging mathematical situation than the integral approach. The advantage is that it enables a treatment of specific problems that cannot reasonably be examined with an integral formulation, e.g. any case where engineering parameters depend not only on boundary values, but also values within the control volume. For example, friction drag within a pipe or on an airplane wing depends on the velocity gradient at the surface. With integral analysis, we can compute the value of the velocity, but its gradient depends on values within the control volume, which we cannot determine. Thus, differential analysis is an important extension in fluid mechanics theory.

6.1. Conservation of Mass

We will now derive the differential equation for conservation of mass for an incompressible fluid. This is the analog for the integral form in Eq. (5.6) on pp. 48. We will proceed along the same lines as the derivation for the differential equation of fluid statics in Chapter 2, i.e. we define an infinitely small control volume[6.1] with a property of interest defined in the center. We then extrapolate the quantity to all boundaries using a truncated Taylor series.

Let us define a differential fluid element of dimensions $\delta x \times \delta y \times \delta z$ (Fig. 6.1). Let the density at the geometric center of the element be ρ and the velocity vector have components (u, v, w) in the (x, y, z) coordinate

[6.1] Once again, the concept of "infinitely small" remains restricted to the continuum assumption introduced in Chapter 1. A term more suitable is then perhaps "differential", which connotes, small enough, but not too small.

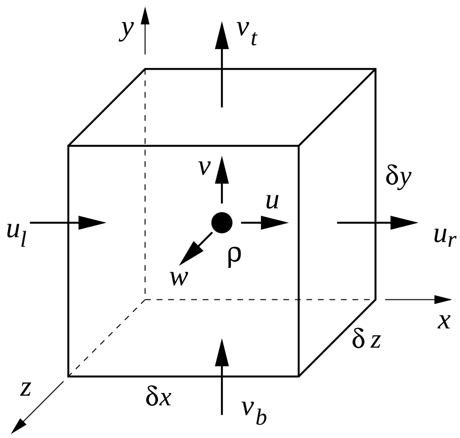

FIGURE 6.1. *Differential volume of fluid with velocity components (u, v, w) and constant density ρ defined at the center.*

directions. We can expand both u and v in terms of Taylor series to obtain horizontal components at the left and right faces of the element and the top and bottom faces, respectively. Although not shown, the same treatment can be applied in the z direction for w. As discussed in Chapter 2, we only use 1–term expansions, which can be written as

$$u_l = u - \frac{\partial u}{\partial x}\frac{\delta x}{2}, \tag{6.1}$$

$$u_r = u + \frac{\partial u}{\partial x}\frac{\delta x}{2}, \tag{6.2}$$

$$v_b = v - \frac{\partial v}{\partial y}\frac{\delta y}{2}, \tag{6.3}$$

$$v_t = v + \frac{\partial v}{\partial y}\frac{\delta y}{2}. \tag{6.4}$$

We already know from our integral formulation in Eq. (5.6) that the mass conservation law for an incompressible fluid pertaining to *any* control volume is that net rate of mass flux must be zero. That is, the sum of mass flow rates over all the boundaries of the control volume must be zero. In two–dimensions, we can write this simply as

$$\dot{m}_l + \dot{m}_r + \dot{m}_b + \dot{m}_t = 0, \tag{6.5}$$

where the mass flow rates are simply the quantities $\rho \mathbf{V} \cdot \hat{n}\, dA$ computed at each face. We then obtain

$$-u_l\,\rho\,\delta y\,\delta z + u_r\,\rho\,\delta y\,\delta z - v_b\,\rho\,\delta x\,\delta z + v_t\,\rho\,\delta x\,\delta z = 0, \tag{6.6}$$

which becomes

$$-\left(u - \frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\rho\,\delta y\,\delta z + \left(u + \frac{\partial u}{\partial x}\frac{\delta x}{2}\right)\rho\,\delta y\,\delta z$$
$$-\left(v - \frac{\partial v}{\partial y}\frac{\delta y}{2}\right)\rho\,\delta x\,\delta z + \left(v + \frac{\partial v}{\partial y}\frac{\delta y}{2}\right)\rho\,\delta x\,\delta z = 0\,, \tag{6.7}$$

The basic u and v terms cancel and the volume term $\delta x \times \delta y \times \delta z$ distributes. We can divide this out since the right hand side is zero. This gives

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{6.8}$$

or, in vector form

$$\nabla \cdot \mathbf{V} = 0\,, \tag{6.9}$$

where ∇ is again the gradient operator defined in Eq. (4.14) on pp. 39. Eq. (6.9) is the differential form of the conservation of mass equation for incompressible flows. The generalization to three dimensions is straightforward and results in

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,, \tag{6.10}$$

We have now derived both the integral form of the law of conservation of mass, Eq. (5.6), and the corresponding differential form of this principle, Eq. (6.9). These two equations are themselves related via the *Divergence Theorem*[6.2] (Kreyszig, 1988), which has the form

$$\iiint_v \nabla \cdot \mathbf{V}\,dv = \iint_A \mathbf{V}\cdot\hat{n}\,dA\,, \tag{6.11}$$

where the left hand side represents the volume integral of Eq. (6.9) over a control volume and the right hand side is our familiar surface integral. Essentially, this theorem provides a means to transform between volume and surface integrals.

<u>EXAMPLE 6.1:</u>

A certain flow is described by the two–dimensional velocity $\mathbf{V} = u\hat{\imath} + v\hat{\jmath}$, where $u = x^2 + y^2$. Determine v such that the conservation of mass is satisfied.

The velocity distribution must satisfy conservation of mass in the form of Eq. (6.8). Evaluating this equation, and in particular $\partial u/\partial x$, we find

$$\frac{\partial u}{\partial x} = 2x = -\frac{\partial v}{\partial y}\cdot$$

We integrate to find the form of v as

$$v = -2xy + f(x)\,.$$

Function $f(x)$ is unconstrained — Conservation of mass will be satisfied regardless of its form. ◇◇◇

[6.2] This is also known as the Gauss Theorem. A proof appears in Kreyszig (1988).

6.2. The Stream Function

Eq. (6.8) immediately suggests that if we define a function ψ such that

$$u = \frac{\partial\psi}{\partial y} \qquad v = -\frac{\partial\psi}{\partial x}\,, \tag{6.12}$$

then conservation of mass is automatically satisfied. This function, which we do not know *a priori* has the form $\psi = \psi(x, y)$. We also have

THEOREM 6.1 (Stream Functions). *Lines (curves) along which ψ is constant are streamlines.*

We can prove this as follows. Recall the concept of a streamline introduced in Chapter 3 (Definition 3.1): Streamlines are curves in the flowfield that are everywhere tangent to the velocity vector. From simple geometry (Fig. 6.2), we see that the slope of a streamline at any point in space is a

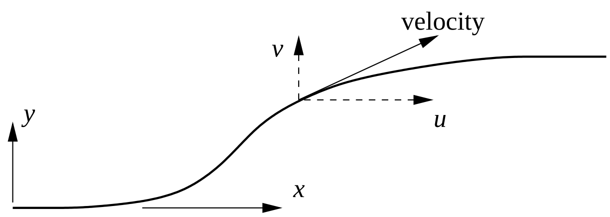

FIGURE 6.2. *Velocity tangent to a streamline, with velocity components.*

function of the local velocity vector, i.e.

$$\frac{dy}{dx} = \frac{v}{u}\,. \tag{6.13}$$

Let us now determine the change in ψ if we move from one point (x, y) to another point close by $(x + dx, y + dy)$ from the Chain Rule:

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy\,, \tag{6.14}$$

from which, after substituting the definition for the stream function in Eq. (6.12), we see

$$d\psi = -\,v\,dx + u\,dy\,. \tag{6.15}$$

Any line (or curve) of constant ψ necessarily means $d\psi = 0$ as we move from point to point. Therefore, lines of constant ψ imply $dy/dx = v/u$ from Eq. (6.15), however, this is equivalent to the slope–related definition of a streamline in Eq. (6.13). It follows that Theorem 6.1 must be valid.

The actual value of ψ for a given streamline is not particularly important, but the differences of streamlines show the remarkable property that:

THEOREM 6.2 (Flow Rates). *The difference in values of two stream functions is equal to the flow rate in the streamtube described by these functions.*

This also is not terribly difficult to prove. Consider two streamlines, 1 and 2, described respectively by two stream functions ψ and $\psi + d\psi$, which are spaced a small distance apart (Fig. 6.3). Let dQ represent the volume

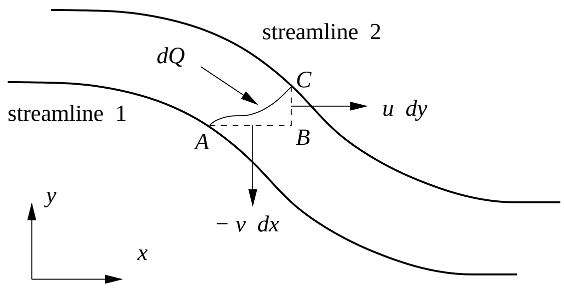

FIGURE 6.3. *The flow in a streamtube bounded by two streamlines ψ and $\psi + d\psi$.*

flow rate in the streamtube, i.e. between the two streamlines. From the concept of mass conservation, we know that the volume rate of flow into the differential control volume ABC must be equal to the volume rate of flow out. Therefore, the volume flow rate crossing surface AC must be equal to the sum of the volume flow rates crossing surfaces AB and BC. Since these have areas represented respectively by dx and dy, we obtain

$$dQ \;=\; u\,dy \;-\; v\,dx\,, \tag{6.16}$$

or, in terms of the stream function

$$dQ \;=\; \frac{\partial \psi}{\partial y}\,dy \;+\; \frac{\partial \psi}{\partial x}\,dx \;\equiv\; d\psi \tag{6.17}$$

Eq. (6.17) follows naturally from Eqs. (6.14) and (6.15). Since $dQ \equiv d\psi$, we can obtain the actual flow rate Q between any two streamlines ψ_1 and ψ_2 by simple integration

$$Q \;=\; \int_{\psi_1}^{\psi_2} d\psi \;=\; \psi_2 \;-\; \psi_1\,. \tag{6.18}$$

If $\psi_2 - \psi_1 > 0$ then Q is positive, otherwise it is negative, i.e. flow is in the negative direction.

<u>EXAMPLE 6.2:</u>

Determine the stream function for the velocity field described in Example 6.1, where $f(x)$ vanishes.

The velocity distribution is $\mathbf{V} = \left(x^2 + y^2\right)\hat{\imath} - 2xy\,\hat{\jmath}$. We proceed by integrating each component as described by Eq. (6.12). Proceeding first with u, we find

$$u \;=\; \frac{\partial \psi}{\partial y} \;=\; x^2 + y^2 \quad\rightarrow\quad \psi \;=\; x^2 y \;+\; \frac{y^3}{3} \;+\; f_1(x)\,.$$

Applying the definition with respect to v, we also find

$$v = -\frac{\partial \psi}{\partial x} = -2\,x\,y \quad \rightarrow \quad \psi = x^2 y + f_2(y)\,.$$

Clearly, the two expressions for ψ that we obtained from integrating u and v, respectively, *must* be equivalent. Comparing these expressions, we see immediately that $f_2(y) = y^3/3$. This leaves $f_1(x)$. This function cannot actually depend upon x because there is nothing "left–over" in x. At most, it could be a constant. We would therefore conclude

$$\psi = x^2 y + \frac{y^3}{3} + C_1\,,$$

where C_1 is a constant. ◊◊◊

6.3. Conservation of Momentum

Once again, we proceed with a derivation of Newton's Second Law, except this time it is applied to a differential fluid element in Fig. 6.4. In this

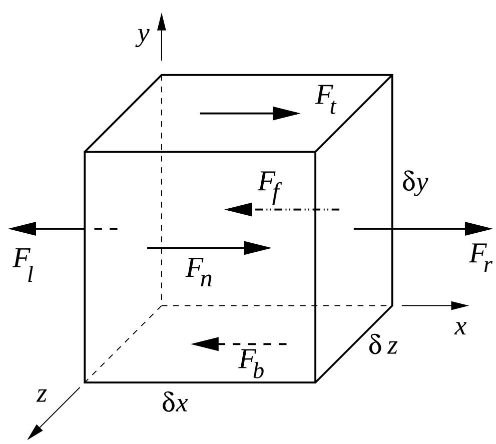

FIGURE 6.4. *Differential element of fluid with all forces in the x direction shown: normal forces are present on the right (r) and left (l) faces, while shear forces act on the top (t), bottom (b), near (n) and far (f) faces. Corresponding forces in the y and z directions are not shown.*

case, we have for a differential element

$$\Sigma\, d\mathbf{F} = dm\,\mathbf{a}\,, \tag{6.19}$$

where we have already derived the vector form of the acceleration $\mathbf{a}$ in Eq. (4.7)[6.3]. The mass is simply $\delta m = \rho\,\delta x \cdot \delta y \cdot \delta z$. Therefore, the main task is to determine the sum of the forces acting on the element. Let us do this

[6.3] The components of $\mathbf{a}$ are given in Eqs. (4.8) through (4.10) on pp. 39.

for the x direction only, since extension to y and z follows the exact same procedure.

Fig. 6.4 shows all the surface forces acting in the x direction. The body force due to gravity is not shown. Surface forces arise from surface stresses acting on specific areas of the differential element. They can be subdivided into stresses which act normal to a surface, e.g. F_l and F_r on the left and right faces, respectively, in Fig. 6.4, and those which act tangentially, e.g. F_b and F_t on the bottom and top faces and F_n and F_f on the near and far faces. We introduce the following nomenclature for these stresses:

- σ_{xx} — This is a stress which acts on a face normal to the x direction (this is the first x) in the x direction. Forces F_l and F_r in Fig. 6.4 represent σ_{xx}.
- τ_{yx} — This describes a stress which acts on a face normal to the y direction in the x direction, e.g. F_t and F_b in Fig. 6.4.
- τ_{zx} — This is a stress which acts on a face normal to the z direction in the x direction, e.g. F_n and F_f in Fig. 6.4.

Note that the first subscript gives the face which the stress acts on, while the second gives the direction[6.4].

To ascertain the collection of forces shown in Fig. 6.4, we resort once again to Taylor Series expansions of the relevant variables from the center of the differential volume to the boundaries[6.5]. That is, we assume that σ_{xx}, τ_{yx}, and τ_{zx} are defined in the center of the differential control volume. We then desire to construct the following expansions:

- F_l and F_r — Expand σ_{xx} in the x direction. The remaining normal components σ_{yy} and σ_{zz} are not relevant to the x direction[6.6].
- F_t and F_b — Expand τ_{yx} in the y direction.
- F_n and F_f — Expand τ_{zx} in the z direction.

According to the standard 1–term truncated series, we find

$$F_r = \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right) \delta y\, \delta z \,, \tag{6.20}$$

$$F_l = \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\frac{\delta x}{2}\right) \delta y\, \delta z \,, \tag{6.21}$$

$$F_t = \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right) \delta x\, \delta z \,, \tag{6.22}$$

[6.4] A convenient memory aid is "*f*ire *d*epartment" for the order of face and direction.

[6.5] We saw this in Chapter 2 for deriving the hydrostatic equation and again in Chapter 3 in the course of obtaining the Bernoulli equation.

[6.6] Note that these quantities, by their very subscripts, are not defined on the faces where F_l and F_r reside. Also, their directions are in y and z, which means they make no contribution to the x momentum equation.

$$F_b = \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\frac{\delta y}{2}\right)\delta x\,\delta z\,, \tag{6.23}$$

$$F_n = \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y\,, \tag{6.24}$$

$$F_f = \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta x\,\delta y\,. \tag{6.25}$$

At this point, let us add back the contribution of a body force, which will usually be gravity[6.7]. Here, the force is the product of mass and acceleration (gravity), which yields $\rho \cdot \delta x \cdot \delta y \cdot \delta z \cdot g_x$, where g_x is the component of the gravity vector in the x direction.

Now, the net force is simply the sum of Eqs. (6.20) through (6.25) plus the body force term, i.e.

$$\Sigma\, F_x = F_r - F_l + F_t - F_b + F_n - F_f + \rho\, g_x\, \delta x\, \delta y\, \delta z\,, \tag{6.26}$$

which gives

$$\Sigma\, F_x = \left(\rho\, g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z\,. \tag{6.27}$$

We can now form the x momentum equation from

$$\Sigma\, F_x = \delta m\, a_x\,, \tag{6.28}$$

where $\Sigma\, F_x$ is given by Eq. (6.27), a_x is given by Eq. (4.8), and $\delta m = \rho\, \delta x \cdot \delta y \cdot \delta z$. We see that the elemental volume $\delta x \cdot \delta y \cdot \delta z$ cancels out, leaving

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho\, g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{6.29}$$

as the final x momentum equation in differential form. By similar procedures, we obtain the y and z direction equations, respectively, as

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho\, g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{6.30}$$

and

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho\, g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{6.31}$$

We notice that Eqs. (6.29) through (6.31) contain too many unknowns to solve, i.e. 3 velocity components and additional stress components. What are still missing are *constitutive relationships* between velocity components and stresses[6.8].

[6.7] Although gravity is typically associated with the y direction, this is actually only a consequence of choosing coordinates in a certain way. In general, the gravity vector does not have to be aligned with any single coordinate direction.

[6.8] Constitutive equations actually relate velocity gradients to stresses, not velocity components directly.

6.4. Inviscid Flow Equations

We will start with the case of *inviscid flow*[6.9], which utilizes the most fundamental set of constitutive relationships, i.e. normal stresses arise only from the contribution of pressure

$$\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -P\,, \tag{6.32}$$

and shear stresses vanish

$$\tau_{xy} = \tau_{yx} = \tau_{yz} = \tau_{zy} = \tau_{xz} = \tau_{zx} = 0\,. \tag{6.33}$$

The negative sign in Eq. (6.32) arises because pressure is compressive. The Eqs. (6.29) through (6.31) then simplify to

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\right) = \rho\,g_x - \frac{\partial P}{\partial x}\,, \tag{6.34}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\right) = \rho\,g_y - \frac{\partial P}{\partial y}\,, \tag{6.35}$$

and

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\right) = \rho\,g_z - \frac{\partial P}{\partial z}\,. \tag{6.36}$$

Eqs. (6.34) through (6.36) are the inviscid form of momentum conservation and are often called the *Euler Equations*[6.10]. These three equations, along with Eq. (6.9) for mass conservation, describe a complete system, which can be solved for velocity and pressure distributions[6.11]. That is, we have four equations, and four unknowns, (u, v, w, P). We can write Eqs. (6.34) through (6.36) collectively in a single vector equation as

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\,\mathbf{V}\right) = \rho\,\mathbf{g} - \nabla P\,, \tag{6.37}$$

where ∇ is gradient operator defined in Eq. (4.14).

While the Euler Equations are appreciably more simple than the equations obtained with more realistic constitutive relationships, they are still not amenable to straightforward analytical solution because of non–linear terms in the acceleration, e.g. $u\partial u/\partial x$. However, we can still obtain useful information for special cases of inviscid flow.

[6.9] Inviscid, frictionless, and non–viscous are all synonymous for describing flows in which shear stresses vanish.

[6.10] Leonhard Euler (1707–1783) — a Swiss mathematician who studied effects of pressure on flow characteristics.

[6.11] Actually, we still require initial and boundary conditions for these differential equations, which we will discuss.

6.5. Relationship of the Euler and Bernoulli Equations

As an exercise, let us show that the steady form of Euler's Equations can be integrated along a streamline to obtain the Bernoulli Equation[6.12], which was first introduced in Chapter 3. For steady flow $\partial \mathbf{V}/\partial t = 0$ and Eq. (6.37) becomes

$$\rho\,\mathbf{V}\cdot\nabla\mathbf{V} \;=\; \rho\,\mathbf{g} \;-\; \nabla P\,. \tag{6.38}$$

First, we will make use of the fact that gravity is a *conservative force*[6.13] so that the gravity vector in Eq. (6.38) can be expressed as the negative gradient of a potential function

$$\mathbf{g} \;=\; -\,g\nabla z\,, \tag{6.39}$$

where g is simply the magnitude of the gravity vector. We must also make use of a special identity for vectors (Hildebrand, 1976)

$$\mathbf{V}\cdot\nabla\mathbf{V} \;\equiv\; \frac{1}{2}\,\nabla\left(\mathbf{V}\cdot\mathbf{V}\right) \;-\; \mathbf{V}\times\left(\nabla\times\mathbf{V}\right) \tag{6.40}$$

Eq. (6.38) can then be written as

$$\frac{\rho}{2}\,\nabla\left(\mathbf{V}\cdot\mathbf{V}\right) \;-\; \rho\,\mathbf{V}\times\left(\nabla\times\mathbf{V}\right) \;=\; -\,\rho\,g\,\nabla z \;-\; \nabla P\,, \tag{6.41}$$

which, using the fact that $\mathbf{V}\cdot\mathbf{V} = V^2$, i.e. the square of the velocity magnitude, can further be written

$$\frac{\nabla P}{\rho} \;+\; \frac{1}{2}\,\nabla V^2 \;+\; g\,\nabla z \;=\; \mathbf{V}\times\left(\nabla\times\mathbf{V}\right) \tag{6.42}$$

Let us now take the dot product of Eq. (6.42) and a differential vector $d\hat{s}$ directed along a streamline[6.14]. This gives

$$\frac{\nabla P}{\rho}\cdot d\hat{s} \;+\; \frac{1}{2}\,\nabla V^2\cdot d\hat{s} \;+\; g\,\nabla z\cdot d\hat{s} \;=\; \mathbf{V}\times\left(\nabla\times\mathbf{V}\right)\cdot d\hat{s}\,, \tag{6.43}$$

which can be simplified as follows:

- $\mathbf{V}\times(\nabla\times\mathbf{V})\cdot d\hat{s}$ — The vector triple product is itself a vector perpendicular to $\mathbf{V}$ (Beyer, 1984). Therefore, the dot product of this vector and $d\hat{s}$, which is *parallel* to $\mathbf{V}$ must be zero. This term vanishes.
- $\nabla P\cdot d\hat{s}$ — This term can be expanded to

$$\left(\frac{\partial P}{\partial x}\,\hat{\imath} \;+\; \frac{\partial P}{\partial y}\,\hat{\jmath} \;+\; \frac{\partial P}{\partial z}\,\hat{k}\right)\cdot\left(dx\,\hat{\imath} \;+\; dy\,\hat{\jmath} \;+\; dz\,\hat{k}\right)$$

[6.12] Various forms of this procedure are shown in texts, e.g. Fox and McDonald (1998) and Munson et al. (2006).

[6.13] This means that work realized from a force is independent of the path followed. Consequently, the force can be expressed in terms of the negative gradient of a potential function. See e.g. Beer and Johnston (1984) for further discussion of conservative forces.

[6.14] This means $\mathbf{V}\parallel d\hat{s}$, i.e. the two vectors are parallel, which is compatible with the original definition of the streamline.

from which we find[6.15]

$$\frac{\partial P}{\partial x}\,dx \,+\, \frac{\partial P}{\partial y}\,dy \,+\, \frac{\partial P}{\partial z}\,dz \;=\; dP$$

- $\nabla V^2 \cdot d\hat{s}$ — By the same analysis as above, this term reduces to the differential $d(V^2)$.
- $\nabla z \cdot d\hat{s}$ — By the same analysis as above, this term reduces to dz.

Thus, Eq. (6.43) simplifies to

$$\frac{dP}{\rho} \,+\, \frac{1}{2}\,d\left(V^2\right) \,+\, g\,dz \;=\; 0\,, \tag{6.44}$$

which is identical to Eq. (3.18) on pp. 26. This equation, in turn, led to the Bernoulli Equation, i.e. Eq. (3.19), when integrated along a streamline. Therefore, we see that the Bernoulli equation is a special case of the Euler equations valid for steady flow along a streamline.

6.6. Irrotational and Potential Flows

The rotation of a fluid particle is described by the *curl* of the velocity field

$$\vec{\zeta} \;=\; \nabla \times \mathbf{V} \;=\; 2\,\vec{\omega}\,, \tag{6.45}$$

where $\vec{\zeta}$ is the fluid *vorticity*, $\vec{\omega}$ is the vector representing angular rotation rate (Munson et al., 2006), and ∇ is the gradient operator defined in Eq. (4.14). This can be expanded in matrix determinant form as fluid vorticity

$$\vec{\zeta} \;=\; \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}\,, \tag{6.46}$$

which evaluates to

$$\vec{\zeta} \;=\; \left(\frac{\partial w}{\partial y} \,-\, \frac{\partial v}{\partial z}\right)\,\hat{i} \,+\, \left(\frac{\partial u}{\partial z} \,-\, \frac{\partial w}{\partial x}\right)\,\hat{j} \,+\, \left(\frac{\partial v}{\partial x} \,-\, \frac{\partial u}{\partial y}\right)\,\hat{k} \tag{6.47}$$

Now suppose we focus on flows that are *irrotational*, that is, their vorticity is zero. In such cases, none of the fluid elements rotates. These are strictly translational flows. How can the flow equations be further simplified under this restriction? irrotational flow

There are two important findings resulting from $\vec{\zeta} = \nabla \times \mathbf{V}$. First, if we look back at Eqs. (6.42) and (6.43), we see that the cross–product term vanished only by virtue of being dotted to a vector along a streamline. For the special case of irrotational flow, this product is already zero by itself, regardless of whether it is integrated along a streamline. In fact, if we take any arbitrary differential direction $d\hat{r}$ (including $d\hat{s}$), we will eventually obtain the Bernoulli equation due to this fact. Therefore, we state

[6.15]The expression $\nabla P \cdot d\hat{s}$ is essentially the directional derivative of P in the $d\hat{s}$ direction (Kreyszig, 1988).

THEOREM 6.3 (Special Application of Bernoulli Equation). *If a flow is both inviscid and irrotational, the Bernoulli Equation is valid between any two points in a flow, not just points which lie along streamlines.*

The second important factor is that this condition allows us to express the velocity distribution as the gradient of a potential function, i.e.

$$\mathbf{V} = \nabla\phi \,, \tag{6.48}$$

in light of the vector identity

$$\nabla \times \nabla\phi \equiv 0 \,. \tag{6.49}$$

In other words, if we assume an irrotational flow, we are saying $\nabla \times \mathbf{V} = 0$. We also know by the vector identity that Eq. (6.49) is true. Therefore, Eq.(6.48) follows as a natural result.

This second point leads to the simplification we are looking for. We will obtain two *linear* equations which describe inviscid irrotational flow[6.16]. Contrast this to the Euler equations given by Eqs. (6.34) through (6.36), which are non–linear, i.e. very difficult to solve except in special cases. To obtain the first equation, we can substitute Eq.6.48 into the vector continuity equation given by Eq. (6.9) to obtain

$$\nabla \cdot \nabla\phi = 0 \,, \tag{6.50}$$

which can be written simply as

$$\nabla^2\phi = 0 \,, \tag{6.51}$$

This is the *Laplace Equation*, which arises in many other areas of mechanics, including conduction heat transfer, elasticity, gravitational problems, and electrostatics[6.17]. We can once again exploit the assumption of irrotational flow by substituting the stream function defined in Eq. (6.12) into the kinematic condition given by Eq. (6.47) that irrotational flow must obey. In two dimensions, the $\hat{\imath}$ and $\hat{\jmath}$ components of Eq. (6.47) vanish because $w = 0$ and $\partial/\partial z = 0$. This leaves only the $\hat{k}$ component

Laplace Equation

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \,, \tag{6.52}$$

which, when switching signs gives

$$\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right) - \frac{\partial}{\partial x}\left(-\frac{\partial\psi}{\partial x}\right) = 0 \,. \tag{6.53}$$

We can simplify Eq. (6.53) to obtain a Laplace equation once again

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \,, \tag{6.54}$$

[6.16] This is also called *potential flow*, e.g. Munson et al. (2006) or *ideal flow*, e.g. Panton (1984).

[6.17] The ∇^2 operator is called the *Laplacian* and has the form $\partial^2/\partial x^2 + \partial^2/\partial y^2$ in two–dimensional Cartesian coordinates. Panton (1984) lists forms for other common orthogonal coordinate systems.

or, using the Laplacian operator notation

$$\nabla^2 \psi \;=\; 0\,. \tag{6.55}$$

Thus, we can effectively recast potential flow problems in the form of Eqs. (6.51) and (6.55). The former derives from combining conservation of mass with irrotationality, while the latter stems from combining the stream function formulation with irrotationality. The entities ϕ and ψ effectively replace our velocity components u and v as the flow variables. Determination of ϕ and ψ implies u and v, from which pressure can then be found by way of Bernoulli's equation (Theorem 6.3).

There is an interesting geometrical property of potential flows cast in the (ϕ, ψ) variables. We showed in Eq. (6.13) that the slope of a streamline is v/u. This can also be proved according to the definition of a streamline. Along a streamline, ψ is a constant, which implies $d\psi = 0$. This gives

$$d\psi \;=\; \frac{\partial \psi}{\partial x}\,dx \;+\; \frac{\partial \psi}{\partial y}\,dy \;=\; 0\,. \tag{6.56}$$

Therefore

$$\left.\frac{dy}{dx}\right|_{\psi} \;=\; -\,\frac{\partial \psi/\partial x}{\partial \psi/\partial y} \;=\; -\,\frac{-v}{u} \;=\; \frac{v}{u}\,. \tag{6.57}$$

By the same procedure, along a potential flow line ϕ is a constant, which implies $d\phi = 0$. This gives

$$d\phi \;=\; \frac{\partial \phi}{\partial x}\,dx \;+\; \frac{\partial \phi}{\partial y}\,dy \;=\; 0\,. \tag{6.58}$$

Eq. (6.48) dictates $u = \partial\phi/\partial x$ and $v = \partial\phi/\partial y$, therefore we solve Eq. (6.58) as

$$\left.\frac{dy}{dx}\right|_{\phi} \;=\; -\,\frac{\partial \phi/\partial x}{\partial \phi/\partial y} \;=\; -\,\frac{u}{v}\,. \tag{6.59}$$

Because the slopes in Eqs. (6.57) and (6.59) are negative reciprocals, ψ and ϕ must be orthogonal.

<u>EXAMPLE 6.3:</u>

Determine the potential function ϕ corresponding to the stream function described in Example 6.2, where constant C_1 vanishes.

Going back to Example 6.1, we recall the stream function $\psi = x^2 y + y^3/3$ corresponds to the velocity distribution $\mathbf{V} = \left(x^2 + y^2\right)\hat{\imath} - 2xy\,\hat{\jmath}$. This example is therefore quite similar to Example 6.2 in that we simply have to integrate according to the definition of ϕ in Eq. (6.48). That is,

$$u \;=\; \frac{\partial \phi}{\partial x} \;=\; x^2 + y^2 \quad\rightarrow\quad \phi \;=\; \frac{x^3}{3} \;+\; xy^2 \;+\; f_1(y)\,.$$

Applying the definition with respect to v, we then find

$$v \;=\; \frac{\partial \phi}{\partial y} \;=\; -\,2\,x\,y \quad\rightarrow\quad \phi \;=\; -\,xy^2 \;+\; f_2(x)\,.$$

However, here we see an inconsistency. The two expressions for ϕ must be equivalent, but in this case they cannot be! Specifically, the sign leading xy^2 in each expression is different. The velocity potential ϕ will only exist if the flow is irrotational, according to Eqs. (6.48) and (6.49), i.e. if $\vec{\zeta} = \nabla \times \mathbf{V} = 0$. Checking $\vec{\zeta}$ for the two–dimensional flow, i.e. Eq. (6.47) simplified to Eq. (6.52), we find

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\Big(-2xy\Big) - \frac{\partial}{\partial y}\Big(x^2 + y^2\Big) = -2y - 2y = -4y \neq 0\,,$$

which in the general case ($y \neq 0$) does *not* vanish. We conclude that there is no potential function for this velocity distribution because the flow is not irrotational. ♢♢♢

6.7. Physical Interpretation of the Curl of Velocity

The curl of the velocity vector, formally introduced in Eq. (6.45) on pp. 67, is a somewhat abstract concept. We briefly stated that this quantity is simply twice the angular rotation rate. Let us examine this assertion more closely from two different perspectives.

First, consider the angular motion and deformation of a fluid element caused by the velocities shown in Fig. 6.5. We define velocity components u

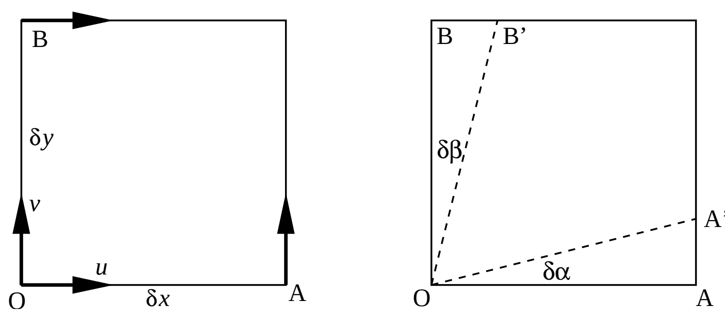

FIGURE 6.5. *Fluid element: instantaneous velocities (left) and angular motion and deformation an infinitesimal time later (right).*

and v at the bottom left corner point O. Velocities at corner points A and B can be obtained by Taylor series expansion in the usual manner, i.e.

$$u_B = u + \frac{\partial u}{\partial y}\,\delta y \quad \text{and} \quad v_A = v + \frac{\partial v}{\partial x}\,\delta x\,.$$

In an infinitesimal time period δt, line segment OB will rotate clockwise through an angle $\delta\beta$ to position OB', while line segment OA will rotate counter–clockwise through an angle $\delta\alpha$ to position OA'. The rate of rotation of these segments about point O is defined as

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} \quad \text{and} \quad \omega_{OB} = \lim_{\delta t \to 0} \frac{\delta\beta}{\delta t}\,. \tag{6.60}$$

The two angles, $\delta\alpha$ and $\delta\beta$ can be measured as the respective tangents $\tan\delta\alpha = A'A/OA$ and $\tan\delta\beta = B'B/OB$. However, since $\delta t \to 0$, these angles will be small enough to use the approximations $\tan\delta\alpha \approx \delta\alpha$ and $\tan\delta\beta \approx \delta\beta$. Relative to corner O, corner B is moving to the right at a rate of $u_B - u = (\partial u/\partial y)\,\delta y$. Likewise, relative to corner O, corner A is moving upward at a rate of $v_A - v = (\partial v/\partial x)\,\delta x$. The respective lengths of BB' and AA' are then the relative velocities of B and A times δt, i.e. $(\partial u/\partial y)\,\delta y\,\delta t$ and $(\partial v/\partial x)\,\delta x\,\delta t$. We can then write angles $\delta\beta$ and $\delta\alpha$ as[6.18]

$$\tan\delta\beta \;\approx\; \delta\beta \;=\; \frac{(\partial u/\partial y)\,\delta y\,\delta t}{\delta y} \;=\; \frac{\partial u}{\partial y}\,\delta t$$

and

$$\tan\delta\alpha \;\approx\; \delta\alpha \;=\; \frac{(\partial v/\partial x)\,\delta x\,\delta t}{\delta x} \;=\; \frac{\partial v}{\partial x}\,\delta t\,.$$

These expressions are now substituted into Eq. (6.60) to obtain the rotation rates

$$\omega_{OA} \;=\; \lim_{\delta t\to 0}\frac{(\partial v/\partial x)\,\delta t}{\delta t} \;=\; \frac{\partial v}{\partial x} \tag{6.61}$$

and

$$\omega_{OB} \;=\; \lim_{\delta t\to 0}\frac{(\partial u/\partial y)\,\delta t}{\delta t} \;=\; \frac{\partial u}{\partial y}\,. \tag{6.62}$$

Now, we define the rotation of the element as a whole to be the average of the two angular rotation rates. Taking clockwise as the positive direction, we find

$$\omega_z \;=\; \frac{1}{2}\left(\frac{\partial v}{\partial x} \;-\; \frac{\partial u}{\partial y}\right)\,,$$

which is exactly half of the vorticity component ζ_z in Eq. (6.47). Components ω_x and ω_y could be shown via similar derivations.

We can also examine the curl of velocity in the completely general sense by way of the following vector operations (Hildebrand, 1976). Suppose the motion of a particular fluid element is purely rotational about a fixed axis (Fig. 6.6). The element's angular velocity (along the circular path) is constant and is defined by $\vec{\omega}$ according to the right–hand rule. If we denote the position P of the element relative to an origin O on the axis by the radius vector $\vec{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then the velocity vector of the element is defined as (Beer and Johnston, 1984)

$$\mathbf{V} \;=\; \vec{\omega} \;\times\; \vec{r}\,,$$

where $\times$ indicates the vector cross product. Using a vector identity from Hildebrand (1976), we can write the curl of velocity as

$$\nabla\times\mathbf{V} \;=\; \nabla\times(\vec{\omega} \;\times\; \vec{r}) \;=\; \vec{r}\cdot\nabla\vec{\omega} \;-\; \vec{\omega}\cdot\nabla\vec{r} \;+\; \vec{\omega}\,(\nabla\cdot\vec{r}) \;-\; \vec{r}\,(\nabla\cdot\vec{\omega})\;.$$

[6.18] Recall from footnote 1.5 on pp. 5 that the tangent of a small angle is approximately equal to the angle itself.

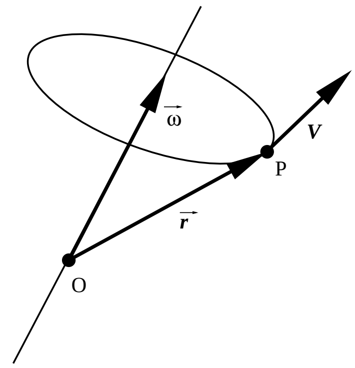

FIGURE 6.6. *Pure rotation of a fluid element about a fixed axis.*

Let us examine each term carefully. The first term can be written

$$\vec{r}\cdot\nabla\vec{\omega} \;=\; \left(x\,\frac{\partial}{\partial x} \,+\, y\,\frac{\partial}{\partial y} \,+\, z\,\frac{\partial}{\partial z}\right)\left(\omega_x\hat{\imath} \,+\, \omega_y\hat{\jmath} \,+\, \omega_z\hat{k}\right)\,,$$

however, this term must vanish since the components of $\vec{\omega}$ are constant. This same observation shows that the last term will also vanish, leaving the non–trivial expression

$$\nabla\times\mathbf{V} \;=\; \nabla\times(\vec{\omega}\,\times\,\vec{r}) \;=\; \vec{\omega}\,(\nabla\cdot\vec{r}) \,-\, \vec{\omega}\cdot\nabla\vec{r}\,. \tag{6.63}$$

Carrying out the remaining operations, we find

$$\nabla\cdot\vec{r} \;=\; \frac{\partial x}{\partial x} \,+\, \frac{\partial y}{\partial y} \,+\, \frac{\partial z}{\partial z} \;=\; 3$$

and

$$\begin{aligned}
\vec{\omega}\cdot\nabla\vec{r} \;&=\; \left(\omega_x\,\frac{\partial}{\partial x} \,+\, \omega_y\,\frac{\partial}{\partial y} \,+\, \omega_z\,\frac{\partial}{\partial z}\right)\left(x\hat{\imath} \,+\, y\hat{\jmath} \,+\, z\hat{k}\right)\\
&=\; \left(\omega_x\,\frac{\partial x}{\partial x} \,+\, \omega_y\,\frac{\partial x}{\partial y} \,+\, \omega_z\,\frac{\partial x}{\partial z}\right)\hat{\imath}\\
&\quad\;\left(\omega_x\,\frac{\partial y}{\partial x} \,+\, \omega_y\,\frac{\partial y}{\partial y} \,+\, \omega_z\,\frac{\partial y}{\partial z}\right)\hat{\jmath}\\
&\quad\;\left(\omega_x\,\frac{\partial z}{\partial x} \,+\, \omega_y\,\frac{\partial z}{\partial y} \,+\, \omega_z\,\frac{\partial z}{\partial z}\right)\hat{k}\\
&=\; \omega_x\,\hat{\imath} \,+\, \omega_y\,\hat{\jmath} \,+\, \omega_z\,\hat{k}\\
&=\; \vec{\omega}\,.
\end{aligned}$$

Thus, we find that Eq. (6.63) simplifies to

$$\nabla\times\mathbf{V} \;=\; \nabla\times(\vec{\omega}\,\times\,\vec{r}) \;=\; 3\,\vec{\omega} \,-\, \vec{\omega} \;=\; 2\,\vec{\omega}\,, \tag{6.64}$$

which is equivalent to the expression in Eq. (6.45) for vorticity $\vec{\zeta}$. Eq. (6.64) provides perhaps the best physical interpretation of the curl of velocity, given our definition at the outset that $\vec{\omega}$ is the rotation rate of the fluid element.

6.8. An Example Using Basic Potential Flow Components

Remarkably, we now have two Laplace equations for ϕ, Eq. (6.51), and ψ, Eq. (6.55), which describe inviscid irrotational (potential) flow. Because these equations are linear, we can apply the principle of *superposition*, whereby we can construct realistic flow problems and their solutions as sums of simple "building block" problems and their respective solutions[6.19]. That is, we can build problems out of simple component flow patterns[6.20]. Many such components are discussed in Fox and McDonald (1998), Panton (1984), and Munson et al. (2006).

Principle of Superposition

The most systematic method for describing such flows utilizes complex numbers, which we will now review briefly. We define the complex variable z as

$$z = x + iy = r\,e^{i\theta}\,, \tag{6.65}$$

where i has the usual definition of $\sqrt{-1}$. Fig. 6.7 shows z on the complex plane and shows graphically the relationship between the rectangular and circular representations of z given by Eq. (6.65).

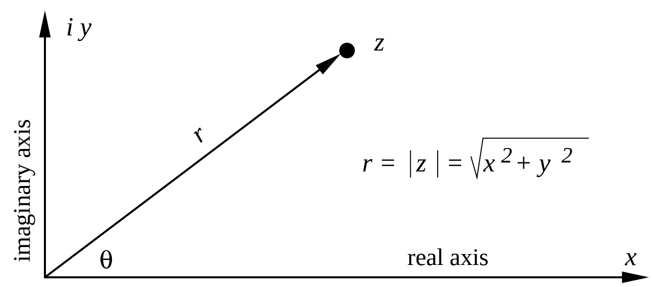

FIGURE 6.7. *A point z on the complex plane.*

Flows are expressed as complex functions of z, notably, the *complex potential function* $F(z)$

$$F(z) = \phi(x,y) + i\,\psi(x,y)\,. \tag{6.66}$$

The real part of F is the velocity potential and the imaginary part is the stream function. The basis for casting problems according to Eq. (6.66) is that any analytic function[6.21] of a complex variable has real and imaginary parts that are each solutions to the Laplace equation (Kreyszig, 1988; Panton, 1984).

[6.19] This is a standard method of study for all *linear systems*, such as DC circuits, etc. We point out here that it is not sufficient for the equations to be linear. The boundary conditions must also be linear.

[6.20] These patterns are basic solutions to the Laplace equation. Such solutions are termed *harmonic functions* in the mathematical literature (Kreyszig, 1988).

[6.21] A function is *analytic* in a domain if it is both defined and differentiable at all points in the domain.

The same rules of calculus apply to F as to real functions, for example differentiation yields the *complex velocity*

$$W(z) = \frac{dF}{dz}, \tag{6.67}$$

which is related to the physical velocity components by

$$W = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} + i\,\frac{\partial \psi}{\partial x} = u - i\,v\,. \tag{6.68}$$

So $W(z)$ is actually the complex conjugate of the velocity vector[6.22]. In polar coordinates, Eq. (6.68) takes the form

$$W = (u_r - i\,u_\theta)\,e^{-i\theta}, \tag{6.69}$$

where $u_r = u_r(r,\theta)$ is the radial velocity component and $u_\theta = u_\theta(r,\theta)$ is the azimuthal component. Let us discuss an example of superimposing two such component flows to derive a more realistic flow.

The first component is simple uniform flow, which is described by the complex potential

$$F_u(z) = u_0\,e^{-i\alpha}\,z\,. \tag{6.70}$$

The complex velocity is

$$W_u = \frac{dF_u}{dz} = u_0\,e^{-i\alpha} = u_0 \cos\alpha - i\,u_0 \sin\alpha\,, \tag{6.71}$$

the latter step of which comes from an identity known as *Euler's formula*

$$e^{i\alpha} = \cos\alpha + i\,\sin\alpha\,. \tag{6.72}$$

From Eq. (6.68), we identify the velocity components as $u = u_0 \cos\alpha$ and $v = u_0 \sin\alpha$. Thus, Eq. (6.70) describes a uniform flow inclined at an angle α to the horizontal (Fig. 6.8). For the time being, we will take $\alpha = 0$ so

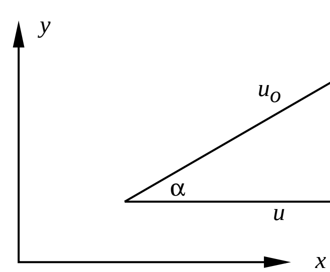

FIGURE 6.8. *Uniform flow described by Eq.* (6.70).

that $u = u_0$ and $v = 0$.

The second component is a source flow having

$$F_s(z) = \frac{m}{2\pi}\,\ln z\,, \tag{6.73}$$

[6.22] The complex conjugate is found by negating the imaginary part of the complex variable.

which can be cast as

$$F_s(z) \;=\; \frac{m}{2\,\pi}\,\ln\left(r\,e^{i\theta}\right) \;=\; \frac{m}{2\,\pi}\left[\ln r \,+\, \ln e^{i\theta}\right] \;=\; \frac{m}{2\,\pi}\left(\ln r \,+\, i\,\theta\right). \tag{6.74}$$

According to Eq. (6.66), we see that

$$\phi \;=\; \frac{m}{2\,\pi}\,\ln r \qquad \psi \;=\; \frac{m}{2\,\pi}\,\theta \tag{6.75}$$

The velocity distribution can be computed using either Eq. (6.69) or the definitions of ψ and ϕ in polar coordinates (Panton, 1984). Per the former method, we find

$$W_s \;=\; \frac{dF}{dz} \;=\; \frac{m}{2\,\pi\,z} \;=\; \frac{m}{2\,\pi\,r}\,e^{-i\theta}\,. \tag{6.76}$$

Comparing this expression to Eq. (6.69), we identify the velocity components as

$$u_r \;=\; \frac{m}{2\,\pi\,r} \tag{6.77}$$

and $u_\theta = 0$. Therefore, Eq. (6.73) represents a purely radial flow emanating from a point, where the velocity decreases as the inverse of the radial distance from the point (Fig. 6.9).

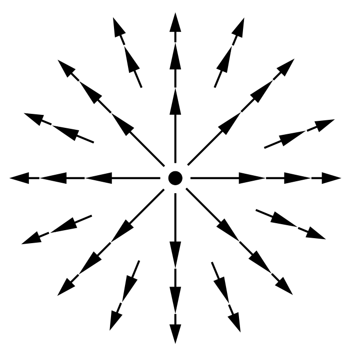

FIGURE 6.9. *Source flow described by Eq.* (6.73).

We can combine these two components to model flow over a generic bluff body (Fig. 6.10), which approximates cases such as oncoming flow for wing sections, struts, cars, and various other blunt cross sections. The complex potential is simply the sum of Eqs. (6.70) and (6.73) according to the principle of superposition

$$F(z) \;=\; u_0\,e^{-i\alpha}\,z \,+\, \frac{m}{2\,\pi}\,\ln z \;=\; u_0\,z \,+\, \frac{m}{2\,\pi}\,\ln z\,. \tag{6.78}$$

The simplification in the latter part of Eq. (6.78) arises because we have assumed $\alpha = 0$, i.e. free–stream flow in the horizontal direction. We can find either by direct calculation, or by adding results above that

$$\psi \;=\; u_0\,y \,+\, \frac{m}{2\,\pi}\,\theta \tag{6.79}$$

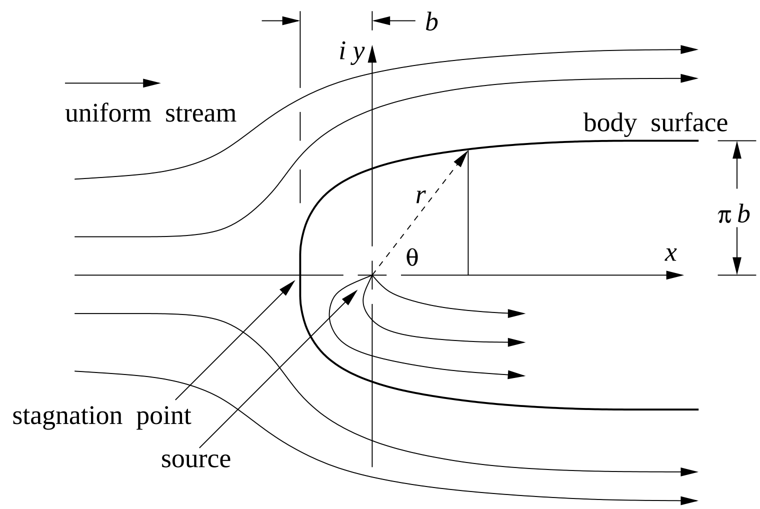

FIGURE 6.10. *Flow over a blunt body can be modeled as the superposition of a uniform stream and a source flow.*

and

$$\phi = u_0\,x + \frac{m}{2\,\pi}\,\ln r \tag{6.80}$$

As shown in Fig. 6.10, at some distance $x = -b$, i.e. to the left of the origin, the left–moving flow from the source will exactly cancel the right–moving flow of the free stream and a stagnation point will result. We can compute the location of this point from the complex velocity

$$W = \frac{dF}{dz} = u_0 + \frac{m}{2\,\pi\,z} \tag{6.81}$$

with the condition that $W = 0$. Solving Eq. (6.81) for $W = 0$, we find

$$z = -\frac{m}{2\,\pi\,u_0}\,. \tag{6.82}$$

Remembering that $z = x + iy$, we can rewrite this equation in a more recognizable form

$$x + iy = -\frac{m}{2\,\pi\,u_0} + 0\,i\,, \tag{6.83}$$

from which it is clear that $y = 0$ and $x = -m/(2\,\pi\,u_0)$. Therefore, the distance b is

$$b = \frac{m}{2\,\pi\,u_0}\,. \tag{6.84}$$

We can determine the value of the stream function at the stagnation point by evaluating Eq. (6.79) at $(x, y) = (-b, 0)$, i.e. the coordinates of the stagnation point. This point lies at $\theta = \pi$ in the polar coordinate system, so that

$$\psi = 0\, u_0 + \frac{m}{2\,\pi}\,\pi = \frac{m}{2}\,. \tag{6.85}$$

We can use this result to determine the parametric equation describing the body from Eq. (6.79). Since $y = r\sin\theta$ in polar coordinates and $m/2 = b\pi u_0$ from Eq. (6.84), we get

$$b\,\pi\,u_0 = u_0\,r\,\sin\theta + b\,u_0\,\theta\,, \tag{6.86}$$

which can be solved for the radius

$$r = \frac{b\,(\pi - \theta)}{\sin\theta}\,, \tag{6.87}$$

where $0 \leq \theta \leq 2\pi$. We note that the "internal" streamlines are of no interest in this case. Only the bounding streamline describing the body and the external flow streamlines are relevant. If we let $r = y/\sin\theta$ in Eq. (6.87), the y coordinate of the body is

$$y = b\,(\pi - \theta)\,, \tag{6.88}$$

which converges to $y \to |b\pi|$ at $\theta = 0$ and $\theta = 2\pi$. Therefore, the body has a thickness of $2b\pi$.

From an applications perspective, this case is a rudimentary aerodynamics problem, e.g. flow around a forward wing section. We are therefore interested in the aerodynamic performance of this object, for example the pressure distribution on its surface. This enables computation of the pressure–related "form" drag. For this, we conveniently resort back to the Bernoulli equation, which can be applied anywhere in the flow domain, i.e. irrespective of considering streamline locations. We recall that this is valid because the flow is irrotational. The first step is calculating V^2 for use in the Bernoulli equation. We could do this "the long way" using ψ or ϕ, or in a more clever way, since the square of the velocity is equal to the complex velocity multiplied by its complex conjugate. In other words from Eq. (6.68), we have

$$W\overline{W} = (u - i\,v)\cdot(u + i\,v) = u^2 + v^2 = V^2 \tag{6.89}$$

where $\overline{W}$ is the complex conjugate of W. Using W from Eq. (6.81), we find

$$W\overline{W} = \left(u_0 + \frac{m}{2\,\pi\,z}\right)\left(u_0 + \frac{m}{2\,\pi\,\overline{z}}\right)\,, \tag{6.90}$$

where $\overline{z}$ in the second term is the complex conjugate of z. We substitute $z = x + iy$ from Eq. (6.65) and its conjugate $\overline{z} = x - iy$ and eliminate imaginary components in the denominator using complex conjugate multiplication. To see this, write

$$W\overline{W} = \left(\alpha + \frac{\beta}{z}\right)\left(\alpha + \frac{\beta}{\overline{z}}\right)\,, \tag{6.91}$$

where we have substituted the constants $\alpha = u_0$ and $\beta = m/(2\,\pi)$. Substituting $z = x + iy$, we obtain

$$W\overline{W} \;=\; \left(\alpha \;+\; \frac{\beta}{x \,+\, iy}\right)\left(\alpha \;+\; \frac{\beta}{x \,-\, iy}\right)\,. \tag{6.92}$$

We extract i from both denominators by

$$W\overline{W} \;=\; \left(\alpha \;+\; \frac{\beta}{x \,+\, iy}\,\frac{x \,-\, iy}{x \,-\, iy}\right)\left(\alpha \;+\; \frac{\beta}{x \,-\, iy}\,\frac{x \,+\, iy}{x \,+\, iy}\right)\,. \tag{6.93}$$

Because $i^2 = -1$, this gives

$$W\overline{W} \;=\; \left(\alpha \;+\; \frac{\beta(x \,-\, iy)}{x^2 \,+\, y^2}\right)\left(\alpha \;+\; \frac{\beta(x \,+\, iy)}{x^2 \,+\, y^2}\right)\,. \tag{6.94}$$

Carrying through the multiplication, we obtain

$$\begin{aligned} W\overline{W} \;&=\; \alpha^2 \;+\; \frac{\alpha\,\beta\,(x \,-\, iy \,+\, x \,+\, iy)}{x^2 \,+\, y^2} \;+\; \frac{\beta^2\,(x^2 \,+\, y^2)}{(x^2 \,+\, y^2)^2} \\ &=\; \alpha^2 \;+\; \frac{2\,\alpha\,\beta\,x}{x^2 \,+\, y^2} \;+\; \frac{\beta^2}{x^2 \,+\, y^2}\,. \end{aligned} \tag{6.95}$$

We can re–substitute the actual parameters for α and β and also the geometric relationships $r^2 = x^2 + y^2$ and $x = r\cos\theta$, which yields

$$W\overline{W} \;=\; V^2 \;=\; u_0^2 \;+\; \frac{u_0\,m\,\cos\theta}{\pi\,r} \;+\; \left(\frac{m}{2\,\pi\,r}\right)^2\,. \tag{6.96}$$

Eq. (6.96) represents V^2 in any location in the flow domain. Let us focus on the velocity *on the surface of the body*, which requires utilizing the equations defining the surface for r and b in Eqs. (6.84) and (6.87). Specifically, we can combine Eqs. (6.84) and (6.87) to obtain

$$r \;=\; \frac{m\,(\pi - \theta)}{2\,\pi\,u_0\,\sin\theta}\,. \tag{6.97}$$

Substituting Eq. (6.97) into Eq. (6.96) yields a result strictly in terms of θ

$$V^2 \;=\; u_0^2\left(1 \;+\; \frac{2\cos\theta\sin\theta}{\pi - \theta} \;+\; \frac{\sin^2\theta}{(\pi - \theta)^2}\right)\,. \tag{6.98}$$

We can write a Bernoulli equation between the oncoming freestream flow and *any* point on the surface of the body according to the assumption of irrotational flow. Thus,

$$P_\infty \;+\; \frac{1}{2}\rho u_0^2 \;=\; P \;+\; \frac{1}{2}\rho V^2\,. \tag{6.99}$$

We now introduce an engineering quantity, the *coefficient of pressure* as (pressure coefficient)

$$C_p \;\equiv\; \frac{P \,-\, P_\infty}{\frac{1}{2}\rho u_0^2}\,, \tag{6.100}$$

which we can derive by substituting Eq. (6.99) to obtain

$$C_p = -\left(\frac{2\cos\theta\sin\theta}{\pi-\theta} + \frac{\sin^2\theta}{(\pi-\theta)^2}\right) . \tag{6.101}$$

This can be plotted as a function of θ as shown in Fig. 6.11. We recall that

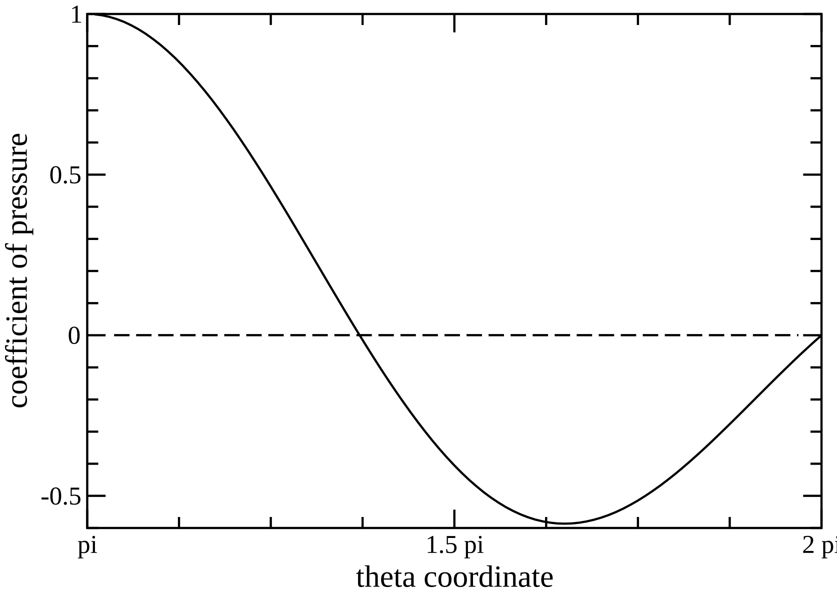

FIGURE 6.11. *Coefficient of pressure around the blunt body as a function of θ starting at the stagnation point $\theta = \pi$.*

there is a stagnation point at the nose of the body where the velocity is zero, $V = 0$. The pressure at this point is the so–called stagnation pressure, i.e. $P_{stag} = P_\infty + \frac{1}{2}\rho\, u_0^2$. Here, Eq. (6.100) defines $C_p = 1$, as shown in the figure[6.23]. As we move around the body on its surface, i.e. for increasing θ, the flow rapidly accelerates from rest and the pressure drops accordingly. The flow speed at the body's surface eventually reaches the freestream speed u_0, which is shown where $C_p = 0$. It continues to accelerate until it reaches a minimum of approximately $C_p \approx -0.59$, which corresponds to a flow speed at the surface of roughly $1.26u_0$. The pressure then starts to recover, eventually reaching P_∞ at $\theta = 2\pi$. We point out here that plotting C_p according to θ as shown in Fig. 6.11 compresses the natural scale of the problem. That is, as $\theta \to 2\pi$, the x coordinate behaves as $x \to \infty$ by simple geometric considerations. It is more complicated, but also more appropriate

[6.23]Note that $C_p = 1$ is not as readily calculated from Eq. 6.101 because zero appears in the denominators. The numerators are also zero. Evidently, Eq. 6.101 is *indeterminate* in the limit $\theta \to \pi$. This paradox can be resolved using L'Hospitals rule, which says essentially that both numerator and denominator are differentiated until the limit of both can be meaningfully evaluated (Kreyszig, 1988).

to use the distance traveled along the body's surface as a natural coordinate system (Panton, 1984).

As a closing, we mention some other component complex functions in Table 6.1. Panton (1984) and Fox and McDonald (1998) examine a number of interesting flows constructed from these components. As a general rule,

TABLE 6.1. Other complex potentials

$F(z)$	Notes	Type of Flow
$A\, z^n$	A and $n \geq 1/2$ are real constants	flow along various wall contours
$-i\Gamma(2\pi)^{-1} \ln z$	Γ is constant	a line vortex
$m(2\pi)^{-1} \ln(z + \epsilon)$	m and ϵ are constant	a source / sink doublet

it is fairly straightforward to assemble basic components and analyze the resulting flow. As we saw in this example, we obtain a flow configuration that we did not necessarily anticipate beforehand, but one which we were able to analyze quite thoroughly. The converse is not true. That is, it is typically much more difficult to specify the flow configuration, e.g. "a blunt body in a freestream flow", and then reverse–engineer the problem to obtain the complex potential. Also, it is emphasized that we are still neglecting viscous effects. This is often acceptable to obtain a first approximation to certain problems, however, in the example we just examined the rising "adverse" pressure gradient on the back side of the shape would tend to cause separation of an actual flow. Such factors cannot generally be accounted for by potential flow theory. We must generalize our treatment to include viscous effects.

6.9. Boundary Conditions for a Viscous Fluid

Up until now we have concentrated primarily on inviscid flows. In generalizing our discussions to viscous fluid, a number of important physical considerations arise. For example, we will now have to concern ourselves explicitly with characterizing fluid behavior at the boundaries of a problem, i.e. *boundary conditions*. Recall in §6.8 that fluid velocity on part of the boundary of the problem, i.e. the body, simply "fell out" naturally in the form of Eq. (6.98). In viscous flows, we will have to *a priori* prescribe fluid velocity at the boundaries in order to obtain a mathematical solution. Often this will mean determining this velocity (both normal and tangential) at the interface between our fluid and a solid boundary. How can this be accomplished?

The normal component of velocity at a solid boundary is fairly obvious from a kinematic standpoint. Flowing fluid does not penetrate the solid, so that it must come to a stop at the surface.

The tangential component is appreciably less straightforward to deduce. In fact, this was a topic of debate for quite some time, as described by Panton (1984). One school held that fluid slipped past the boundary, while another advocated the *no–slip* theory that fluid essentially adhered to the wall. It is certainly true that interaction at the wall is complicated at the microscopic scale. However, a large body of experimental evidence confirms that in most practical cases the no–slip model is essentially correct. For stationary solid boundaries, we can therefore write the velocity boundary condition as

no–slip boundary conditions

$$\mathbf{V}|_{boundary} = 0 . \tag{6.102}$$

That is, both normal and tangential components of velocity vanish. If the boundary is moving, then Eq. (6.102) would be modified so that $\mathbf{V}|_{boundary}$ is set equal to the boundary velocity, i.e. so that the fluid would still "stick" to the boundary. Panton (1984) discusses both the history and physics of this phenomenon in greater detail.

6.10. Viscous Flow

The last few sections relied on simple model forms for the constitutive relations, i.e. Eqs. (6.32) and (6.33). Here we introduce the appropriate relations for *Newtonian fluids*, where stresses are linearly related to deformation rates. For the normal stresses, the relations are

Newtonian fluids

$$\sigma_{xx} = -P + 2\mu\frac{\partial u}{\partial x} , \tag{6.103}$$

$$\sigma_{yy} = -P + 2\mu\frac{\partial v}{\partial y} , \tag{6.104}$$

and

$$\sigma_{zz} = -P + 2\mu\frac{\partial w}{\partial z} . \tag{6.105}$$

and for shear stresses they take the form

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) , \tag{6.106}$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) , \tag{6.107}$$

and

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) . \tag{6.108}$$

The derivation of Eqs. (6.103) through (6.108) is beyond the scope we cover here. Advanced texts such as Schlichting (1979) and Panton (1984) can be consulted.

As we did for inviscid flow, we can now substitute these constitutive relations into the equations we derived for the conservation of momentum, i.e. Eqs (6.29) through (6.31). Several terms in each equation group in the form of the continuity equation, Eq. (6.9), and therefore can be dropped. For example, in the x direction, we find

$$LHS = \frac{\partial}{\partial x}\left(-P + 2\mu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right], \tag{6.109}$$

where LHS is the "left hand side", which includes all acceleration and gravity terms. This simplifies to

$$LHS = -\frac{\partial P}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial y \partial x} + \mu\frac{\partial^2 w}{\partial z \partial x} + \mu\frac{\partial^2 u}{\partial z^2}. \tag{6.110}$$

Note that we have assumed μ is constant, so that it can be taken outside the differentials. We now use the assumption that the order of differentiation is irrelevant, that is for example

$$\frac{\partial^2}{\partial y\, \partial x} = \frac{\partial^2}{\partial x\, \partial y}. \tag{6.111}$$

We can then write

$$LHS = -\frac{\partial P}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x \partial y} + \mu\frac{\partial^2 w}{\partial x \partial z} + \mu\frac{\partial^2 u}{\partial z^2}, \tag{6.112}$$

which, although visually similar to Eq. (6.110), allows significant simplification that is not obvious in the former equation. Let us group the following terms under $\partial/\partial x$

$$LHS = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 u}{\partial z^2} + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \tag{6.113}$$

We see that the last term in brackets *is* the continuity equation, which equals zero according to Eq. (6.9). Therefore, this group of terms drops out entirely. We are left with a final x momentum equation of

$$\begin{aligned} \rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\right) &= \\ \rho\, g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)&. \end{aligned} \tag{6.114}$$

By similar procedures, we obtain the y and z direction equations, respectively, as

$$\begin{aligned} \rho\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\right) &= \\ \rho\, g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)&. \end{aligned} \tag{6.115}$$

and

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\right) = \tag{6.116}$$

$$\rho\, g_z \;-\; \frac{\partial P}{\partial z} \;+\; \mu \left(\frac{\partial^2 w}{\partial x^2} \;+\; \frac{\partial^2 w}{\partial y^2} \;+\; \frac{\partial^2 w}{\partial z^2} \right) .$$

Equations (6.114) through (6.116) are known as the *Navier–Stokes equations* for incompressible flow[6.24]. Taken with the Eq. (6.9) for the conservation of mass, we have 4 equations for the 4 unknown quantities (u, v, w, P). Navier–Stokes equations

This system is by far the most general we have yet discussed. It is valid for constant–property viscous incompressible flow. However, it is difficult to the point that no general solution has been developed[6.25]. We shall therefore restrict our study to idealized cases that can be solved exactly and other simplified configurations that can be solved approximately. For example, we consider the steady laminar flow between two infinite parallel plates (Fig. 6.12). The fluid flows in the x direction parallel to the plates, so $v =$

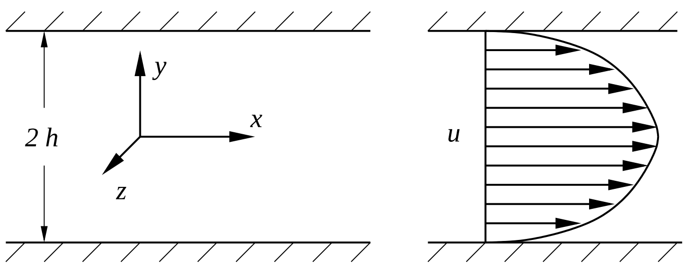

FIGURE 6.12. *Steady laminar viscous flow between infinite parallel plates: coordinates (left) and parabolic velocity profile (right).*

$w = 0$. By conservation of mass, e.g. Eq. (6.10), we find $\partial u/\partial x = 0$. In other words, the flow is *fully developed*, meaning that the rate of change of the u velocity profile with respect to x is zero. This is synonymous with saying the u profile is independent of the length traveled along the x coordinate[6.26]. Since we have assumed steady flow, $\partial/\partial t = 0$. Let us also ignore gravity terms for the moment[6.27]. If we apply these simplifications to Eqs. (6.114) through (6.116), we obtain the governing equations characteristic of this problem, i.e. fully developed flow

$$0 \;=\; -\,\frac{\partial P}{\partial x} \;+\; \mu\, \frac{\partial^2 u}{\partial y^2}\,, \tag{6.117}$$

[6.24]These equations were derived by the mathematician L. Navier in 1827 and, independently, by the mechanician Sir G. Stokes in 1845. Sabersky et al. (1999) provide a brief history of the development of these equations in their first chapter.

[6.25]Significant progress toward a general solution of the Navier–Stokes equations is one of the seven great unsolved problems of mathematics according to the Clay Institute. Each problem carries a US $1 million prize.

[6.26]In later chapters, we will also study *developing flows*, whose u profile is not constant with respect to x.

[6.27]More specifically, let us assume that density is small so that there is no significant hydrostatic consideration for the problem, vis–à–vis Eq. (2.25).

$$0 \;=\; -\,\frac{\partial P}{\partial y}\,, \tag{6.118}$$

and

$$0 \;=\; -\,\frac{\partial P}{\partial z}\,, \tag{6.119}$$

Of course, Eq. (6.117) is the most interesting because the other two equations simply imply that the pressure distributions in the y and z directions are constant. If pressure is constant in y and z, P can be, at most, a function of x, i.e. $P(x,y,z) \rightarrow P(x)$. Therefore,

$$\frac{\partial P}{\partial x} \;\rightarrow\; \frac{dP}{dx}\,. \tag{6.120}$$

The pressure gradient represents the driving force for the flow and we take it to be a prescribed constant. Since it is clear that u is only a function of y, we have by similar argument

$$\frac{\partial^2 u}{\partial y^2} \;\rightarrow\; \frac{d^2 u}{dy^2}\,, \tag{6.121}$$

which yields

$$\frac{d^2 u}{dy^2} \;=\; \frac{1}{\mu}\,\frac{dP}{dx}\,. \tag{6.122}$$

Eq. (6.122) is a linear ordinary differential equation that can be twice integrated to obtain

$$u \;=\; \frac{1}{2\mu}\left(\frac{dP}{dx}\right) y^2 \;+\; C_1\, y \;+\; C_2\,, \tag{6.123}$$

where C_1 and C_2 are constants of integration. These must be determined from the boundary conditions, which are implied by Fig. 6.12 as

$$u|_{y=\pm h} \;=\; 0 \tag{6.124}$$

from the no–slip condition of viscous flow in Eq. (6.102). Solving and substituting, we find the final exact solution to be

$$u \;=\; \frac{1}{2\mu}\left(\frac{dP}{dx}\right)(y^2 \;-\; h^2)\,. \tag{6.125}$$

This expression describes a parabolic velocity profile as depicted in Fig. 6.12. Note that the pressure gradient is *negative*, i.e. the flow proceeds from high pressure to low pressure. Therefore, u in Eq. (6.125) will always be positive. From this expression we can derive various other properties of interest including the maximum velocity at $y = 0$

$$u_{max} \;=\; -\,\frac{h^2}{2\mu}\left(\frac{dP}{dx}\right) \tag{6.126}$$

and the average velocity

$$u_{avg} \;=\; \frac{1}{2h}\int_{-h}^{h} u\,dy \;=\; \frac{1}{2h}\int_{-h}^{h}\frac{1}{2\mu}\left(\frac{dP}{dx}\right)(y^2\;-\;h^2)\,dy\,, \tag{6.127}$$

which is evaluated as

$$u_{avg} \;=\; -\,\frac{h^2}{3\mu}\left(\frac{dP}{dx}\right)\,. \tag{6.128}$$

We then have $u_{max} = 1.5u_{avg}$.

There are many other idealized problems for which the Navier–Stokes equations can be solved. For example, lubricant flow between a bearing and journal can be modeled by allowing one of the boundaries in Fig. 6.12 to translate along the x direction. There are a number of variations for the one–dimensional problem (Munson et al., 2006). Panton (1984) and Schlichting (1979) discuss time–dependent generalizations for this problem and Wendl (1999) addresses the two–dimensional spatial form. Panton (1984), Schlichting (1979), and Berker (1963) have particularly good discussions of other viscous flow solutions.

CHAPTER 7

Dimensional Formulations

Up until now, we have focused on describing conservation laws for fluid flows. In this sense, fluid mechanics has likely been quite similar to other engineering and physics courses you have taken thus far, e.g. most of the material in statics is devoted to deriving and applying laws governing forces and moments on bodies at rest. The existence of appropriate scales for measuring and describing these dimensional quantities (force, length, velocity, etc.) was taken completely for granted! Here, we will study the concept of dimensionality on a more systematic level. Dimensional analysis can extend our ability to handle difficult problems having many variables and is critical in both experimental and theoretical approaches.

The primary observation prompting such study is that there are no natural units of measure relevant to all applications of a physical variable. For example, the speed of light in a vacuum, c, is a constant. Therefore, it is a good scale for comparing other velocities in problems of astrophysics and relativity, i.e. it is a natural basis of measure. For example, one can speak of $0.9c$ as a velocity. However, the speed of light is not relevant to fluid mechanics problems we are interested in. We are therefore left with the fact that we must construct our own relevant measuring scales that are appropriate for problems we are interested in. This is a common task in all subjects of mechanics. As a matter of fact, appropriate scales often depend on individual problems. The payoff for this process will be a more concise and systematic problem formulation that can yield significant insights, even if the problem is too difficult to solve exactly. Moreover, it is the basis for *similitude*, i.e. the principle that allows measurements and results for a model flow problem to be extrapolated to other problems of interest that are *dimensionally similar*[7.1]. This can take the form, for example, of measuring performance of a wind tunnel model of a wing. If the experiment is constructed properly, the results can be extrapolated, according to certain rules, to a real wing section that might be too large to test in a tunnel and too difficult to outfit with real–time measurement capability.

similitude

There are a number of additional, and quite important advantages in correctly casting a problem in a dimensional framework. We will see these in the coming chapters. Briefly, dimensionless numbers often describe the

[7.1]The term *dimensionally similar* does not necessarily imply geometric dimensions. It implies relations between relevant dimensionless groups of variables, which we discuss throughout the remainder of this chapter.

physical nature of a flow configuration, for example whether it is laminar or turbulent (Chapters 8 and 9). Also, dimensional formulation leads to a net reduction in the number of variables that describe a problem. The information content of a problem increases rapidly with the number of variables (Fig. 7.1), so any means of collapsing variables without loosing inherent information is valuable.

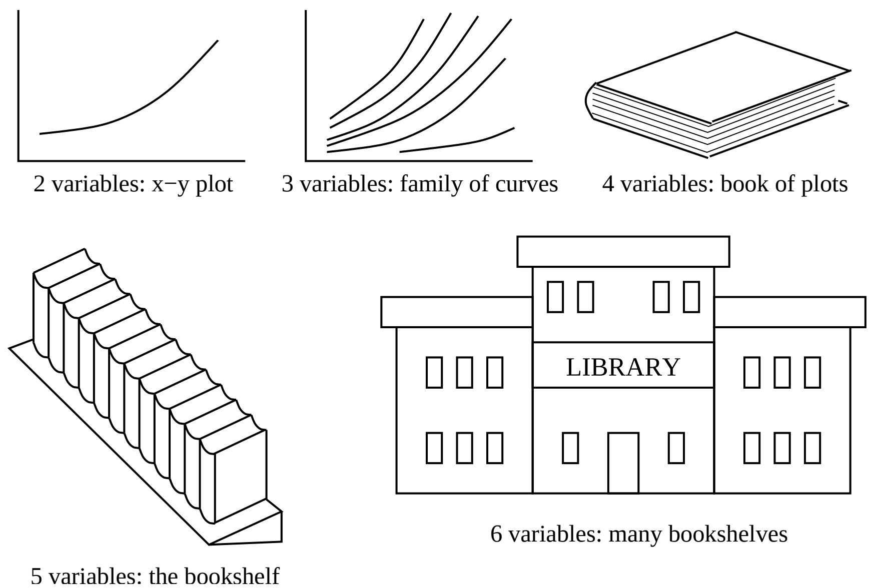

FIGURE 7.1. *Information content increases rapidly as the number of variables associated with a problem grows.*

7.1. Measurement and Dimensions

There are two classes of quantification: counting and measuring. A counted quantity is dimensionless, e.g. the number of water molecules in a lake. Conversely, a quantity that is measured has units associated with it. The fundamental procedure of measurement is simply to compare what we want to measure with a defined scale having a rigorous definition of an appropriate unit. Thus, the basic problem is defining units. Take the task of measuring length as an example. First, we require a standard reference unit of length, e.g. a meter, a light year, etc. Let us abstract this fundamental unit of length as L, without specifying it explicitly. We know simply that it is a real number. Let us now define the following: We measure a particular length that has an absolute size of $\hat{l}$. Our measurement indicates l units of the fundamental basis L. We can therefore write

$$\hat{l} = l \times L \,. \tag{7.1}$$

As a numerical example of Eq. (7.1), we might have a fundamental unit size of 1 meter. An object of length 20 meter units would be equivalent to

20 of the 1 meter units. This may seem trivially obvious, but it illustrates the fundamental difference between the magnitude of a variable l and its physical dimension L. We can write down similar relationships for mass and time as

$$\hat{m} = m \times M \tag{7.2}$$

and

$$\hat{t} = t \times T \,. \tag{7.3}$$

The equations we have discussed so far are all relations among l, m, and t type variables, i.e. among variables having man–made scales. However, we would expect these equations to be valid in any arbitrary system of measuring units we prescribe.

7.2. Bridgman's Equation

Rather than being formulated exclusively in terms of one of the "basic" units of length, mass, or time, many variables exist as combinations of these units. For example, because it is defined as the rate of change of a length with respect to time, velocity is made of both length and time units

$$\hat{v} = v \times \frac{L}{T} = v \times L\,T^{-1}\,M^0 \,. \tag{7.4}$$

Values change if we alter the length or the time unit, or both. Force can be represented as

$$\hat{F} = F \times L\,T^{-2}\,M^1 \,, \tag{7.5}$$

and so on. This implies that we can describe any arbitrary parameter $\hat{\xi}$ as

$$\hat{\xi} = \xi \times L^a\,T^b\,M^c \,, \tag{7.6}$$

where a, b, and c are always fractions[7.2] and L, T, and M are the sizes of the length, time, and mass scales, respectively. Eq. (7.6) is a special case of *Bridgman's Equation*. Here L, T, and M are the primary dimensions, or scales, by which we measure other things. A more general formulation admits other combinations of primary dimensions (Panton, 1984), but we will focus only on the LMT set. We now state, without proof, the following theorem (see e.g. Panton, 1984)

Bridgman's Equation

THEOREM 7.1 (Minimum Dimensions). *The minimum number of primary dimensions necessary to describe all physical variables is 3.*

It is emphasized that this is valid for *any* physical variable. An non–obvious example is temperature, which, if interpreted as energy per unit mole, has units of[7.3] ML^2/T^2. In a more general formulation, it is possible to have *more* than 3 primary dimensions, however one must then prescribe extra relationships to account for redundancy (Panton, 1984).

[7.2] Often these parameters are whole numbers, but they are never irrational numbers.

[7.3] Remember that here T is the time unit, not temperature!

Let us look at a preview of dimensional analysis. Suppose it is known that the pressure P at any point $\vec{x}$ depends also upon velocity u_0, density ρ, viscosity μ, a characteristic length d, and a freestream pressure P_∞. We can write in functional form

$$P = f(\vec{x},\, u_0,\, \rho,\, \mu,\, d,\, P_\infty)\ . \tag{7.7}$$

Let us now assume we know in advance that the proper measuring scale for pressure is ρu_0^2 so that

$$\Pi = \frac{P}{\rho\, u_0^2} \tag{7.8}$$

is dimensionless. Here, Π can be thought of as a non–dimensional form of pressure. It follows that all the exponents of the primary dimensions in Bridgman's equation are zero

$$\hat{\Pi} = \Pi \times L^0\, T^0\, M^0\ , \tag{7.9}$$

which implies

$$\hat{\Pi} = \Pi\, . \tag{7.10}$$

Here, we make a remarkable observation: The value of this non–dimensional variable is completely independent of any system of measurement. It is the natural scale of the problem. Of course, the natural scale is problem dependent. For example, there are certain aerodynamics problems where ρu_0^2 is *not* the proper scale.

A simplistic recasting of Eq. (7.7) yields

$$\Pi = \frac{P}{\rho\, u_0^2} = \frac{1}{\rho\, u_0^2}\, f(\vec{x},\, u_0,\, \rho,\, \mu,\, d,\, P_\infty)\ . \tag{7.11}$$

The Buckingham–Pi theorem, which we discuss momentarily, provides a systematic procedure to collapse an expression such as Eq. (7.11) into a more concise form

$$\Pi = \frac{P}{\rho\, u_0^2} = F\left(\frac{\vec{x}}{d},\, \frac{\rho\, d\, u_0}{\mu},\, \frac{P_\infty}{\rho\, u_0^2}\right)\ . \tag{7.12}$$

Or, rewriting symbolically

$$\Pi_1 = F(\Pi_2,\, \Pi_3,\, \Pi_4)\ . \tag{7.13}$$

Here, a function of 6 variables in Eq. (7.7) has been reduced to a function of 3 variables in Eqs. (7.12) and (7.13). This has substantial implications in, for example, the amount of experimental data which must be gathered to formulate f (or F), as pointed out in Fig. 7.1.

7.3. The Buckingham Pi Theorem

The *Buckingham Pi* theorem allows us to determine, in a systematic fashion, how many dimensionless variables are associated with a set of dimensional variables. It relies on the assumption that all variables obey Bridgman's Equation, i.e. Eq. (7.6). We state the theorem formally as

Buckingham Pi Theorem

THEOREM 7.2 (Buckingham Pi Theorem). *A dimensionally homogeneous function of k variables can be reduced to a corresponding function of $k - r$ independent variables, where r is the minimum number of reference dimensions required to describe all the variables.*

Let us dissect this theorem a bit. A *dimensionally homogeneous function* simply means that the dimension of any equation, or function, must be consistent. For example, in the function

$$u_1 = f(u_2, u_3, \ldots, u_k) \, , \tag{7.14}$$

the dimensions of the variable on the left hand side of the equal sign must match the dimensions of any and all terms that stand by themselves on the right hand side. Specifically, we cannot mix, for example, meters per second and miles per day for velocity in the same equation. Reference dimensions were discussed in the previous section. The number of reference dimensions r is often 3, but it can be fewer in certain instances. We shall introduce a systematic way to determine this parameter. If we perform the analysis correctly, we should be able to transform Eq. (7.14) into a new dimensionless form with r fewer variables

$$\Pi_1 = F(\Pi_2, \Pi_3, \ldots, \Pi_{k-r}) \, , \tag{7.15}$$

7.4. Determination of Pi Terms

We can now formulate a general procedure for transforming a dimensional equation or function into a dimensionless one using the Buckingham Pi theorem. The following steps are recommended:

(1) List *all* variables that are involved in your problem. This would include all variables related to geometry, fluid properties, physical constants, and any variable associated with external effects, such as freestream pressure, etc. However, make sure that all variables are independent! For example, the geometry for a pipe flow problem could be stated in terms of either the pipe radius, diameter, or cross–sectional area. Use only one of these. Any single one is independent, and the other two then depend on the first one. Using more than one will complicate the problem. The final number of variables is then k.

(2) Determine the number of reference dimensions, r. This will often be 3, but can be less if there are linear dependencies. We will show an example of this momentarily. The way to systematically determine r is as follows:

(a) Construct a dimensional matrix using *all* variables determined in the previous step, where row designations are primary dimensions and column designations are the physical variables. For example, if we have a problem encompassing 4 variables: a length d, density ρ, a viscosity μ, and a velocity u_0, the dimensional matrix would be as shown. This means, for example,

	μ	d	ρ	u_0
M	1	0	1	0
L	-1	1	-3	1
T	-1	0	0	-1

that density has the dimensionality of mass over a length dimension to the third power.

(b) The dimensional matrix is used to check for linear independence of the variable dimensions in terms of the primary dimensions. This is done by evaluating the *rank* of the matrix[7.4]. The rank of the matrix is equal to r, which for the above example is 3.

(3) Determine the number of dimensionless numbers (or groups) as $k - r$. In the above example, we have $4 - 3 = 1$ dimensionless number.

(4) Select a number of repeating variables equal to r. Choose them such that *all of the primary dimensions are represented* among these r variables. Also, the group of repeating variables cannot be linearly dependent. It is therefore typically a good choice to use the set of variables corresponding to the rank calculation above. In the example given, we would choose d, ρ, and u_0 as our $r = 3$ repeating variables.

(5) For each of the $k - r$ remaining variables (i.e. not the repeating variables), write a Bridgman equation and solve the exponents such that the result is dimensionless. In the above example, we would have only 1 Bridgman equation, written as

$$\Pi = \mu \, d^a \, \rho^b \, u_0^c \, .$$

Since this group must be dimensionless, we find

$$M^0 \, L^0 \, T^0 = \left(M^1 L^{-1} T^{-1}\right) \left(M^0 L^1 T^0\right)^a \left(M^1 L^{-3} T^0\right)^b \left(M^0 L^1 T^{-1}\right)^c ,$$

[7.4]The rank of a matrix is the size of the largest *square* submatrix having a non–zero determinant. For dimensional analysis problems, the largest possible rank of any problem (using 3 primary dimensions) will be 3. Therefore, start by checking all 3×3 sub–matrices in the dimensional matrix. If the determinants of *any* of these are non–zero, then the rank is 3. If the determinants of *all* of these are zero, then start checking all 2×2 sub–matrices. If the determinants of *any* of these are non–zero, then the rank is 2. Continue until the rank is determined. In the example above, the rank is 3 since the sub–matrix formed by d, ρ, and u_0 has a non–zero determinant. Kreyszig (1988) contains a more formalized discussion of rank.

which gives us a set of equations to solve for exponents a, b, and c, i.e.

$$\begin{aligned}0 &= 1 + 0a + 1b + 0c\\ 0 &= -1 + 1a - 3b + 1c\\ 0 &= -1 + 0a + 0b - 1c\end{aligned}$$

We find $a = b = c = -1$, therefore, our dimensionless group is

$$\Pi = \frac{\mu}{d\,\rho\,u_0}\,.$$

(6) Check to confirm that each group is actually dimensionless.
(7) Express dimensionless groups in their final forms.

The last step is not entirely intuitive. For example, the dimensionless group we have just formulated would be better utilized in its inverse form. That is, we define a new dimensionless variable Π' as the inverse of Π above

$$\Pi' = \frac{1}{\Pi} = \frac{d\,\rho\,u_0}{\mu}\,.$$

This is of course the Reynolds number, which we will discuss momentarily. This example leads to another rule of dimensional analysis:

Theorem 7.3 (Rearrangement of Dimensionless Groups). *Products, quotients, and powers of dimensionless groups remain valid dimensionless groups.*

This is somewhat intuitive for two reasons. First, the beginning and ending sets of groups are both dimensionless. Second, we could obtain the alternate groups by starting with different combinations of variables at the beginning of the problem. That is, a set of dimensionless groups we obtain from one particular analysis is not unique!

7.5. An Example Having Reduced Reference Dimensions

For many dimensional analyses, the number of reference dimensions is $r = 3$, as with the example above. The exception is when there are linear dependencies among the group of variables. However, the systematic method we have presented using the rank evaluation handles such cases without any special treatment.

For example, assume that a cylindrical storage tank of diameter D is filled with liquid of specific weight γ up to a height h. The tank is supported around its perimeter such that there is a vertical deflection in the center of δ, where d is the thickness of the bottom and E is its modulus of elasticity[7.5]. We can then formulate an expression for the deflection as a function of the other variables, i.e.

$$\delta = f\,(d,\ h,\ D,\ \gamma,\ E)\ . \tag{7.16}$$

Listing the variables in this manner fulfills the first step of dimensionless analysis. We see that there are $k = 6$ relevant dimensional variables.

[7.5] Elastic modulus is given in terms of force per units area, e.g. N/m^2.

The second step is to construct the dimensional matrix, shown in Table 7.1. We must now determine the number of primary dimensions by

TABLE 7.1. The Dimensional Matrix

	δ	d	h	D	γ	E
M	0	0	0	0	1	1
L	1	1	1	1	-2	-1
T	0	0	0	0	-2	-2

evaluating the rank of this matrix. We start by evaluating all 3×3 sub–matrices, proceeding arbitrarily from the left. The first matrix, i.e. the one having columns δ, d, and h, is

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0\,, \tag{7.17}$$

which has a vanishing determinant. The next matrix to the right, i.e. columns d, h, and D, is identical. These two trials do not establish the rank since their determinants vanish. The third matrix, having columns h, D, and γ, is

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & -2 \\ 0 & 0 & -2 \end{vmatrix} = 0 - 0 + 1(0-0) = 0\,, \tag{7.18}$$

also does not establish the rank. Finally, we see that the last matrix

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 0 & -2 & -2 \end{vmatrix} = 0 - 1(-2-0) + 1(-2-0) = 0 \tag{7.19}$$

also has a zero determinant.

Since we have examined all 3×3 sub–matrices and found that all have vanishing determinants, the number of reference dimensions must be less than 3. We therefore start evaluating all 2×2 sub–matrices. We see by inspection that sub–matrices associated strictly with the "length"–type variables will all have zero determinants, for example, taking columns δ and d and rows M and L, we find

$$\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} = 0\,. \tag{7.20}$$

At the 2×2 level, we can evaluate many of the possibilities by inspection. Let our next pick be one that we know will work, e.g. the 2×2 sub–matrix having columns D and γ and rows M and L. We find

$$\begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1\,. \tag{7.21}$$

We have now found a non–zero determinant for a 2×2 sub–matrix. Our system therefore has a rank of 2, which by definition means that there are 2 reference dimensions required, i.e. $r = 2$.

For the next step, we determine how many dimensionless parameters will be specified by our analysis as $k - r = 6 - 2 = 4$. We require $r = 2$ repeating variables and we choose the variables D and γ since we already know from our above analysis that they are linearly independent. We now write $k - r = 4$ Bridgman's equations for δ, h, d, and E to derive the dimensionless groups. For δ, we obtain

$$\Pi_1 \;=\; \delta \, D^a \, \gamma^b \,, \tag{7.22}$$

which implies

$$M^0 \, L^0 \, T^0 \;=\; \left(M^0 L^1 T^0\right) \, \left(M^0 L^1 T^0\right)^a \, \left(M^1 L^{-2} T^{-2}\right)^b \,. \tag{7.23}$$

We equate exponents, giving 1 equation for each of M, L, and T.

$$0 \;=\; 0a \;+\; 1b \tag{7.24}$$

$$0 \;=\; 1 \;+\; 1a \;-\; 2b \tag{7.25}$$

$$0 \;=\; 0a \;-\; 2b \tag{7.26}$$

Notice that have 3 equations but only 2 unknowns! However, in this special case where $r = 2$, we see that of these 3 equations, two of them are *not* independent of each other. That is, Eqs. (7.24) and (7.26) *both* require $b = 0$. Eq. (7.25) is independent of the other two. From this we find $a = -1$. Therefore, our first dimensionless group is

$$\Pi_1 \;=\; \frac{\delta}{D} \,. \tag{7.27}$$

It is probably not too difficult to see by inspection that, because the dimensions for h and d are the same as those for δ, we obtain two more dimensionless groups that are the same form as for Π_1. Specifically,

$$\Pi_2 \;=\; \frac{h}{D} \tag{7.28}$$

and

$$\Pi_3 \;=\; \frac{d}{D} \,. \tag{7.29}$$

Eqs. (7.28) and (7.29) are written down directly in lieu of writing two of the Bridgman's equations.

We now have 3 of the 4 required dimensionless groups. The last Bridgman's equation is written for E as

$$\Pi_4 \;=\; E \, D^a \, \gamma^b \,, \tag{7.30}$$

which implies

$$M^0 \, L^0 \, T^0 \;=\; \left(M^1 L^{-1} T^{-2}\right) \, \left(M^0 L^1 T^0\right)^a \, \left(M^1 L^{-2} T^{-2}\right)^b \,. \tag{7.31}$$

We again equate exponents, giving 1 equation for each of M, L, and T.

$$0 = 1 + 0a + 1b \tag{7.32}$$

$$0 = -1 + 1a - 2b \tag{7.33}$$

$$0 = -2 + 0a - 2b \tag{7.34}$$

Again, two of the equations are dependent, that is, Eq. (7.32) multiplied by -2 yields Eq. (7.34). We find $b = -1$. Plugging this into Eq. (7.33), we get $a = -1$. Our final dimensionless group is therefore

$$\Pi_4 = \frac{E}{D\,\gamma}\,. \tag{7.35}$$

We can therefore recast the 6–variable dimensional characterization of the problem in Eq. (7.16) as the following 4–variable dimensionless one

$$\frac{\delta}{D} = F\left(\frac{d}{D}, \frac{h}{D}, \frac{E}{D\,\gamma}\right)\,, \tag{7.36}$$

which can also be written directly as

$$\Pi_1 = F\left(\Pi_2, \Pi_3, \Pi_4\right)\,, \tag{7.37}$$

7.6. Some Dimensionless Numbers

Dimensionless analysis is important in all branches of science and engineering. There are several hundred in common usage (Weast and Astle, 1982). Here we introduce a few that are common to many problems in fluid mechanics.

- Reynolds Number : This parameter is perhaps the most famous one in fluid mechanics and has the form

$$Re = \frac{u_0\,L}{\nu}\,, \tag{7.38}$$

  where u_0 and L are velocity and length scales, respectively, and ν is kinematic viscosity. It gives the ratio of inertia forces to viscous forces and is an indicator of the transition point between laminar and turbulent flow.
- Froude Number : This parameter is important in flows having a free surface and has the form

$$Fr = \frac{u_0}{\sqrt{g\,L}}\,, \tag{7.39}$$

  where u_0 and L are again velocity and length scales, respectively, and g is the acceleration of gravity[7.6]. It is an indicator of the relative size of inertia forces versus gravity forces.

[7.6] Fr is also often defined as the square of the expression in Eq. (7.39).

7.7. Further Study

We have only introduced the rudiments of dimensional analysis here. There are many good texts for further study, especially Panton (1984), which formally proves the results we have used here, notably Bridgman's equation and the Buckingham Pi theorem. We close this chapter with a practical example for pipe flow, which we will study in greater depth in Chapter 8.

<u>Example 7.1:</u>

Pressure drop along a pipe, ΔP, is an important aspect of the piping design problem because it governs the required pumping power. For laminar flow in a long, straight pipe section, ΔP depends upon the diameter and length of the pipe, D and L, respectively, the fluid properties μ and ρ, and the average velocity of fluid in the pipe, $\bar{u}$. Cast this problem in terms of the relevant dimensionless variables.

The problem states that there is the following functional relationship among the variables

$$\Delta P \;=\; f\left(\bar{u},\, L,\, D,\, \mu,\, \rho\right),$$

which is written in a similar form to Eqs. (7.7) and (7.16). In other words, ΔP depends on all the variables in $f\,(\,)$. There are clearly $k = 6$ variables, altogether. That is, there are 5 independent variables and 1 dependent variable.

To determine the number of reference dimensions, r, we construct the dimensional matrix. We see by inspection that the columns for ΔP, $\bar{u}$,

	ΔP	$\bar{u}$	L	D	μ	ρ
M	1	0	0	0	1	1
L	-1	1	1	1	-1	-3
T	-2	-1	0	0	-1	0

and L yield a matrix whose determinant does not vanish. The rank of this matrix is therefore 3, so that $r = 3$. We therefore expect $k - r = 6 - 3 = 3$ dimensionless variables and require $r = 3$ repeating variables to determine them. We select ΔP, $\bar{u}$, and L as the repeaters. For the 3 remaining (non–repeating) variables, D, μ, and ρ, we write dimensionless Bridgman's equations and solve for the powers. For example, we cast the first Pi group Π_1 in terms of D as $\Pi_1 = D\,\Delta P^a\,\bar{u}^b\,L^c$, which expands to

$$M^0\,L^0\,T^0 \;=\; \left(M^0 L^1 T^0\right)\,\left(M^1 L^{-1} T^{-2}\right)^a\,\left(M^0 L^1 T^{-1}\right)^b\,\left(M^0 L^1 T^0\right)^c\,.$$

We have three unknowns, a, b, and c, and three equations for the $M/L/T$ system. Solving, we find $(a, b, c) = (0, 0, -1)$, which implies

$$\Pi_1 \;=\; \frac{D}{L}\,.$$

This is clearly a valid dimensionless group, and in retrospect is probably an obvious one. We perform the same operations for the two remaining Pi groups: $\Pi_2 = \mu\, \Delta P^a\, \bar{u}^b\, L^c$ and $\Pi_3 = \rho\, \Delta P^a\, \bar{u}^b\, L^c$. Solving the resulting equations yields

$$\Pi_2 = \frac{\mu\, \bar{u}}{\Delta P\, L} \quad \text{and} \quad \Pi_3 = \frac{\rho\, \bar{u}^2}{\Delta P}\,.$$

We check that Π_2 and Π_3 are actually dimensionless.

Now we come to the last, and most non–intuitive step, which is to express the Pi groups in their final form. The Π_1 group seems acceptable at first, but if we examine it more closely, a more relevant group would be

$$\Pi_1' = \frac{1}{\Pi_1} = \frac{L}{D}\,.$$

This arrangement physically suits the problem somewhat better because pressure drop along the pipe can be approximated as

$$\frac{dP}{dx} \approx \frac{\Delta P}{L}\,,$$

which we can re–arrange as

$$\Delta P \approx L\, \frac{dP}{dx}\,.$$

The implication is that ΔP will grow with L for a given value of dP/dx. In other words, L/D is a an appropriate (in fact the natural) dimensionless pipe "length", which is associated with pressure drop. Recall that it is perfectly valid to take the inverse of Π_1 to obtain a new group Π_1'.

The other two groups are not immediately recognized quantities. Let us form a new group Π_2' from Π_2 by operating as

$$\Pi_2' = \frac{\Pi_3\, \Pi_1}{\Pi_2} = \frac{\rho\, \bar{u}^2}{\Delta P} \times \frac{D}{L} \times \frac{\Delta P\, L}{\mu\, \bar{u}} = \frac{\rho\, \bar{u}\, D}{\mu} = Re\,,$$

which we readily recognize as the Reynolds number, Re. Thus, the ratio of inertial forces to viscous forces, given in the form of the Reynolds number, is another natural parameter of the problem.

Lastly, let us recast Π_3 as

$$\Pi_3' = \frac{2}{\Pi_3} = \frac{\Delta P}{\frac{1}{2}\, \rho\, \bar{u}^2}\,.$$

The numerator is the pressure drop, while the denominator is the dynamic pressure introduced back in Chapter 3. Dynamic pressure is thus the natural measuring scale for pressure drop. Another way to interpret Π_3' is to notice that the quantity in the denominator looks very much like kinetic energy. However, instead of mass, this term contains density, i.e. mass per unit volume. Thus, Π_3' quantifies pressure drop with respect to the kinetic energy *per unit volume* of the flow.

We thus write the final formulation of the problem as

$$\frac{\Delta P}{\frac{1}{2}\, \rho\, \bar{u}^2} = F\left(Re,\, \frac{L}{D}\right)\,.$$

That is, dimensionless pressure drop is a function of the Reynolds number and the L/D ratio. Note that we have reduced a "library" size problem to one of a family of curves (Fig. 7.1).

$\diamondsuit\diamondsuit\diamondsuit$

CHAPTER 8

Viscous Pipe Flow

Our treatment of fluid mechanics has so far concentrated on building a theoretical foundation via deriving conservation laws and casting them in useful forms. In this chapter, we will focus on a specific, but very important application: internal flow in pipes. While we will confine our study to the type of "pipes" we generally recognize as e.g. copper pipes in residential water systems, the concepts and principles introduced here are applicable to the general case of internal flow in conduits. For example, internal flows are important in

- Industrial Applications
  - steam, petroleum, chemical, and other factory piping
  - industrial cooling, filtering, pumping, and hydraulic systems
  - large–scale transport, e.g. Alaskan pipeline, city water system
  - hydroelectric applications
  - wind tunnels, water tunnels, and other test equipment
- Commercial and Residential Applications
  - HVAC, plumbing, and sewer systems
  - fluid and liquid handling
  - automotive cooling, fuel, brake, and exhaust systems
- Biomedical Applications
  - human cardiovascular and respiratory systems
  - medical equipment, e.g. renal dialysis
  - capillary and hypodermic flows

Problems such as these are generally too complicated to yield to theoretical approaches alone. Therefore, in addition to our theoretical tools, we will utilize empirical data characterized by the dimensional treatment introduced in Chapter 7.

8.1. General Description of Pipe Flow

For our purposes, we will define a "pipe" as any closed conduit for transporting a fluid which can withstand an internal pressure without deforming. A common example is the average copper water pipe, which moves water by the action of an internal pressure gradient. The pipe cross–section is often circular, but this is not a necessity. For example, HVAC ducts usually have a rectangular section and specialty equipment, such as heat exchanger tubes and pipette capillaries, may have internal fins or beads (Figure 8.1). In all

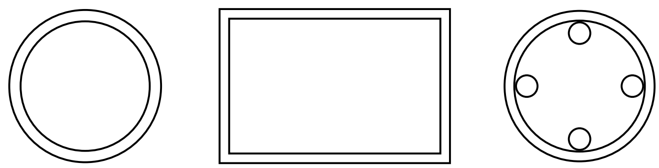

FIGURE 8.1. *Common pipe cross sections, including from left: circular section found in water pipes, hoses, etc.; rectangular section found in wind tunnels and HVAC systems; finned circular section found in specialty equipment.*

cases, we assume that the fluid fills the entire section of the pipe. This will actually exclude some systems named above in certain cases. For example, water flowing in a half–full concrete storm sewer is driven by gravity alone. A pressure gradient cannot be sustained in such systems.

Pipe flows fall into two major classifications: laminar or turbulent[8.1]. Laminar flow is typically defined as smooth streamline flow, where the fluid moves in orderly layers free of random fluctuations. Conversely, turbulent flow is loosely defined as a flow in which the fluid moves in an erratic disordered manner having significant random fluctuation. Perhaps the best way to illustrate the difference is to observe the streamwise velocity component u at a given (x, y, z) location in our pipe. For "steady" flow, Fig. 8.2 shows that u is constant. This is what we would intuitively expect for truly steady

laminar flow

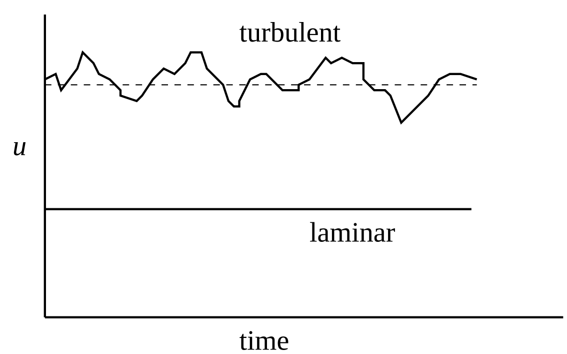

FIGURE 8.2. *Instantaneous velocities for laminar and turbulent flows as a function of time at a specific single location (x, y, z) in a pipe.*

[8.1]Many investigators consider the transition regime between laminar and turbulent flows as a third flow type. This is technically correct, although its complexity is such that it is not ordinarily introduced in a first course in fluid mechanics. We will not discuss it further here. Likewise, turbulence is a complicated phenomenon and we have not developed a sufficient mathematical framework to examine it analytically. We shall therefore rely in large degree on empirical information for studying turbulence. For detailed discussions of transition and turbulence see e.g. Batchelor (1953); Tennekes and Lumley (1972); Hinze (1975); Schlichting (1979); Drazin and Reid (1981); Panton (1984).

flow. On the other hand, turbulent flow is, by definition, never steady. The instantaneous value of u is very erratic. However, it fluctuates about a mean velocity $\bar{u}$, which is essentially steady (the dashed line in Fig. 8.2). This implies that the remaining velocity components v and w are non–zero. Therefore, turbulence is unsteady and multi–dimensional by nature.

The Reynolds number Re was introduced in Chapter 7, (Eq. (7.38) on pp. 95), and it was said that Re is, in many situations, the main parameter that characterizes whether flow is laminar or turbulent. For pipe flow, the characteristic length dimension is usually taken as the diameter, D, and the characteristic velocity scale is the average velocity u_{avg}. Therefore, the Reynolds number has the form

$$Re = \frac{u_{avg}\, D}{\nu}, \tag{8.1}$$

where ν is the kinematic viscosity. Notice that none of these variables taken alone describes the flow type. Their combination, as grouped in Eq. (8.1), is the important parameter. That said, it is not possible to give an exact Re which delineates laminar from turbulent flow regimes because the situation depends upon the degree to which flow is disturbed by vibrations, the smoothness of the approach flow, etc. For pipe flow in a general engineering application, i.e. one in which no special treatment is used to eliminate the effects, we can define laminar flow as[8.2]:

DEFINITION 8.1. *The flow in a round pipe is laminar for $Re \leq 2100$.*

If pains are taken to minimize building vibration and to ensure a very smooth approach flow, the transition value of 2100 can be raised at least by a factor of two, however, we will not consider such special cases here.

Another general aspect of pipe flow is the concept of the *entrance region* and flow development (Fig. 8.3). Suppose that flow enters a pipe section

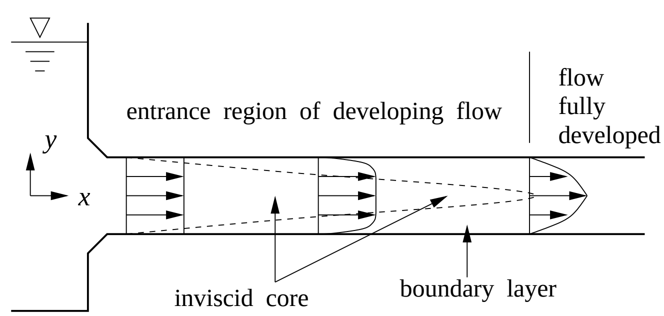

FIGURE 8.3. *Flow development in the entrance region of a pipe.*

with a uniform velocity profile. As the fluid moves along the pipe, viscous

[8.2]Some texts quote a transition Reynolds number of 2300 rather than 2100, e.g. Fox and McDonald (1998), however, for our treatment, we will use the value 2100.

effects and the no–slip condition at the inner pipe wall begin to modify the velocity profile. Two regions become apparent. Close to the wall is a *boundary layer*, where viscous effects are important, but toward the center, flow remains largely inviscid and the profile remains uniform. As x increases, the flow continues to develop until the boundary layer finally fills the entire cross section of the pipe. After this point, the profile does not change as a function of x and the flow is said to be *fully developed*. The length over which this phenomenon takes place, l_e, is called the *entrance length*.

boundary layer

Entrance length is difficult to compute from the theories we have developed thus far. For example, l_e depends upon whether the flow is laminar or turbulent. If we cast this problem in dimensionless terms, as discussed in Chapter 7, and correlate the resulting dimensionless groups via empirical data, we find

$$\frac{l_e}{D} \approx 0.06\, Re \tag{8.2}$$

for laminar flow and

$$\frac{l_e}{D} \approx 4.4\, Re^{1/6} \tag{8.3}$$

for turbulent flow. Because there is enhanced mixing in turbulent boundary layers, the boundary layer thickness grows more quickly and the entrance length is shorter in a relative sense, i.e. it goes as the one–sixth power in Re as opposed to the first power in Re for laminar flow[8.3].

8.2. Fully Developed Laminar Flow

When the flow no longer changes as a function of the distance x traveled along a pipe, the flow is fully developed. In other words, fully developed flow implies $\partial/\partial x = 0$ for all quantities[8.4]. Here, viscous "drag" effects exactly balance the driving force supplied by the pressure gradient. Were viscous forces to be absent, pressure would be constant along the pipe, except for any changes due to hydrostatic variation. Thus, there is a natural pressure drop along pipes in the x direction. Viewed from an energy standpoint, the mechanical work done by pressure effects is dissipated by the frictional losses of viscosity.

If we assume laminar flow in a long straight section of pipe, we can derive parameters of engineering interest strictly via theoretical treatment. Such flows are governed by the Navier–Stokes system discussed in Chapter 6. However, Eqs. (6.114) through (6.116) on pp. 82 are cast in Cartesian

[8.3] However, it is important to remember that the *values* of the Reynolds numbers in Eqs. (8.2) and (8.3) are not the same! For our purposes, the maximum Re in Eq. (8.2) would be $Re = 2100$, which implies a maximum entrance length of $l_e \approx 0.06\,Re\,D \approx 126D$. That is, l_e is, at most, about 126 pipe diameters long for laminar flow. For turbulent pipe flows, we typically have $10^4 < Re < 10^5$, which gives roughly $20D < l_e < 30D$.

[8.4] Except for $\partial P/\partial x$ which drives the flow. That is $\partial P/\partial x < 0$ is required to sustain a flow from high pressure to lower pressure.

form, whereas cylindrical coordinates are the natural system for pipe flow (Fig. 8.4). The differential analysis in cylindrical coordinates can be ap-

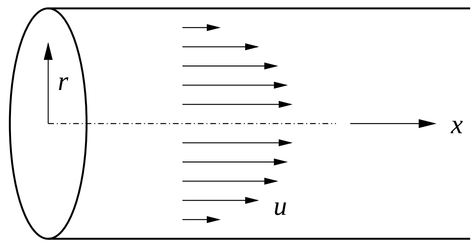

FIGURE 8.4. *The cylindrical coordinate system and dominant streamwise flow direction for pipes.*

plied to derive the appropriate Navier–Stokes momentum equation in the streamwise direction as[8.5]:

$$\rho\left(\frac{\partial u}{\partial t} + u_r\frac{\partial u}{\partial r} + \frac{u_\theta}{r}\frac{\partial u}{\partial \theta} + u\frac{\partial u}{\partial x}\right) = \tag{8.4}$$

$$-\frac{\partial P}{\partial x} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial x^2}\right],$$

where u_r and u_θ are velocity components in the r and θ coordinate directions, respectively. This is the general form of the equation, which simplifies substantially for this case. Specifically, we assume steady flow, so that $\partial u/\partial t = 0$. We also make the same observation as we did for flow between flat plates shown in Fig. 6.12 on pp. 83, i.e. the fluid flows strictly parallel to the walls in the x direction so that $u_r = u_\theta = 0$. Therefore, the second and third terms on the left hand side drop out. Since the flow is fully developed, i.e. $\partial u/\partial x = 0$, the fourth term on the left hand side also drops out. As a result, the entire left hand side of the equation vanishes. On the right hand side, we assume a symmetric profile so that $\partial/\partial\theta = 0$. Also, $\partial^2 u/\partial x^2 = 0$, again by the condition of fully developed flow. These simplifications leave

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = \frac{1}{\mu}\frac{\partial P}{\partial x} \tag{8.5}$$

as the final equation governing laminar pipe flow. We can formally convert the partial differentials in Eq. (8.5) into ordinary differentials in the same manner as for the infinite parallel plate problem. Specifically, we note that $u_r = u_\theta = 0$ imply that pressure is constant in the r and θ directions, so that P is not a function of r or θ. This gives $P(x, r, \theta) \to P(x)$. Therefore

$$\frac{\partial P}{\partial x} \to \frac{dP}{dx}, \tag{8.6}$$

[8.5]See Panton (1984).

which is identical to Eq. (6.120). The pressure gradient represents the driving force for the flow and we take it to be a prescribed constant. If this is the case, then Eq. (8.5) implies that u is only a function of r, meaning

$$\frac{\partial u}{\partial r} \rightarrow \frac{du}{dr} \,. \tag{8.7}$$

We can then formally recast Eq. (8.5) as

$$\frac{1}{r}\frac{d}{dr}\left(r\,\frac{du}{dr}\right) \;=\; \frac{1}{\mu}\frac{dP}{dx} \,. \tag{8.8}$$

Eq. (8.8) is a simple ordinary differential equation that can be integrated twice to find the velocity profile, u as

$$u(r) \;=\; \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{4} \;+\; C_1 \,\ln r \;+\; C_2 \,, \tag{8.9}$$

where C_1 and C_2 are constants of integration.

Constants C_1 and C_2 must be evaluated using boundary conditions. Unlike the parallel plate problem which had two obvious boundary conditions, i.e. Eq (6.124), here we have only one obvious condition:

$$u|_{r=r_0} \;=\; 0 \,, \tag{8.10}$$

where r_0 is the radius of the pipe. Eq. (8.10) is basically a statement of the no–slip condition at the inner wall of the pipe. Unfortunately, as a second order equation, Eq. (8.8) yields two constants, so we are missing a means of determining both constants.

Such situations arise often in problems cast in the cylindrical coordinate system. The second "boundary" is clearly at $r = 0$, i.e. the pipe centerline, but the boundary condition here is not obvious. However, we can deduce the proper relationship physically in at least two ways. The first way is essentially by an "order of magnitude" argument. Specifically, the term $C_1 \ln r \rightarrow -\infty$ as $r \rightarrow 0$ in Eq. (8.9). Since this behavior is not physically admissible, we conclude $C_1 = 0$. Conversely, we can draw the same conclusion from a "symmetry" argument. Because the flow profile is symmetric, the first derivative of the profile must vanish at the centerline, i.e. $u'(0) = 0$. This gives

$$u'(0) \;=\; \left[\frac{1}{\mu}\frac{dP}{dx}\frac{r}{2} \;+\; \frac{C_1}{r}\right]_{r=0} \;=\; 0 \,, \tag{8.11}$$

where u' is the first derivative. The only way Eq. (8.11) can be satisfied is if $C_1 = 0$. The final solution for the velocity profile is

$$u(r) \;=\; \frac{1}{4\mu}\frac{dP}{dx}\,(r^2 - r_0^2) \,. \tag{8.12}$$

Eq. (8.12) enables us to calculate quantities of engineering interest for laminar pipe flow directly from their definitions. For example, according to

Eq. (1.5) on pp. 5, shear stress is

$$\tau \;=\; \mu\,\frac{du}{dr} \;=\; \mu\,\left(2r\frac{1}{4\mu}\,\frac{dP}{dx}\right) \;=\; \frac{r}{2}\,\frac{dP}{dx}\,. \tag{8.13}$$

Thus shear varies linearly from $\tau = 0$ at the centerline to a maximum at the inner wall of the pipe. Volume flow rate is

$$Q = \int_A \mathbf{V}\cdot\hat{n}\,dA = \int_0^{r_0} u\,2\pi r\,dr = -\,\frac{\pi\,r_0^4}{8\,\mu}\,\frac{dP}{dx} = -\,\frac{\pi\,D^4}{128\,\mu}\,\frac{dP}{dx}\,. \tag{8.14}$$

In the last term, we've written the solution in terms of the diameter D, rather than radius r_0. The negative sign in Eq. (8.14) seems out of place until one recalls that the pressure gradient dP/dx is negative along the direction of flow. We immediately have the average velocity $\bar{u}$ as

$$\bar{u} \;=\; \frac{Q}{A} \;=\; \frac{Q}{\pi\,r_0^2} \;=\; -\,\frac{r_0^2}{8\,\mu}\,\frac{dP}{dx} \;=\; -\,\frac{D^2}{32\,\mu}\,\frac{dP}{dx}\,. \tag{8.15}$$

Also, the maximum velocity is

$$u_{max} \;=\; u(0) \;=\; -\,\frac{r_0^2}{4\,\mu}\,\frac{dP}{dx} \;=\; 2\,\bar{u}\,. \tag{8.16}$$

8.3. Dimensional Analysis of Pipe Flow

For actual engineering work, internal flows in pipes are usually characterized non–dimensionally. Of special interest are metrics for frictional losses of pressure head. For laminar flow in the previous section, a theoretical approach yielded a complete solution of the problem. We can therefore not only formulate the flow problem non–dimensionally, we can also solve the functional relationship between the parameters. We will assume that pressure loss, ΔP, is a function of the average flow velocity, $\bar{u}$, the length traveled along the pipe, L, the diameter, D, and the fluid viscosity, μ, and density, ρ. We postulate a functional relationship

$$\Delta P \;=\; \phi\,(\bar{u},\; L,\; D,\; \mu,\; \rho)\;, \tag{8.17}$$

where function ϕ is unknown. At the end of Chapter 7, we found that this configuration could be formulated in the dimensionless context

$$\frac{\Delta P}{\frac{1}{2}\,\rho\,\bar{u}^2} \;=\; \Phi\left(Re,\;\frac{L}{D}\right)\;, \tag{8.18}$$

where Φ remains an unknown function. Eq. (8.18) represents the limit of what dimensional analysis yields.

Experiments show that pressure drop is essentially proportional to the pipe length. In other words, L/D is actually a constant factor multiplying Φ. We can therefore bring it outside of the old function and write the relationship in terms of a new function f as

$$\frac{\Delta P}{\frac{1}{2}\,\rho\,\bar{u}^2} \;=\; \frac{L}{D}\;f\,(Re)\;. \tag{8.19}$$

The unknown function f is defined as the *Darcy friction factor*[8.6]. This equations should be read as: "The Darcy friction factor is a function of the Reynolds number". Note that Eq. (8.19) implies that we can calculate the pressure drop if we know the friction factor, i.e.

Darcy friction factor

$$\Delta P \;=\; f\,\frac{L}{D}\,\frac{\rho\,\bar{u}^2}{2}\,. \tag{8.20}$$

We now utilize our exact solution from the previous section to complete the analysis. First, we assume that the pressure gradient dP/dx is equal to the pressure loss along our pipe of fixed length L, i.e.

$$\frac{dP}{dx} \;\approx\; \frac{\Delta P}{L}\,, \tag{8.21}$$

which implies

$$\Delta P \;=\; L\,\frac{dP}{dx}\,. \tag{8.22}$$

Substituting dP/dx from the exact solution in Eq. (8.15), we find

$$\Delta P \;=\; -\,\frac{32\,L\,\bar{u}\,\mu}{D^2}\,, \tag{8.23}$$

which is interpreted as a pressure drop of $32L\bar{u}\mu/D^2$. We can substitute this entire expression into Eq. (8.19) to obtain

$$\frac{\Delta P}{\frac{1}{2}\,\rho\,\bar{u}^2} \;=\; \frac{32\,L\,\bar{u}\,\mu}{D^2}\,\frac{1}{\frac{1}{2}\,\rho\,\bar{u}^2}\,, \tag{8.24}$$

which simplifies to

$$\frac{\Delta P}{\frac{1}{2}\,\rho\,\bar{u}^2} \;=\; 64\,\frac{L}{D}\,\frac{\mu}{\rho\,\bar{u}\,D}\,. \;=\; \frac{L}{D}\,\frac{64}{Re}\,. \tag{8.25}$$

Comparing Eqs. (8.19) and (8.25), we see that

$$f \;=\; \frac{64}{Re}\,, \tag{8.26}$$

i.e. the Darcy friction factor for laminar flow is a constant proportional to the inverse of the Reynolds number.

Unfortunately, turbulent pipe flow is altogether too complex to evaluate f analytically. Therefore, we adopt a more empirical approach. It is known from experiments that turbulent flow depends not only on all the parameters above for laminar flow in Eq. (8.17), but also on the roughness of the inner wall of the pipe, ε. This parameter has units of length and represents the average altitude to which surface irregularities and microscopic "bumps" protrude into the flow cross section (Figure 8.5). Thus, the analog of Eq. (8.17) for turbulent flow is

$$\Delta P \;=\; \phi\,(\bar{u},\, L,\, D,\, \mu,\, \rho,\, \varepsilon\,)\;, \tag{8.27}$$

[8.6] Note that the Darcy friction factor f is different from the seldom used Fanning friction factor, which is defined as $f/4$. Our discussions are based strictly on the Darcy factor.

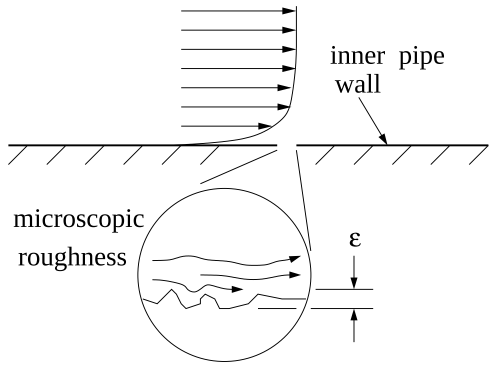

FIGURE 8.5. *The concept of roughness at the inner surface of pipes.*

The experimental observation of pressure loss per length of pipe holds true for turbulent flow as well. Dimensional analysis then yields the analog to Eq. (8.19) for the friction factor

$$\frac{\Delta P}{\frac{1}{2}\,\rho\,\bar{u}^2} = \frac{L}{D}\,f\left(Re,\,\frac{\varepsilon}{D}\right)\,. \tag{8.28}$$

Thus, the friction factor for turbulent flow is a function of both the Reynolds number and a dimensionless roughness factor ε/D. We cannot easily evaluate f using an analytical treatment. Rather, measurements provide the basis of estimating f via correlation. Nikuradse (1933) provided extensive data, which was correlated by Colebrook (1939) and Moody (1944) in terms of ε/D of commercially available pipes. Their results lead to the so–called *Moody Diagram* (Figure 8.6), which is valid for steady fully developed turbulent pipe flow over a large range of Re and ε/D. The laminar result given by Eq. (8.26) is shown toward the left, while results for turbulent flow are the curves to the right. The region between laminar and turbulent flow, $2100 \le Re \le 4000$, is left blank because there are additional factors that govern f in the more complicated transition range. According to Fig. 8.6, f tends to become independent of the Reynolds number for very large Re. In these cases, termed *completely turbulent flow*, the viscous layer at the wall is so thin that the roughness essentially governs the flow. It is intuitive then that a flattening occurs earlier for higher roughnesses, as shown in the diagram. Also note that even as complete smoothness is approached, $\varepsilon/D \to 0$, the friction factor does not go to zero. There are losses in all viscous pipe flow, regardless of the degree of smoothness of the pipe wall. This is a direct result of the no–slip condition.

Empirical equations for $f\,(Re,\,\varepsilon/D)$ have also been derived. The Colebrook correlation is perhaps the best known instance and is given by

$$\frac{1}{\sqrt{f}} = -\,2.0\,\log_{10}\left(\frac{\varepsilon/D}{3.7} + \frac{2.51}{Re\,\sqrt{f}}\right)\,, \tag{8.29}$$

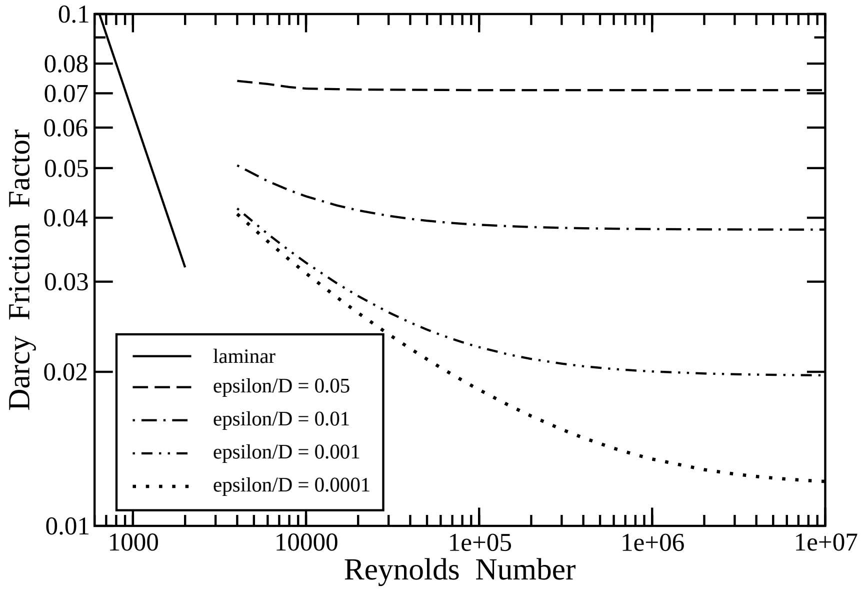

FIGURE 8.6. *Abbreviated Moody Diagram showing Darcy friction factor as a function of Reynolds number and relative roughness.*

where $\log_{10}$ is the base 10 logarithm[8.7]. Eq. (8.29) is valid for approximately $4000 \leq Re \leq 10^8$. Its main drawback is that it cannot be solved explicitly for f, that is, the friction factor must be evaluated iteratively. According to Fox and McDonald (1998), a single iteration will give the answer to within $\pm$ 1 % for the initial guess

$$f_{guess} \;=\; 0.25 \,\left[\log_{10}\left(\frac{\varepsilon/D}{3.7} \;+\; \frac{5.74}{Re^{0.9}}\right)\right]^{-2} \;. \tag{8.30}$$

Munson et al. (2006) suggest that Eq. (8.30) can be used directly[8.8] without iteration for $10^{-6} < \varepsilon/D < 10^{-2}$ and $5000 < Re < 10^8$.

There are special cases in which Eq. (8.29) simplifies. For example, if pipes are hydrodynamically "smooth", i.e. $\varepsilon/D = 0$, we obtain

$$\frac{1}{\sqrt{f}} \;=\; 2.0\,\log_{10}\left(Re\,\sqrt{f}\right) \;-\; 0.8\,, \tag{8.31}$$

[8.7]Some texts give Eq. (8.29) in terms of the natural logarithm, e.g. Potter et al. (1997). Here, the argument of the log is the same, but the leading constant changes to roughly -0.87, i.e. $f^{-1/2} = -0.87\log_e(\dots)$, based upon the conversion $\log_e x = \log_{10} x/\log_{10} e \approx 2.30258\log_{10} x$ (Beyer, 1984).

[8.8]This is shown in Munson et al. (2006) as homework problem 8.37, however, although they introduced Eq. (8.29) in terms of a base 10 logarithm, they show Eq. (8.30) in a natural log form. Therefore, since the logarithmic term is squared, their constant appears as $0.25 \times 2.30256^2 = 1.325$.

while for "fully turbulent" flow, i.e. $Re \to \infty$, we find

$$\frac{1}{\sqrt{f}} = -2.0 \log_{10} \left(\frac{\varepsilon / D}{3.7} \right) . \tag{8.32}$$

It should be understood that Eqs. (8.29) through (8.32) are subject to the standard uncertainties associated with experimental correlations.

8.4. Characterization of Frictional Losses

The concept of "head loss" was introduced in Eq. (5.34) on pp. 55, which can be written

$$\frac{P_1}{\gamma} + \frac{u_1^2}{2\,g} + z_1 - h_L = \frac{P_2}{\gamma} + \frac{u_2^2}{2\,g} + z_2 , \tag{8.33}$$

where h_L is a head loss representing the dissipation of energy due to frictional effects. Recall, this is an "extended" version of Bernoulli's equation. For steady flow in a horizontal constant cross–section pipe, we can deduce the form of h_L as

$$h_L = \frac{\Delta P}{\gamma} , \tag{8.34}$$

which, when combined with the form of pressure drop in Eq. (8.20) yields

$$h_L = f \, \frac{L}{D} \, \frac{u^2}{2\,g} . \tag{8.35}$$

Eq. (8.35) is the *Darcy–Weisbach* equation, which is valid for *any* fully developed steady incompressible pipe flow. Darcy Weisbach equation

While Eq. (8.35) is useful for computing losses in long straight sections of pipe, *systems* of pipes generally have many other components, such as valves, bends, nozzles, and expansions. Losses are also associated with these components[8.9]. These losses are, by convention, characterized by a loss coefficient defined as

$$K_L = \frac{h_L}{u^2/2g} = \frac{\Delta P}{\rho u^2/2} , \tag{8.36}$$

so that pressure drop across components is

$$\Delta P = K_L \, \frac{1}{2} \rho u^2 , \tag{8.37}$$

and head loss is

$$h_L = K_L \, \frac{u^2}{2g} . \tag{8.38}$$

[8.9]Classical engineering terminology labels these losses as *minor losses*, with the implication that losses in long straight sections of pipe are *major losses*. This convention is regrettable because the relative sizes of these contributions are system dependent. We prefer the more descriptive term *component losses*.

We can also classify component losses by an "equivalent length", that is, a length of pipe that would produce the same loss as the component

$$h_L = K_L \frac{u^2}{2g} = f \frac{L_{equiv}}{D} \frac{u^2}{2\,g}\,, \tag{8.39}$$

from which we deduce

$$L_{equiv} = \frac{K_L\,D}{f}\,. \tag{8.40}$$

Eq. (8.40) is used less frequently than the method of loss coefficient. *MY&O Ex. 8.8 pp 451*

In actuality, flows in such components are incredibly complex, for example, flow separation often occurs. Therefore, loss coefficients are determined empirically. Standard texts contain numerous tables and graphical results which characterize loss coefficients for various components (Potter et al., 1997; Fox and McDonald, 1998; Munson et al., 2006).

8.5. Non–Circular Pipes

We have developed the material in this chapter predominantly on the assumption of pipes having circular cross–sectional areas. Fortunately, many of our circular pipe results can be carried over to other types of conduits if minor modifications are made. For laminar flow, obtaining a closed–form solution and thereby determining an analytical form for f is beyond the mathematics we have discussed thus far. For example, for a rectangular cross section, the momentum equation cannot be reduced to an ordinary differential equation like what we obtained for circular pipes in Eq. (8.8). We are instead faced with a partial differential equation, however, this has been solved for many types of cross section[8.10]. We now consider the problem such that f can be written as

$$f = \frac{C}{Re_h}\,, \tag{8.41}$$

where C is a geometry–specific shape factor[8.11] and Re_h is the Reynolds number computed on the basis of the *hydraulic diameter*. The hydraulic diameter is defined as

$$D_h = \frac{4\,A}{P}\,, \tag{8.42}$$

where A is the cross–sectional area of the conduit and P is the wetted perimeter[8.12]. Of course, the value of D_h can be computed directly from

[8.10] See for example Berker (1963).

[8.11] According to the form of Eq. (8.41), $C = 64$ for circular pipes.

[8.12] Eq. (8.42) contains a factor of 4 so that the hydraulic diameter of a circular pipe is equal to its geometric diameter, i.e. $D_h = 4\left(\pi D^2/4\right)/\left(\pi D\right) = D$.

geometry, for example, for a rectangular duct of height a and width b, we obtain

$$D_h = \frac{4\,a\,b}{2(a+b)} = \frac{2\,a\,b}{a+b}\,, \tag{8.43}$$

Standard texts contain tables of shape factors for various cross sections (Munson et al., 2006).

CHAPTER 9

External Flow

Whereas in Chapter 8 we considered internal flow in pipes, we now focus on *external* flows. This is usually manifested as a body moving through a fluid[9.1], rather than the other way around as we saw in the previous chapter. This topic is obviously relevant to the aerodynamics of vehicles, structural aerodynamics, submarine dynamics, projectile flight, etc.

The main concepts of interest from a purely engineering standpoint are *lift* and *drag*. Lift is a force normal to the direction of movement[9.2], which is caused by the interaction of the fluid with the vehicle. Drag is a force exerted along the negative direction of the body, which tends to impede movement. It too arises because of interaction with the surrounding fluid. These surface forces are the result of the fluid stresses at the surface of the vehicle: shear stresses due to viscous effects and normal stresses due to pressure. This implies, of course, that we must once again determine velocity and pressure distributions in order to compute the quantities.

We probably should already be intuitively expecting that external flows in real engineering situations are too complex for us to approach on a purely theoretical basis. Like pipe flow, we will place some reliance on empirical data and dimensional analysis for these cases. However, we will also examine from a theoretical perspective a fundamental model problem which incorporates important concepts of external flow.

9.1. The Boundary Layer

The concept of the boundary layer is one of the fundamental foundations of fluid mechanics and was introduced by Ludwig Prandtl (1904). Prior to this time, the study of fluid mechanics was actually fragmented into two sub–disciplines. The specialty of *theoretical hydrodynamics* was based mainly upon Euler's equations for inviscid flow, i.e. Eqs. (6.34) through (6.36) on pp. 65 in Chapter 6. However, as we are now aware, these equations cannot resolve phenomenon that arise from the existence of viscous effects. Hence, engineers were unable to provide accurate analyses based upon theory for

[9.1]Here we will only consider the problem where the body is fully immersed in a fluid, e.g. an airplane or car surrounded by air or a submarine surrounded by water. This explicitly excludes problems such as ships, where the hydrodynamics involve two different fluids and a free surface between them.

[9.2]For example, this would be vertically upward forced for a wing–shaped vehicle traveling in a horizontal direction.

important problems of the day such as drag on a ship's hull[9.3]. The other branch of study, *hydraulics*, was based strictly on empirical data. While this provided accurate results for specific problems, it did not permit systematic generalizations nor conceptual understanding of the kind permitted by theory.

Prandtl (1904) showed that many flows can be divided into two regions: one close to solid boundaries, and the other encompassing the rest of the domain. The so–called boundary layer is the thin region in the neighborhood of solid boundaries. Here, gradients are significant and consequently viscous effects are important. This also allows the no–slip boundary condition to be satisfied. Outside of the boundary layer, gradients are minimal and viscous effects can be neglected[9.4]. In other words, the outer region can be considered inviscid. This concept provided the bridge between theory and empirics and also enabled many viscous flow problems to be solved that would otherwise have been too difficult using the full Navier–Stokes equations. The introduction of the boundary layer idea can be considered the beginning of modern fluid mechanics.

We have qualitatively described the boundary layer in Chapter 8. It can be thought of essentially as a layer of flow that melds the outer region, where the freestream velocity $u = u_\infty$ is uniform, with the no–slip region at the boundary, where $u = 0$. In actuality, there is a tremendous diversity in the types of boundary layer flows which occur, e.g. depending upon surface shape and roughness, Reynolds number, turbulence, etc. The canonical model problem is the development of a boundary layer in the flow over a flat plate. Here, the Reynolds number is based upon the distance traveled along the plate (Figure 9.1), that is

$$Re = \frac{\rho\, u_\infty\, x}{\mu}\,. \tag{9.1}$$

We specified the onset of turbulence in pipe flow as $Re \approx 2100$ in Chapter 8. Boundary layer flow on the flat plate also can undergo a transition from a laminar to a turbulent state. This occurs at a Reynolds number of roughly[9.5] $Re \approx 5 \times 10^5$.

The boundary layer is characterized mainly by its local "thickness" denoted by $y = \delta$. In actuality, there's no sharp interface between the boundary layer and the outer flow region. Mathematically, the interface is more akin to a limiting process, i.e. $u \to u_\infty$, rather than $u = u_\infty$. There is then,

---

[9.3] The Navier–Stokes equations for viscous flow, Equations (6.114) through (6.116) on pp. 82 in Chapter 6, were already known, but because of their extreme difficulty, investigators had difficulty working with them in a meaningful way before the advent of high–speed computers.

[9.4] Of course, the actual viscosity of the fluid remains the same as that in the boundary layer.

[9.5] We say "roughly" in the context of a typical application where no special pains are taken to minimize disturbances, smooth the approach flow, etc. Transition Reynolds numbers can be greatly increased if such precautions are taken.

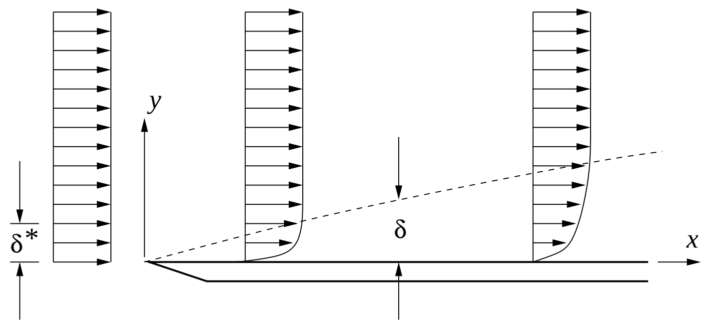

FIGURE 9.1. *Boundary layer development on a flat plate from an approach flow having uniform velocity u_∞. The vertical thickness of the layer, δ, is exaggerated for clarity. Also, the so–called displacement thickness δ^* is shown. The Reynolds number is based upon the distance traveled along the plate, x.*

by necessity, some level of arbitrariness in defining δ. The primary definition is:

DEFINITION 9.1. *The edge of the boundary layer $y = \delta$ is the location where the local velocity reaches 99 % of the freestream velocity, $u = 0.99u_\infty$.*

Another way to quantify the boundary layer is the so–called *displacement thickness* δ^*, which is the amount by which a uniform flow profile would have to be constricted so that its flow rate equals the true flow rate of the profile. For example, assume bottom part of the left–most (uniform) profile in Fig. 9.1 was cut off starting at $y = 0$ up to the displacement thickness $y = \delta^*$. What is left over, i.e. the profile for $y > \delta^*$, must be equal to the flow rate of the right–most profile. This will be the case if what we removed on the left is equal to the deficit on the right, i.e.

$$\delta^* \, b \, u_\infty \ = \ \int_0^\infty \ (u_\infty - u) \, b \, dy \ , \tag{9.2}$$

where b is the width of the flat plate. Eq. (9.2) gives

$$\delta^* \ = \ \int_0^\infty \ \left(1 - \frac{u}{u_\infty}\right) \ dy \ , \tag{9.3}$$

Thus, δ^* can be thought of as the amount of thickness by which the plate should be increased so that a uniform flow profile has the same flow rate as the real profile on the real plate. Another measure, the *momentum thickness* Θ, can be derived using a similar argument based upon momentum deficit. We find

$$\Theta \ = \ \int_0^\infty \ \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) \ dy \ . \tag{9.4}$$

The entire boundary layer concept is based upon the presumption that the layer itself is *thin*, i.e. $\delta \ll x$, $\delta^* \ll x$, and $\Theta \ll x$. Since the Reynolds number varies directly with x, these conditions are synonymous with high Reynolds number flows. Note that this argument further implies that the boundary layer approximation is not valid near the beginning of the plate, where x is small.

9.2. Boundary Layer Equations

Let us assume a steady two–dimensional flow, i.e. $w = 0$, so that the Navier–Stokes Equations (6.114) through (6.116) simplify to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{9.5}$$

and

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) . \tag{9.6}$$

Also, the mass conservation equation (6.10) simplifies to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{9.7}$$

Prandtl made the following *order of magnitude* arguments to simplify these equations[9.6]:

- Assume in continuity equation (9.7) that $\partial u/\partial x$ is of order unity, which we write as $\partial u/\partial x \sim [1]$. Since there are no parameters in this equation and we know that the flow must develop along the axis, i.e. $\partial/\partial x \neq 0$, the term $\partial v/\partial y$ must be the same order of magnitude. That is $\partial v/\partial y \sim [1]$.
- The freestream flow is of order unity, i.e. $u \sim [1]$.
- The flow remains predominantly oriented along the plate so that the vertical velocity component is much smaller than the horizontal component, $v \ll u$. Therefore, we write $v \sim [\epsilon]$, where $\epsilon \ll 1$. That is, ϵ is a small number.
- Since changes in the vertical direction happen over a very short distance, i.e. the boundary layer thickness, then gradients in the vertical direction must be much larger than gradients along the plate, i.e. $\partial/\partial x \ll \partial/\partial y$. Since $u \sim [1]$ and $\partial u/\partial x \sim [1]$, we can say that $\partial/\partial x \sim [1]$. Since $\partial/\partial x \ll \partial/\partial y$, we can write $\partial/\partial y \sim [1/\epsilon]$. Note that we could also infer this from the fact that $\partial v/\partial y \sim [1]$, but that $v \sim [\epsilon]$. Thus, we see $\partial^2 u/\partial^2 x \sim [1]$, $\partial v/\partial x \sim [\epsilon]$ and $\partial^2 v/\partial^2 x \sim [\epsilon]$. Moreover, $\partial u/\partial y \sim [1/\epsilon]$, $\partial^2 u/\partial^2 y \sim [1/\epsilon^2]$, and $\partial^2 v/\partial^2 y \sim [1/\epsilon]$.
- In the boundary layer, inertial and viscous effects must be of the same overall order of magnitude. If we were to rewrite the equations (9.5) through (9.7) in dimensionless form, i.e. using the

[9.6] See (Schlichting, 1979) for a more detailed analysis of this process.

Reynolds number as defined in equation (9.1), we would see that $Re \sim [1/\epsilon^2]$ for this to be the case.

Following the last point, let us rewrite Eqs. (9.5) through (9.7) in dimensionless form along with the orders of magnitude for each term we have deduced above. Showing the orders of magnitude in square brackets beside each term, we find

$$u\frac{\partial u}{\partial x}[1]\cdot[1] + v\frac{\partial u}{\partial y}[\epsilon]\cdot[\epsilon^{-1}] = -\frac{\partial P}{\partial x} + \frac{1}{Re}[\epsilon^2]\left(\frac{\partial^2 u}{\partial x^2}[1] + \frac{\partial^2 u}{\partial y^2}[\epsilon^{-2}]\right) \tag{9.8}$$

and

$$u\frac{\partial v}{\partial x}[1]\cdot[\epsilon] + v\frac{\partial v}{\partial y}[\epsilon]\cdot[1] = -\frac{\partial P}{\partial y} + \frac{1}{Re}[\epsilon^2]\left(\frac{\partial^2 v}{\partial x^2}[\epsilon] + \frac{\partial^2 v}{\partial y^2}[\epsilon^{-1}]\right) \tag{9.9}$$

for the momentum equations. As we said above, the entire continuity equation is of order unity, i.e.

$$\frac{\partial u}{\partial x}[1] + \frac{\partial v}{\partial y}[1] = 0\,. \tag{9.10}$$

We did not write down the orders of magnitude for the pressure terms in Eqs. (9.8) and (9.9). We can infer that the pressure gradient in Eq. (9.9) should be the same order of magnitude as the equation itself. In other words, changes in pressure across the boundary layer should be of comparable size to the other terms in the equation. Specifically, $\partial P/\partial y \sim [\epsilon]$, since Eq. (9.9) is of overall order $[\epsilon]$. This leads us to conclude $P \sim [\epsilon^2]$, since $\partial P/\partial y \sim [\epsilon]$ and $\partial/\partial y \sim [1/\epsilon]$. We then see $\partial P/\partial x \sim [\epsilon^2]$.

We can now make some simplifying observations. First, $\partial^2 u/\partial x^2$ and $\partial P/\partial x$ in Eq. (9.8) are *very* small compared to the other terms in that equation, so they can be dropped. Also, the entire Eq. (9.9) is small compared Eq. (9.8). That is Eq. (9.9) is order $[\epsilon]$, while Eq. (9.8) is order $[1]$. Since, $P \sim [\epsilon^2]$ pressure variation in the flow is extremely small. We can say that pressure is essentially a constant. These deductions show that Eq. (9.8) through (9.10) become

flat plate boundary layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{9.11}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,. \tag{9.12}$$

Eqs. (9.11) and 9.12 are the boundary layer equations for a flat plate. Note that the mass equation is unchanged, but Eq. (9.9) has been eliminated and Eq. (9.8) has been simplified considerably. Pressure has essentially dropped out as a variable so that there are now only two unknowns, u and v, and two equations. In other words, pressure is constant throughout the flow domain[9.7]. The boundary conditions are no–slip

$$u\big|_{y=0} \;=\; v\big|_{y=0} \;=\; 0 \tag{9.13}$$

and uniform outer flow profile

$$u\big|_{y\to\infty} \;\to\; u_\infty \,. \tag{9.14}$$

9.3. Blasius Laminar Boundary Layer Solution

Eqs. (9.11) and (9.12) remain non–linear and are therefore difficult to solve. One of Prandtl's students, H. Blasius, employed a *similarity transformation* whereby these partial differential equations are transformed to an ordinary, albeit non–linear equation[9.8]. This equation can, in turn, be solved by series expansion, or it can be numerically solved on a computer in a fairly straightforward fashion. The transformation technique is beyond the scope of our discussion here[9.9], so we will simply give the results so as to be able to compare them to the approximate solution discussed in the next section.

Blasius found the boundary layer thickness to be

$$\delta \;=\; \frac{5.0\,x}{\sqrt{Re_x}}\,, \tag{9.15}$$

where Re_x is the Reynolds number based on the distance traveled along the plate x given by Eq. (9.1). Shear stress at the plate's surface $y = 0$ is

$$\tau_w \;=\; \frac{0.332\,\rho\,u_\infty^2}{\sqrt{Re_x}}\,, \tag{9.16}$$

which can also be written in terms of the wall shear stress coefficient C_f as

$$C_f \;=\; \frac{\tau_w}{\rho\,u_\infty^2/2} \;=\; \frac{0.664}{\sqrt{Re_x}}\,. \tag{9.17}$$

This quantity is more commonly called the *skin friction factor*. Note once again that the scaling factor is the dynamic pressure, which is also the kinetic energy per unit volume. Many experimental studies have confirmed the solution.

[9.7] Be careful to remember that this is valid for the special case of a flat plate. It is not true in general!

[9.8] The similarity technique only works in special (fairly restrictive) cases. It is a good rule of thumb that if there are no identifiably finite dimensions or length scales to the problem, a similarity transformation may work. In this case, it is assumed that the velocity profile u is self–similar. In other words, if we observe the shape of u at a particular x and compare that to another u further downstream, the shapes will be identical. The difference will only be one of scale.

[9.9] See Schlichting (1979) for extensive discussion of the similarity technique.

9.4. Karman–Pohlhausen Integral Solution

The fact that the exact solution in the previous section is difficult from a mathematical point suggests that an approximate solution might be useful. T. von Kármán (1921) and K. Pohlhausen (1921) derived just such a solution in the form of an integral method that can be solved to various degrees of accuracy. First, start with the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.18}$$

and integrate it across the boundary layer:

$$\int_0^\delta \frac{\partial u}{\partial x}\, dy + \int_0^\delta \frac{\partial v}{\partial y}\, dy = 0\,. \tag{9.19}$$

Now note that $v = 0$ at $y = 0$ (as part of the no–slip boundary condition) and that the second part of the equation is simply dv because of the chain rule. This gives:

$$v\,|_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x}\, dy\,. \tag{9.20}$$

Now, we also integrate the boundary layer form of the momentum equation (9.11) over the boundary layer

$$\int_0^\delta u\,\frac{\partial u}{\partial x}\, dy + \int_0^\delta v\,\frac{\partial u}{\partial y}\, dy = \nu\int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy\,, \tag{9.21}$$

which gives, after integrating the second term on the left hand side by parts[9.10] and recognizing the right hand side can be integrated directly,

$$\int_0^\delta u\,\frac{\partial u}{\partial x}\, dy + uv\,|_{y=0}^{y=\delta} - \int_0^\delta u\,\frac{\partial v}{\partial y}\, dy = \nu\frac{\partial u}{\partial y}\bigg|_{y=0}^{y=\delta}\,. \tag{9.22}$$

The first term remains the same, however, to further develop Eq. (9.22), we make the following observations. The second term can be evaluated using Eq. (9.20) for v and observing that $u = 0$ at $y = 0$ and $u = u_\infty$ at $y = \delta$. In the third term, we utilize the original continuity equation (9.18) to swap $\partial v/\partial y = -\,\partial u/\partial x$. The fourth term is evaluated using the fact that $\partial u/\partial y$ is zero at $y = \delta$. These modifications give

$$\int_0^\delta u\frac{\partial u}{\partial x}dy - u_\infty\int_0^\delta \frac{\partial u}{\partial x}dy + \int_0^\delta u\frac{\partial u}{\partial x}dy = -\nu\frac{\partial u}{\partial y}\bigg|_{y=0}\,, \tag{9.23}$$

which can be simplified to

$$u_\infty\int_0^\delta \frac{\partial u}{\partial x}\, dy - \int_0^\delta 2u\frac{\partial u}{\partial x}\, dy = \nu\frac{\partial u}{\partial y}\bigg|_{y=0}\,. \tag{9.24}$$

[9.10] If integrating by parts, we can consider $\frac{\partial u}{\partial y}\, dy$ as simply du by the Chain Rule, which means we are evaluating $\int v\, du$. Applying integration by parts, we see $\int v\, du = u\,v - \int u\, dv$. However, $\frac{\partial v}{\partial y}\, dy = dv$, again by the Chain Rule. This gives the second and third terms on the left hand side of Eq. (9.22).

The u_∞ can be taken directly under the integral and the u can be "integrated" in[9.11], which yields:

$$\int_0^\delta \frac{\partial}{\partial x}\left(u_\infty \cdot u - u \cdot u\right) dy \;=\; \nu \frac{\partial u}{\partial y}\bigg|_{y=0} \,. \tag{9.25}$$

Now, the right hand side is basically a term that is evaluated, and the differential on the left hand side can be taken outside the integral (since the term is an integral w.r.t. y) and done later, therefore $\partial/\partial x \to d/dx$, which yields:

$$\frac{d}{dx}\int_0^\delta (u_\infty - u) \cdot u \, dy \;=\; \nu \frac{\partial u}{\partial y}\bigg|_{y=0} \,. \tag{9.26}$$

Eq. (9.26) is an integral form of the boundary layer equations. If we examine this equation carefully, we see that the right hand side is basically the shear stress at the surface of the plate. That is,

$$\frac{d}{dx}\int_0^\delta (u_\infty - u) \cdot u \, dy \;=\; \tau_w \,. \tag{9.27}$$

This quantity is unknown because we do not know the velocity profile u. Similarly, the left hand side also cannot be evaluated because it contains u. We apparently have an implicit equation in u that cannot be solved in closed form.

However, we can apply reasonable approximations for u. For example, let us assume that u is described by a polynomial[9.12]. We know that the flow is developing, therefore it is a function of both x and y, i.e. $u = u(x, y)$. However, we would like to use a polynomial that is a function of just a single variable. Let us then assume the following: choose a third–order polynomial having 4 constants[9.13]

$$\frac{u}{u_\infty} \;=\; a_1 \;+\; a_2\left(\frac{y}{\delta}\right) \;+\; a_3\left(\frac{y}{\delta}\right)^2 \;+\; a_4\left(\frac{y}{\delta}\right)^3 \,. \tag{9.28}$$

Eq. (9.28) appears to depend only upon y, however, since the boundary layer thickness depends upon x, i.e. $\delta = \delta(x)$ as shown in Eq. (9.15), our solution for u/u_∞ is a function of both x and y. Let us describe the boundary conditions that we will use to evaluate the coefficients in Eq. (9.28):

- No–slip boundary conditions at the wall give

$$u\,\big|_{y=0} \;=\; 0 \,. \tag{9.29}$$

[9.11] i.e. $2u\frac{\partial u}{\partial x} = \frac{\partial u^2}{\partial x}$

[9.12] Here is where the idea of various degrees of accuracy enters into the problem. Specifically, lower–order polynomials will, in general, yield less accurate results as compared to higher–order polynomials. The order is limited by how many boundary conditions we can identify because these must be used to evaluate the polynomial coefficients.

[9.13] There are three boundary conditions that can be easily identified, and a fourth that can be deduced without too much difficulty, as we shall see shortly.

- The gradient of velocity vanishes at the edge of the boundary layer. In other words, the rate of change of u with respect to y goes to zero at $y \geq \delta$. This gives

$$\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0\,. \tag{9.30}$$

- The velocity at the edge of the boundary layer is known to be u_∞, which gives

$$u\,\big|_{y=\delta} = u_\infty\,. \tag{9.31}$$

- While these 3 boundary conditions are fairly obvious, a fourth can be deduced with a little effort. Looking back to the boundary layer momentum equation (9.11) we see that this can be *evaluated* at $y = 0$ using, once again, the concept that there is no–slip at the plate surface. Therefore, plugging in $u = v = 0$ at $y = 0$, we find

$$\left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} = 0\,. \tag{9.32}$$

Now, we have four unknowns: $a_1 \ldots a_4$, and four equations with which to evaluate them: (9.29) through (9.32). Carrying this out, we find $a_1 = a_3 = 0$, $a_2 = 1.5$, and $a_4 = -0.5$, which gives

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\,. \tag{9.33}$$

We now know the *form* of u/u_∞, but since we do not know how δ varies, we still do not know the explicit solution for u/u_∞! This is where we will now apply the integral form of the boundary layer equations we derived: i.e. Eq. (9.26).

We use Eq. (9.33) to now evaluate the terms in Eq. (9.26). Let us show this explicitly:

- From Eq. (9.33), we write u as

$$u = u_\infty\left(\frac{1.5}{\delta}y - \frac{0.5}{\delta^3}y^3\right)$$

- Starting inside the integral on the left hand side we can then write

$$u_\infty - u = u_\infty\left(1 - \frac{1.5}{\delta}y + \frac{0.5}{\delta^3}y^3\right)$$

so that

$$(u_\infty - u)\cdot u = u_\infty^2\left(1 - \frac{1.5}{\delta}y + \frac{0.5}{\delta^3}y^3\right)\left(\frac{1.5}{\delta}y - \frac{0.5}{\delta^3}y^3\right)$$

- We can then evaluate the integral on the left hand side of Eq. (9.26) as

$$\int_0^\delta (u_\infty - u)\cdot u\,dy = \int_0^\delta u_\infty^2\left(\frac{1.5}{\delta}y - \frac{2.25}{\delta^2}y^2 - \frac{0.5}{\delta^3}y^3 + \frac{1.5}{\delta^4}y^4 - \frac{0.25}{\delta^6}y^6\right)dy$$

which is

$$u_\infty^2 \left(\frac{3}{4\delta} y^2 - \frac{3}{4\delta^2} y^3 - \frac{1}{8\delta^3} y^4 + \frac{3}{10\delta^4} y^5 - \frac{1}{28\delta^6} y^7 \right) \Big|_{y=0}^{\delta}$$

which simplifies to

$$\frac{39}{280} \, u_\infty^2 \, \delta$$

- The right hand side of Eq. (9.26) is easily evaluated from Eq. (9.33) as

$$\frac{3}{2} \, \frac{\nu u_\infty}{\delta}$$

- Eq. (9.26) can then be written as

$$\frac{d}{dx} \left(\frac{39}{280} u_\infty^2 \delta \right) \; = \; \frac{3}{2} \, \frac{\nu u_\infty}{\delta}$$

- We note that δ is the only term on the left hand side that is a function of x. Everything else is constant with respect to x, therefore we can write the equation as

$$\left(\frac{39}{280} u_\infty^2 \right) \, \frac{d\delta}{dx} \; = \; \frac{3}{2} \, \frac{\nu u_\infty}{\delta}$$

which is a simple separable differential equation that can be written as

$$\delta \, d\delta \; = \; \frac{140}{13} \, \frac{\nu}{u_\infty} \, dx$$

- This equation can be integrated along the flow direction, that is, in x as

$$\int \delta \, d\delta \; = \; \int \frac{140}{13} \, \frac{\nu}{u_\infty} \, dx$$

from which we find

$$\frac{\delta^2}{2} \; = \; \frac{140}{13} \, \frac{\nu \, x}{u_\infty} \; + \; C_0$$

where C_0 is a constant.

- Recalling the fact that the boundary layer thickness is zero at the leading edge of the plate, that is $\delta(0) = 0$, we find $C_0 = 0$ so that

$$\frac{\delta^2}{2} \; = \; \frac{140}{13} \, \frac{\nu \, x}{u_\infty}$$

which can be written as

$$\delta \; = \; \sqrt{\frac{280}{13} \, \frac{\nu \, x}{u_\infty}} \; = \; \sqrt{\frac{280}{13} \, \frac{\nu \, x}{u_\infty \, x^2} \, x^2} \; = \; 4.640955 \, \sqrt{\frac{\nu}{u_\infty \, x}} \, x \tag{9.34}$$

We see that the second part of Eq. (9.34) is simply the inverse of the square root of the Reynolds number based on x. We can therefore write the boundary layer solution as

$$\delta \; \approx \; \frac{4.64 \, x}{Re^{1/2}} \tag{9.35}$$

We have now solved the problem, since Eq. (9.35), along with Eq. (9.33) describe the complete velocity profile. Note that the constant 4.64 is remarkably close to the value of 5.0 that was obtained from the full non–linear (and much more difficult) exact solution! Using Eq. (9.27), we can also evaluate the shear stress and the wall shear stress coefficient C_f, for example

$$C_f \;=\; \frac{\tau_w}{\rho\, u_\infty^2/2} \;=\; \frac{0.646}{\sqrt{Re_x}}\,, \tag{9.36}$$

which is, once again, quite comparable to the exact solution in Eq. (9.17).

We assumed a third–order polynomial for the velocity profile in Eq. (9.28). Of course, other forms could have been assumed as well. Table 9.1 compares the exact solution to several approximate ones, including the cubic polynomial we have used here.

Table 9.1. Laminar boundary layer solutions

solution	$\delta\sqrt{Re_x}/x$	$C_f\sqrt{Re_x}$
exact	5.0	0.664
linear	3.46	0.578
parabolic	5.48	0.730
cubic	4.64	0.646
sinusoidal	4.79	0.655

9.5. Turbulent Boundary Layers

What we have discussed in the previous sections is one of the simplest cases of external flow: a laminar boundary layer on a flat plate having no pressure gradient. We already mentioned in Eq. (9.1) that the Reynolds number based on x characterizes the flow, so any plate that is long enough to experience a critical Reynolds number will exhibit turbulent flow. As you might expect, this encompasses most cases of traditional engineering interest. Unfortunately, we are in the same position here as we were for studying turbulence in internal (pipe) flows in Chapter 8. The phenomenon is too complicated to study from first principles, so we must resort to empirics. In general, we can say that turbulent boundary layer profiles have larger wall gradients than laminar profiles, and produce larger boundary layer thicknesses (Figure 9.2). The approximate momentum integral boundary layer equation can one again be applied, at least for a portion of the flow in the y direction. Here, a typical empirical power law profile is often used

$$\frac{u}{u_\infty} \;=\; \left(\frac{y}{\delta}\right)^{1/7}\,. \tag{9.37}$$

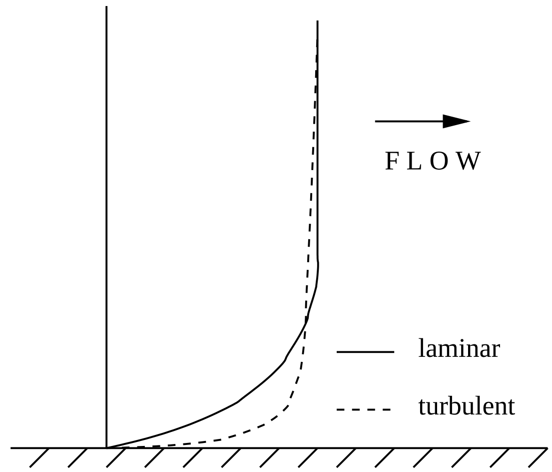

FIGURE 9.2. *Qualitative comparison of laminar versus turbulent boundary layer velocity profiles.*

However, this profile is not valid in the immediate neighborhood of the plate[9.14]. Using a mixture of empirical results, some from pipe flow, it can be shown that

$$\delta = \frac{0.382\, x}{Re_x^{1/5}}\,, \tag{9.38}$$

and

$$C_f = \frac{\tau_w}{\rho\, u_\infty^2/2} = \frac{0.0594}{Re_x^{1/5}} \tag{9.39}$$

(Fox and McDonald, 1998). Thus, turbulent flow goes inversely as the Reynolds number to the one–fifth power, rather than the one–half power that was found for laminar flow. Experiments show that Eqs. (9.38) and (9.39) are reasonably accurate for turbulent flows up to $Re_x < 10^7$.

9.6. Lift and Drag

Lift and drag were mentioned at the beginning of the chapter as the two resultant forces of primary interest for external flows over bodies. To describe these forces, we must know the shear and pressure forces acting on the body. The flat plate we just discussed is one of the few cases in which these distributions can be determined analytically. For the more complicated configurations of engineering interest, we can once again turn to empirics to deduce some practical results. However, to sensibly formulate such results, we should apply a proper dimensional treatment. For example,

[9.14] Eq. (9.37) predicts an infinite shear stress at $y = 0$. Specifically, τ at $y = 0$ depends upon $\partial u/\partial y$. Differentiating Eq. (9.37), we find that $\partial u/\partial y \sim y^{-6/7}$, which is infinite when evaluated at $y = 0$.

consider the drag on a moving sphere. Let us assume[9.15] that the drag force F_D experienced by this sphere is a function of its diameter d, its velocity u_∞ and the fluid density μ and viscosity ρ. According to the conventions of Chapter 7, we would write the functional relationship as

$$F_D \;=\; f_1\,(d,\; u_\infty,\; \mu,\; \rho)\;. \tag{9.40}$$

Applying dimensional analysis, we would find two dimensionless groups, e.g.

$$\frac{F_D}{\rho\,u_\infty^2\,d^2} \;=\; f_2\left(\frac{\rho\,u_\infty\,d}{\mu}\right)\;. \tag{9.41}$$

Let us develop Eq. (9.41) with the following observations: d^2 is proportional to the cross–sectional area of the sphere A, so we can replace it by A without changing the meaning of the dimensional equation. Also, let us add a factor of one–half to the left hand side so that the denominator matches the definition of dynamic pressure introduced in Chapter 3. We can then write Eq. (9.41) as

$$\frac{F_D}{\frac{1}{2}\,\rho\,u_\infty^2\,A} \;=\; f_3\left(\frac{\rho\,u_\infty\,d}{\mu}\right)\;. \tag{9.42}$$

We see clearly that the dimensionless quantity on the right hand side is the Reynolds number, while the one on the left hand side is defined as the *coefficient of drag* drag coefficient

$$C_D \;=\; \frac{F_D}{\frac{1}{2}\,\rho\,u_\infty^2\,A}\;. \tag{9.43}$$

Thus, Eq. (9.42) can be written

$$C_D \;=\; f_3\,(Re)\;, \tag{9.44}$$

in other words, the drag coefficient is a function of the Reynolds number. We could go through a similar dimensional analysis for lift. Here, we define the *coefficient of lift* as lift coefficient

$$C_L \;=\; \frac{F_L}{\frac{1}{2}\,\rho\,u_\infty^2\,A_p}\;, \tag{9.45}$$

where F_L is the lifting force and A_p is the *planform area* whose normal vector is transverse to the flow.

Let us formulate a specific description of how these forces depend upon the pressure and shear distributions. Figure 9.3 shows a differential surface element for a body moving horizontally in the $-x$ direction with local contributions of pressure and shear. The component of force in the x direction is the drag force and is given by

$$dF_D \;=\; P\,dA\cos\theta \;+\; \tau\,dA\sin\theta\;. \tag{9.46}$$

[9.15] In actuality, drag might also depend upon other factors, such as compressibility (supersonic vehicles) or free–surface effects (ships), however, we will not consider these more complicated cases.

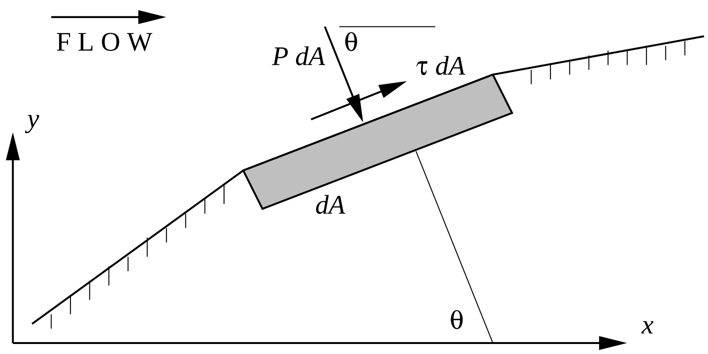

FIGURE 9.3. *Pressure and shear forces on a differential element of a moving body.*

Likewise, the component of force in the y direction is the lift and can be expressed as

$$dF_L \;=\; -\,P\,dA \sin\theta \;+\; \tau\,dA \cos\theta\,. \tag{9.47}$$

The overall lift and drag forces can be calculated by integrating Eqs. (9.46) and 9.47 around the body.

Let us take drag as an example. Drag has two components, *friction drag*, which is due to shear forces, and *pressure drag*[9.16], which arises from the pressure distribution. The magnitude of these forces clearly depends upon the orientation of the body surface θ as is obvious from Eq. (9.46). For example, in the laminar flat plate boundary layer flow we studied previously, $\theta = \pi/2$ and the entire drag experienced by the plate is friction drag. Thus, from Eq. (9.16), we can calculate

$$F_D \;=\; \int_A \tau\,dA \;=\; \int_0^L \frac{0.332\,\rho\,u_\infty^2}{\sqrt{Re_x}}\,w\,dx\,, \tag{9.48}$$

where w is the width and L is the length of the plate. Eq. (9.48) can be evaluated as

$$F_D \;=\; \frac{0.332\,\rho\,u_\infty^2\,w}{\sqrt{u_\infty/\nu}} \int_0^L x^{-1/2}\,dx \;=\; \frac{0.664\,\rho\,u_\infty^2\,w}{\sqrt{u_\infty/\nu}}\,x^{1/2}\Big|_0^L \tag{9.49}$$

This simplifies to

$$F_D \;=\; \frac{0.664\,\rho\,u_\infty^2\,w\,\sqrt{L}}{\sqrt{u_\infty/\nu}}\,\frac{L}{\sqrt{L^2}} \;=\; \frac{0.664\,\rho\,u_\infty^2\,w\,L}{\sqrt{u_\infty L/\nu}}\,. \tag{9.50}$$

Since the area of the plate is $A = w \cdot L$

$$F_D \;=\; \frac{0.664\,\rho\,u_\infty^2\,A}{\sqrt{Re_L}}\,, \tag{9.51}$$

[9.16] This is also commonly called *form drag* because it typically depends very highly on the shape or form of the moving object.

where Re_L is the Reynolds number based on the length of the plate. Thus, applying the definition in Eq. (9.43), we find

$$C_D = \frac{1.328}{\sqrt{Re_L}}, \tag{9.52}$$

which is entirely due to friction drag.

If we were instead to turn the plate normal to the flow so that $\theta = 0$, then the drag would given by

$$dF_D = P\,dA \tag{9.53}$$

according to Eq. (9.46). This would be entirely form drag. In actuality, flow would separate behind the plate, leaving a low–pressure wake. This case cannot be studied analytically with the tools we have at our current disposal. Thus, we cannot determine C_D in closed form as we did for the case of friction drag. We can only qualitatively observe that for moderate to high Reynolds numbers ($Re > 10^3$) the drag coefficient for bodies with sharp edges is largely independent of the Reynolds number because locations of separation are fixed by the geometry of the body. In general, most engineering cases of interest will involve some mixture of these two extremes. Specifically, the relative sizes of form and friction drag will vary over the body. Standard textbooks contain considerable empirical results for various body shapes and flow conditions (Fox and McDonald, 1998; Munson et al., 2006).

One remarkable phenomenon that bears at least a qualitative description is the difference in drag realized for streamlined versus blunt bodies. We saw in Figure 9.2 that turbulent boundary layers have larger gradients at the body surface than laminar ones. Since τ is proportional to the velocity gradient, turbulent boundary layers lead to higher friction drag. Streamlined bodies, by their very design, attempt to minimize flow separation, so the primary drag force they realize is due to friction. A laminar boundary layer is obviously more desirable for these configurations. Streamlined bodies are therefore designed to minimize the chances that boundary layers will become turbulent. In particular, surfaces are smooth in order to prevent disturbances near the surface that might "trip" the boundary layer to transition to turbulence. Airplane wing sections are a good example of this.

Conversely, blunt bodies, such as a golf ball, realize most of their resistance as form drag (Mehta, 1985). As the boundary layer develops around the ball, the pressure distribution is such that the flow will eventually separate, leaving a low–pressure wake behind the ball (Figure 9.4). There is some friction drag, but it is small compared to the form drag. However, there is a remarkable difference between laminar and turbulent boundary layers for this case. Perhaps contrary to our intuition (at least based on what we just established for streamlined bodies), the turbulent boundary layer causes significantly lower drag coefficients. How can that be? Essentially, the turbulent boundary layer has more momentum, so it is better able to follow the

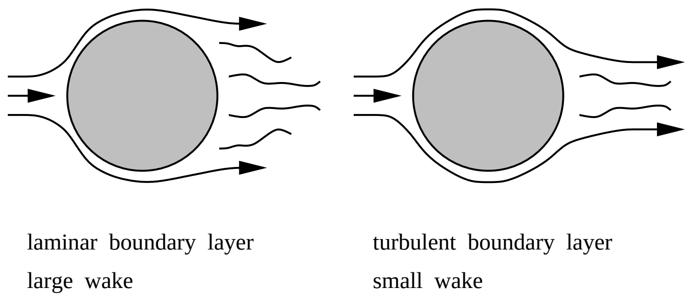

FIGURE 9.4. *Qualitative flow patterns around a golf ball.*

contour of the body shape without separating. This means the wake left behind the ball will be smaller for the turbulent boundary layer (Figure 9.4), so the drag will be less. This phenomenon is shown clearly by plotting the experimentally–determined drag coefficient versus the Reynolds number (Figure 9.5). At a Reynolds number of roughly 300,000 laminar–turbulent

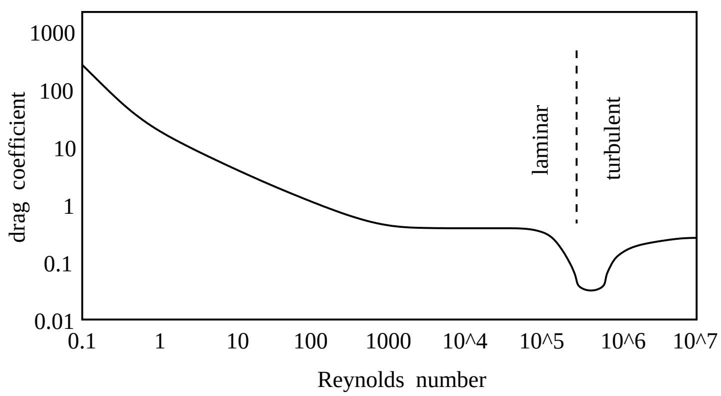

FIGURE 9.5. *Drag coefficient for smooth sphere as a function of Reynolds number.*

transition occurs so that the boundary layer on the forward portion of the ball becomes turbulent. The separation point moves downstream, leaving a smaller wake. The drag coefficient drops by roughly a factor of five. The "dimples" on a golf ball are designed to cause flow disturbances so as to trip the boundary layer into becoming turbulent.

CHAPTER 10

Open Channel Flow

In this chapter, we introduce a rather different flow configuration than what we have studied up to this point. Specifically, unlike the internal and external flows discussed in Chapters 8 and 9, *open channel flows* have a free–surface that divides a liquid side from a gas side. Almost always (at least for our purposes), the liquid is water and the gas is atmospheric air. Recall, for example, that in pipe flow, a streamwise pressure gradient drives the flow. Here, gravity is the driver — open channel flows are driven by the weight of the liquid itself. Examples of such flows are:

- rivers, streams, canals, etc.
- drainage channels, ditches, ravines, and gutters
- flumes, aqueducts, and man–made water transportation systems
- storm sewers and culverts

The designation "open channel" is actually a bit misleading, because we tend to think of "open" in the geometric sense. Storm sewers, for example, would not fit this designation, since they are underground. "Open" actually refers to the fact that the cross–sections of such configurations are not fully occupied by the liquid, i.e. they are open. Fig. 10.1 shows a storm sewer as an example. The channel has a rectangular cross–section and is partially

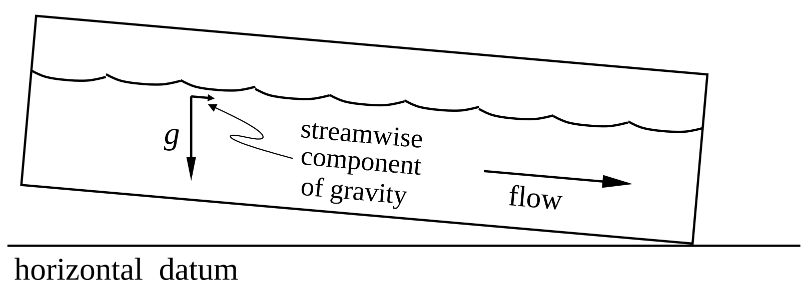

FIGURE 10.1. *Open–channel flow is driven by gravity, i.e. it flows "down–hill".*

filled with water. The remainder of the section above the water is open. If we were to try to impose a pressure gradient between the inlet and outlet, we would simply realize "blow–by" of air above the water. This configuration is unable to sustain a pressure gradient! Because of a slight incline, there is a small component of the gravity vector acting in the streamwise direction and this is what drives the flow.

10.1. Flow Classification

Open–channel flows, like the other configurations we have examined up to this point, are classified in a number of ways. One of the most important characteristics is whether the flow is laminar or turbulent. This question is, once again, quantitatively addressed by the Reynolds number

$$Re = \frac{\bar{u}\, R_h}{\nu}\,, \tag{10.1}$$

where $\bar{u}$ is the average streamwise velocity in the channel, R_h is the *hydraulic radius*, and ν is the liquid kinematic viscosity. The hydraulic radius, a close relative to the hydraulic diameter in Eq. (8.42) on pp. 110, is defined as

hydraulic radius

$$R_h = \frac{A}{P}\,, \tag{10.2}$$

where A is the cross–sectional area of the flow and P is the wetted perimeter. We note a few subtle differences between R_h in Eq. (10.2) used for open–channel flow and the hydraulic diameter, D_h, in Eq. (8.42) used for pipe flow:

- Although D_h is based on the geometric cross–sectional area of the pipe, R_h is based on the cross–sectional area of the *flow itself*. For example, in Fig. 10.2, we would use h in calculating D_h, but a in calculating R_h.
- The wetted perimeter P in Eq. (10.2) does not include the top part or the un–wetted upper part of the sides of the conduit in Fig. 10.2. That is, $P = a + b + a = 2a + b$ for Eq. (10.2). Recall that for pipe flow, we would have computed D_h in terms of a wetted perimeter of $P = h + b + h + b = 2\,(h + b)$, as in Eq. (8.43).

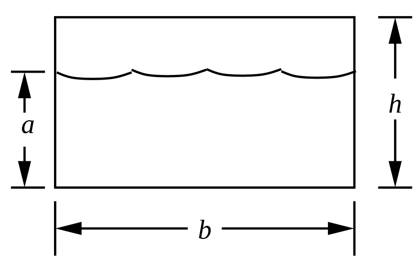

FIGURE 10.2. *Cross–section of open–channel flow in a conduit of width b. Conduit height is h and fluid height is a, where a < h.*

Thus, for the conduit in Fig. 10.2 we find

$$R_h = \frac{b\,a}{2a + b}\,,$$

which simplifies[10.1] to $R_h \approx a$ if $b \gg a$.

These subtleties are mostly relevant to conduits, such as the storm sewer depicted in Figs. 10.1 and 10.2, whose geometry is enclosed. For open geometries, such as rivers and drainage channels, correctly identifying A and P in Eq. (10.2) is more straightforward.

For open–channels, flow classification according to the Reynolds number is somewhat less exact than for other configurations because geometry can be highly irregular, e.g. a riverbed. Cases where $Re < 500$ are laminar, while $Re > 12,500$ is taken to indicate turbulent flow (Munson et al., 2006). However, the laminar regime is not too common with the open–channel configuration. Consider that most of the these cases have water as the working fluid (liquid). The kinematic viscosity of water is actually quite low (Table 10.1), which increases the Reynolds number. Also, R_h is usually

TABLE 10.1. Kinematic viscosity of water and air at approximately 20^o C.

fluid	$\nu\ (m^2/sec)$
water	1×10^{-6}
air	1.5×10^{-5}

fairly large for these flows. For example, in a river having $R_h = 3\,m$ and an average velocity of $\bar{u} = 0.5\,m/sec$, we find $Re = 0.5 \times 3/1 \times 10^{-6} = 1.5 \times 10^6$. This flow is well into the turbulent regime.

Another classification made for open–channel flows involves the depth. Let us represent the generic open–channel configuration by the schematic shown in Fig. 10.3. The upstream and downstream velocities are V_1 and V_2, respectively. Furthermore, the channel itself slopes: the channel bed has an upstream elevation of z_1 and a downstream elevation of z_2, where $z_2 < z_1$. The average channel slope S_0 is taken as the positive quantity[10.2] channel slope

$$S_0 = \frac{dz}{dx} \approx \frac{z_1 - z_2}{L}\,. \tag{10.3}$$

For our problems of interest, S_0 will be constant.

[10.1] If $b \gg a$, then $2a + b \approx b$, so that

$$R_h = \frac{b\,a}{2a + b} \approx \frac{b\,a}{b} = a\,.$$

[10.2] Note this is opposite to the typical sign convention for gradients. That is, we would *usually* represent the slope as

$$S_0 = \frac{dz}{dx} \approx \frac{z_2 - z_1}{L}\,,$$

which is a negative quantity because $z_2 < z_1$. However, for open–channel flow, the sign convention is that shown in Eq. (10.3).

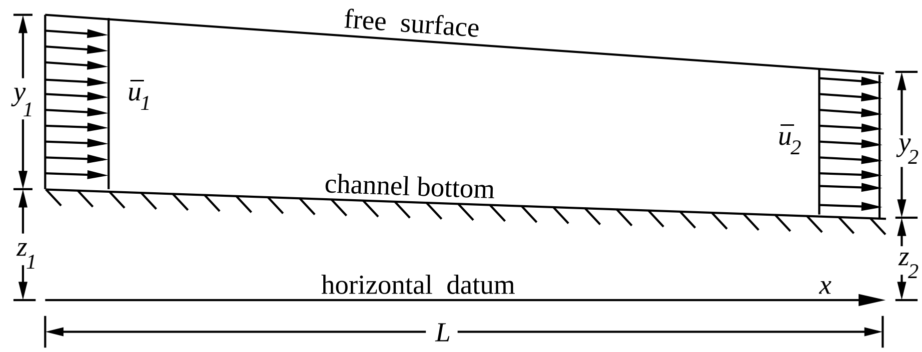

FIGURE 10.3. *Schematic of open–channel flow.*

According to Fig. 10.3, there are actually two coordinate systems, the x–y system based on the horizontal datum and another "tilted" system that is aligned with the bottom of the channel. However, because S_0 is almost always very small, it is customary to measure stream depth in terms of y, rather than the coordinate normal to the channel bottom. In other words, the upstream and downstream depths are y_1 and y_2, respectively, to within negligible error[10.3]. The local slope of the free surface is then dy/dx. We classify flows as *uniform* if $dy/dx = 0$ and *non–uniform* if $dy/dx \neq 0$.

10.2. An Energy–Based Analysis Methodology

For our examination of open–channel flow, we will make the assumption that the velocity profile is oriented in the streamwise direction only, i.e. $\vec{V} = \bar{u}\hat{i} + 0\hat{j} + 0\hat{k}$. Moreover, we assume that $\bar{u}$ is uniform over the fluid cross–section at any particular x. These assumptions permit us to write the extended form of Bernoulli's equation[10.4] between the upstream and downstream locations, i.e.

$$\frac{P_1}{\rho\, g} + \frac{\bar{u}_1^2}{2\, g} + z_1 - h_L = \frac{P_2}{\rho\, g} + \frac{\bar{u}_2^2}{2\, g} + z_2\,. \tag{10.4}$$

Recall that h_L represents the viscous losses as the fluid moves from upstream to downstream locations.

According to our assumptions about the velocity distribution, streamlines are all straight, so that the pressure distribution is essentially hydrostatic, i.e.

$$P_1 = \rho\, g\, y_1 \quad \text{and} \quad P_2 = \rho\, g\, y_2\,.$$

[10.3] For example, the Mississippi River is about 1.24×10^7 feet long, over which it drops approximately 1470 feet. Its average slope is then about $S_0 \approx 1.2 \times 10^{-4}$.

[10.4] See for example Eqs. (5.34) and (8.33).

Using this observation, and the result in Eq. (10.3) for the channel slope, which we write $z_1 - z_2 = S_0 L$, we can re–cast Eq. (10.4) as

$$y_1 + \frac{\bar{u}_1^2}{2\,g} + S_0 L = y_2 + \frac{\bar{u}_2^2}{2\,g} + h_L \,. \tag{10.5}$$

Let us take one further step. Notice that the form of Eq. (10.5) is such that its units are given in terms of a length scale[10.5], i.e. "height". Therefore, the loss term h_L is also a height, specifically a change of height. If we consider this quantity in the same sense as the change of height of the channel bed, $z_1 - z_2$, it should be clear that we can write h_L in the form of a "slope", i.e.

$$S_f = \frac{h_L}{L} \,. \tag{10.6}$$

This quantity is called the *friction slope* and it is quite similar to the fashion that we wrote $z_1 - z_2$ via S_0 in Eq. (10.3). Thus, we can finally write the Bernoulli equation as friction slope

$$y_1 + \frac{\bar{u}_1^2}{2\,g} = y_2 + \frac{\bar{u}_2^2}{2\,g} + (S_f - S_0)\,L \,, \tag{10.7}$$

or, equivalently

$$y_1 - y_2 = \frac{\bar{u}_2^2 - \bar{u}_1^2}{2\,g} + (S_f - S_0)\,L \,. \tag{10.8}$$

Eq. (10.8) reports the change in depth of the flow from the upstream to the downstream location in terms of the velocities and the associated slopes of the flow.

Let us consider Eq. (10.7) further. First, define a quantity known as the *specific energy* as specific energy

$$E = y + \frac{\bar{u}^2}{2\,g} \,. \tag{10.9}$$

[10.5] Recall that we can cast Bernoulli–type equations in terms of various units, all of which are representative of energy in some fashion. For example, our first realization of the Bernoulli equation in Eq. (3.19) on pp. 27, was cast in terms of pressure. Working this through, i.e.

$$\frac{N}{m^2} = \frac{N}{m^2}\,\frac{m}{m} = \frac{N\,m}{m^3} = \frac{J}{m^3} \,,$$

we find this form represents energy per unit volume. Conversely, we can do the same sort of analysis to find that Eq. (5.34) on pp. 55 is written in terms of energy per unit mass. That is, we have divided through by the density, which has units of mass per unit volume. Eq. (10.5) is written in terms of a length scale, in this case height. If we consider, for example, the form of potential energy, i.e.

$$\text{mass} \times \text{gravity} \times \text{height} = \text{weight} \times \text{height} \,,$$

it is clear that Eq. (10.5) is cast in terms of energy per unit weight.

In each of these cases we considered h_L as a generic loss, represented in terms of units appropriate for each particular equation.

Eq. (10.7) can be recast as

$$E_1 \;=\; E_2 \;+\; (S_f - S_0)\, L\,, \tag{10.10}$$

which is a statement of the conservation of specific energy.

Suppose we apply our equations to a simple channel of rectangular cross–section (Fig. 10.4). According to our assumption of a uniform velocity profile, the volume flow rate is $Q = \bar{u}\, b\, y$. Conservation of mass dictates that Q is constant. The flow rate per unit width of the channel is $q = Q/b = \bar{u}\, y$.

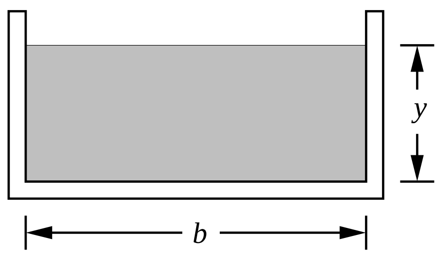

FIGURE 10.4. *Simple channel having a rectangular cross–section and fluid depth of y.*

This clearly means that we can write the local velocity $\bar{u}$ in terms of the local depth y as

$$\bar{u} \;=\; \frac{q}{y}\,. \tag{10.11}$$

Substituting this equation into the expression for specific energy, i.e. Eq. (10.9), we find

$$E \;=\; y \;+\; \frac{(q/y)^2}{2\,g} \;=\; y \;+\; \frac{q^2}{2\,g\,y^2}\,. \tag{10.12}$$

Eq. (10.12) governs the relationship between specific energy and depth of the flow for a given (constant) per unit width flow rate of q.

At any particular location x, we can examine a diagram for the specific energy as quantified by Eq. (10.12). From the form of this equation, we might consider plotting E as a function of y for a specific flow rate q. However, Eq. (10.12) is *multi–valued*, i.e. for a given value of E, there are several (actually up to three) values of y that will satisfy the equation. Therefore, we will plot y as a function of E. Fig. 10.5 shows such a plot, called a "specific energy diagram", for $q = 0.1\, m^2/s$. In general, two of the solutions for y will be positive, while one will be negative[10.6]. A special case is the *cusp* of the curve at E_{min}, where there is only one solution for y.

For the general case, we have two distinct flow regimes: sub–critical and super–critical. We will denote the two associated solutions for y as $y_{sub-crit}$ and $y_{super-crit}$, respectively, where $y_{sub-crit} > y_{super-crit}$. In other words,

[10.6]The negative value has no physical significance and can be ignored.

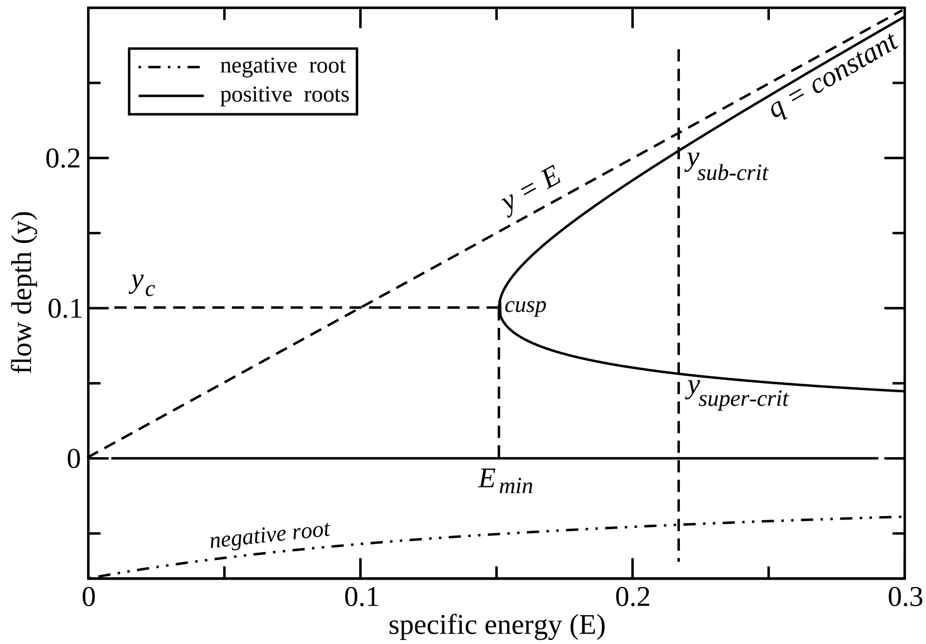

FIGURE 10.5. *Specific energy as a function of depth for* $q = 0.1\, m^2/s$.

there are generally two distinct depths that are possible for the flow in our rectangular channel. Sub–critical flow is "deeper" than super–critical flow.

Recall that q is constant along the solution contour. That is, the rate of flow is the same for both flow regimes. According to Eq. (10.11), we see that sub–critical flow can be characterized as "slow–moving" since y in the denominator is large, while super–critical flow is "fast–moving" since y is small. The two depths have limiting behavior $y_{sub-crit} \to E \to \infty$ and $y_{super-crit} \to 0$.

An important aspect of the problem is the location of the cusp at $(E, y) = (E_{min}, y_c)$. In other words, we want to determine the values of E_{min} and y_c. Let us first note that the rate of change of E with respect to y vanishes at the cusp, i.e.

$$\left.\frac{dE}{dy}\right|_{cusp} = \frac{d}{dy}\left.\left(y + \frac{q^2}{2\,g\,y^2}\right)\right|_{cusp} = \left.\left(1 - \frac{q^2}{g\,y^3}\right)\right|_{cusp} = 0\,. \tag{10.13}$$

Solving Eq. (10.13), we find

$$y|_{cusp} = y_c = \left(\frac{q^2}{g}\right)^{1/3}. \tag{10.14}$$

Here, y_c is the so–called *critical* depth. Thus, in our example in Fig. 10.5, we find

$$y_c = \left(\frac{0.1^2}{9.8}\right)^{1/3} = 0.101\, m\,.$$

Sub–critical flow will be deeper than $0.101\, m$, while super–critical flow will be shallower.

Determining the corresponding value of E_{min} simply requires substituting the expression for y_c into Eq. (10.12), i.e.

$$\begin{aligned}
E_{min} &= y_c + \frac{q^2}{2\, g\, y_c^2} \\
&= y_c + \frac{q^2}{2\, g\, \left(q^2/g\right)^{2/3}} \\
&= y_c + \frac{q^2\, g^{2/3}}{2\, g\, q^{4/3}} \\
&= y_c + \frac{q^{2/3}}{2\, g^{1/3}} \\
&= y_c + \frac{y_c}{2} \\
E_{min} &= 1.5\, y_c\,. \qquad (10.15)
\end{aligned}$$

Thus, E_{min} is always 50% greater than y_c. In our example in Fig. 10.5, we find $E_{min} = 0.151\, m$.

Flow velocity at the critical condition can also be determined by substitution into Eq. (10.11). For example, Eq. (10.14) can be re–cast as $q = g^{1/2} y_c^{3/2}$, so that we can write

$$\bar{u}_c = \frac{q}{y_c} = \frac{g^{1/2}\, y_c^{3/2}}{y_c} = \sqrt{g\, y_c}\,. \qquad (10.16)$$

Eq. (10.16) is informative from a physical standpoint in that it formally defines the *critical point* of open–channel flow. Specifically, from Eq. (10.14) we see y_c is fixed according to q and is, therefore, constant for a given flow configuration. Eq. (10.16) reveals that $\bar{u}_c$ is therefore also a fixed constant. Thus, if $\bar{u} > \bar{u}_c = \sqrt{g\, y_c}$, the flow will be super–critical, but if $\bar{u} < \bar{u}_c = \sqrt{g\, y_c}$, the flow is sub–critical.

This concept provides a ready–made dimensionless parameter for the above context called the *Froude number*, defined as Froude number

$$Fr = \frac{\bar{u}}{\sqrt{g\, y}}\,. \qquad (10.17)$$

Eq. (10.16) indicates that the critical condition is defined as $Fr = 1$. According to the above observations, super–critical flow occurs for $Fr > 1$, while sub–critical flow is defined for $Fr < 1$. *MY&O Ex. 10.2 pp 572*

10.3. Variation in Channel Depth

We can use the above concepts to determine the nature of depth variation as flow proceeds along a channel, i.e. how dy/dx varies. According to the assumptions that were made at the outset, we will only be able to examine gradually–varying flows, which are those where $dy/dx \ll 1$. Rapidly–varying flows involve multi–dimensional motion that is beyond our scope and analytical capability.

Once again, we can write the extended Bernoulli equation in a form that is convenient for the task at hand, in this case

$$y_1 + \frac{\bar{u}_1^2}{2\,g} + z_1 = y_2 + \frac{\bar{u}_2^2}{2\,g} + z_2 + h_L\,. \tag{10.18}$$

Defining the total energy, or *total head*, as $H = 0.5\,\bar{u}^2/g + y + z$, we write Eq. (10.18) simply as

total head

$$H_1 = H_2 + h_L\,. \tag{10.19}$$

In turn, we write Eq. (10.19) into the more useful form

$$\frac{H_1 - H_2}{L} = \frac{h_L}{L}\,, \tag{10.20}$$

where L is the distance between points 1 and 2. According to Eq. (10.6), the right–hand side is simply the friction slope S_f. The left–hand side is approximately the gradient in H along the channel, $dH/dx \approx (H_1 - H_2)/L$ *written according to the special sign convention discussed in footnote 10.2 (pp. 130)*. We can therefore rewrite Eq. (10.20) as

$$\frac{dH}{dx} = S_f\,. \tag{10.21}$$

Returning to Eq. (10.18) and the concept of total head, we can also write dH/dx out as

$$\frac{dH}{dx} = \frac{d}{dx}\left(y + \frac{\bar{u}^2}{2\,g} + z\right) = \frac{dy}{dx} + \frac{\bar{u}}{g}\,\frac{d\bar{u}}{dx} + \frac{dz}{dx}\,. \tag{10.22}$$

This expression can readily be re–cast in the form

$$\frac{dy}{dx} = -\,\frac{\bar{u}}{g}\,\frac{d\bar{u}}{dx} - \frac{dz}{dx} + \frac{dH}{dx}\,. \tag{10.23}$$

We immediately recognize the last two terms on the right–hand side as S_0 and S_f from Eqs. (10.3) and (10.21), respectively. However, the first term on the right–hand side needs additional development. The problem with this term is that we do not know how to explicitly treat the velocity gradient $d\bar{u}/dx$.

Let us go back to Eq. (10.11), which relates velocity $\bar{u}$ to channel depth y, i.e. $\bar{u} = qy^{-1}$. We can take the derivative $d\bar{u}/dy$, finding

$$\frac{d\bar{u}}{dx} = \frac{d\bar{u}}{dy}\,\frac{dy}{dx} = -\,\left(q\,y^{-2}\right)\,\frac{dy}{dx} = -\,\frac{1}{y}\,\frac{q}{y}\,\frac{dy}{dx} = -\,\frac{1}{y}\,\bar{u}\,\frac{dy}{dx}\,. \tag{10.24}$$

Notice that in the last step, we have re–applied Eq. (10.11).

We can now substitute all these results back into Eq. (10.23).

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\bar{u}}{g}\left(-\frac{1}{y}\,\bar{u}\,\frac{dy}{dx}\right) - S_0 + S_f \\ &= \frac{\bar{u}^2}{g\,y}\,\frac{dy}{dx} - S_0 + S_f \\ &= Fr^2\,\frac{dy}{dx} - S_0 + S_f \qquad (10.25)\end{aligned}$$

In the last step, we have used the definition of the local Froude number from Eq. (10.17). Solving for the slope dy/dx, we find

$$\frac{dy}{dx} = \frac{S_f - S_0}{1 - Fr^2}\,. \qquad (10.26)$$

Eq. (10.26) indicates that channel depth varies with the Froude number, the friction slope, and the geometric slope. For given values of S_0 and S_f, the behavior will be opposite for sub–critical versus supercritical flow because the denominator changes signs. Note that Eq. (10.26) is not defined for critical conditions at $Fr = 1$. The sign convention for Eq. (10.26) is $dy/dx > 0$ if flow gets deeper with x and $dy/dx < 0$ if flow gets shallower with x.

10.4. Channel Flows of Constant Depth

An interesting configuration that is frequently used in practice is constant–depth flow, where $dy/dx = 0$. Eq. (10.26) shows that this can be obtained by adjusting the channel slope S_0 to match the friction slope S_f. The physical interpretation of this configuration is that the loss of potential energy as fluid flows downward equals the viscous energy dissipation resulting from frictional shear stresses.

Consider the constant–depth configuration depicted in Fig. 10.6. We can

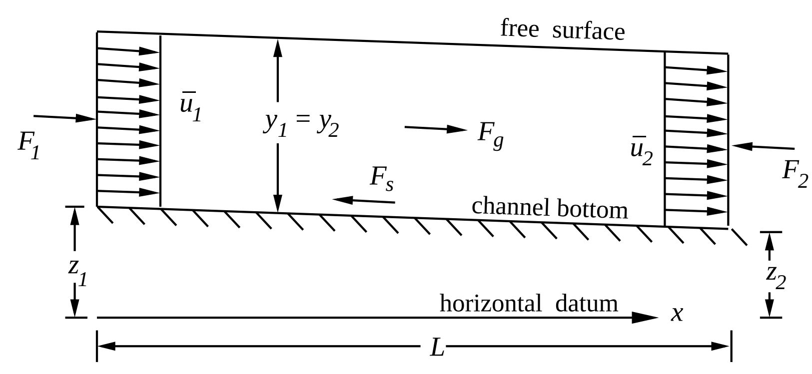

FIGURE 10.6. *Constant–depth open–channel flow.*

write an integral conservation of momentum equation according to Eq. (5.14) on pp. 50 as

$$\begin{aligned} \Sigma F_x &= \frac{\partial}{\partial t}\int_{cv} u\,\rho\,dv + \int_{cs} u\,\rho\,\mathbf{V}\cdot\hat{n}\,dA \\ &= -\,\bar{u}_1\,\rho\,Q_1 + \bar{u}_2\,\rho\,Q_2 \\ &= \rho\,Q\,(\bar{u}_2 - \bar{u}_1)\ . \end{aligned} \tag{10.27}$$

The second step arises from the fact that the flow is assumed steady, so $\partial/\partial t = 0$, while the third step is a consequence of the constant volume flow rate Q that we have assumed for open–channel flows.

We can simplify Eq. (10.27) further by writing a simple conservation of mass equation

$$\rho\,\bar{u}_1\,A_1 = \rho\,\bar{u}_2\,A_2\ , \tag{10.28}$$

or, on a per unit width basis

$$\rho\,\bar{u}_1\,y_1 = \rho\,\bar{u}_2\,y_2\ , \tag{10.29}$$

and noticing that the constant–depth condition means that $y_1 = y_2$, and consequently $\bar{u}_1 = \bar{u}_2$. Substituting back into Eq. (10.27), we find that the conservation of momentum reduces to the simple condition that

$$\Sigma F_x = 0\ . \tag{10.30}$$

We can now set about quantifying all the forces acting on the control volume in Fig. 10.6. There are the hydrostatic forces, F_1 and F_2, a body force F_g that arises from the component of gravity in the flow direction, and finally the surface force F_s that is a consequence of shear stress acting wherever the fluid is in contact with the channel surface. Expanding Eq. (10.30), we find

$$F_1 - F_2 + F_g - F_s = 0\ . \tag{10.31}$$

Once again, we obtain simplification because the constant–depth condition $y_1 = y_2$ means that the hydrostatic components will be equal, i.e. $F_1 = F_2$, and will therefore cancel each other. The flow is then a simple balance between the gravitational "driving" force and the frictional (shearing) "drag" force, i.e.

$$F_g = F_s\ . \tag{10.32}$$

These two forces are straightforward to quantify. The gravitational force is simply the component of the fluid weight W in the streamwise direction, i.e. $F_g = W\sin\theta$, as shown in Fig. 10.7. Since the total weight is simply the volume $V = AL$ multiplied by the specific weight of the fluid $\gamma = \rho g$, we find

$$\begin{aligned} F_g &= W\sin\theta \\ &= A\,L\,\rho\,g\sin\theta \\ &= R_h\,P\,L\,\rho\,g\sin\theta\ , \end{aligned} \tag{10.33}$$

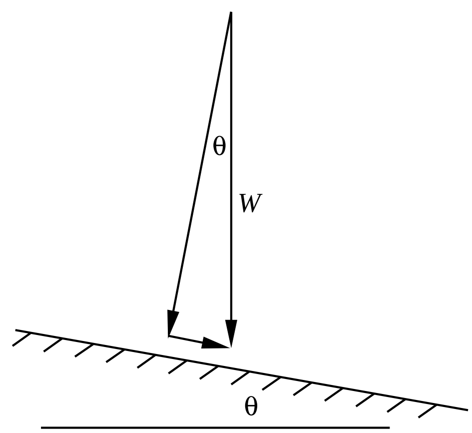

FIGURE 10.7. *Gravitational component in the streamwise direction.*

where we have used the definition of the hydraulic radius from Eq. (10.2) in the last step. The total shearing force is simply the shear stress at the channel wall, τ_w, multiplied by the total wall area in contact with the fluid, PL. Thus,

$$F_s = P\, L\, \tau_w \,. \tag{10.34}$$

Substituting Eqs. (10.33) and (10.34) into Eq. (10.32), we find

$$R_h\, P\, L\, \rho\, g \sin\theta = P\, L\, \tau_w \,, \tag{10.35}$$

or, more simply

$$\tau_w = R_h\, \rho\, g \sin\theta \,. \tag{10.36}$$

Taking this one step further, we notice that $\sin\theta \to S_0$ for small angles[10.7], so that

$$\tau_w = R_h\, \rho\, g\, S_0 \,. \tag{10.37}$$

Eq. (10.37) is a simple algebraic expression that gives wall shear stress in terms of fluid density and the channel hydraulic radius and slope.

It also represents the limit beyond which theory alone no longer suffices. Recall that most open–channel flows are turbulent, so we do not have an

[10.7]Sine expands as $\sin\theta = \theta - \theta^3/3! + \theta^5/5! - \cdots$, while tangent expands as $\tan\theta = \theta + \theta^3/3! + 2\theta^5/15 + \cdots$, so that the two are equal for small angles (Beyer, 1984). That is $\sin\theta \approx \tan\theta$, since the higher–order terms tend to vanish as θ becomes small. However, $\tan\theta$, as shown in Fig. 10.7, is precisely S_0, as defined in Eq. (10.3). That is,

$$S_0 = \tan\theta = \frac{z_1 - z_2}{L} \,.$$

For small angles, we can then invoke $\sin\theta = S_0$.

analytical expression that relates flow velocity to the wall shear stress[10.8]. As a first approximation, we can apply the following empirical model. We will assume dynamics similar to pipe flow at very high Reynolds numbers, i.e. losses are essentially independent of the Reynolds number and only depend on the surface roughness, e.g. Eq. (8.32) (pp. 109). Wall shear stress is therefore assumed to be proportional to the dynamic pressure. That is

$$\tau_w \;=\; K\,\frac{\rho\,\bar{u}^2}{2}\,, \tag{10.38}$$

where K is an empirically–determined proportionality constant, which depends upon the channel roughness.

Equating Eqs. (10.37) and (10.38), we find

$$R_h\,\rho\,g\,S_0 \;=\; K\,\frac{\rho\,\bar{u}^2}{2}\,, \tag{10.39}$$

from which we can solve for the channel velocity as

$$\bar{u} \;=\; \sqrt{\frac{2\,g}{K}}\,\sqrt{R_h\,S_0} \;=\; C_c\,\sqrt{R_h\,S_0}\,. \tag{10.40}$$

Eq. (10.40) is the *Chezy equation*. All the constants in Eq. (10.40) have been folded into a single entity C_c called the *Chezy coefficient*, which must be empirically determined. Notice that the form of this equation implies that the units of C_c are square–root of length over time[10.9]. Chezy equation

We note that the Chezy equation depends strongly on our assumption in Eq. (10.38) that wall shear stress is proportional to the dynamic pressure, and, therefore, *not* a function of fluid viscosity. Experiments have shown that the dependence of $\bar{u}$ in Eq. (10.40) on $\sqrt{S_0}$ is actually quite a good model for such flows. Conversely, the dependence on $\sqrt{R_h}$ is not terribly realistic. A better model is given by the *Manning equation*, which models the dependence upon the hydraulic radius according to a "2/3" power law, i.e. Manning equation

$$\bar{u} \;=\; \frac{R_h^{2/3}\sqrt{S_0}}{n}\,, \tag{10.41}$$

where n is the *Manning coefficient*. Like the Chezy coefficient, n is determined empirically and, moreover, is not dimensionless[10.10].

[10.8] Probably the most convenient example that we have already studied is pipe flow, where for the laminar case we could, in fact, derive an analytical relationship between velocity and shear stress, e.g. Eqs. (8.12) and (8.13) on pp. 104. For turbulent flows, we could derive no such relationship.

[10.9] The slope S_0 is unitless and R_h has units of length, but it falls within the square root. Therefore, C_c must have units of length$^{1/2}$ over time in order for the overall expression to have units of length over time.

[10.10] Again, the slope S_0 is unitless and, since $R_h^{2/3}$ has units of length to the 2/3 power, n must have units of time per length$^{1/3}$.

Manning's equation is a much better model than the Chezy equation, and should be used for regular calculations[10.11]. According to the points made in footnotes 10.9 and 10.10, you have probably already noticed that there will be a problem with using Manning's equation in a practical capacity. Specifically, the value of the coefficient n will depend upon the units in which R_h is expressed. Therefore, Manning's equation is often expressed in the slightly more general form

$$\bar{u} \;=\; \kappa\, \frac{R_h^{2/3}\sqrt{S_0}}{n}\,, \tag{10.42}$$

where $\kappa = 1$ if R_h is expressed in SI units (meters) and $\kappa = 1.49$ if using English units[10.12]. Some values of the Manning coefficient *for water* are given in Table 10.2.

MY&O Ex. 10.3 pp 579

TABLE 10.2. Some Manning coefficients for open–channel water flows

Channel Type	n
riverbed	0.035
flood plain having light brush	0.05
flood plain having heavy brush	0.075
smooth earthen channel	0.022
gravelly earthen channel	0.025
earthen channel with weeds and plants	0.03
cast iron	0.013
finished concrete	0.012
clay tile	0.014
brick–lined channel	0.015
corrugated metal sewer pipe	0.022

10.5. Hydraulic Jumps

In §10.3, we indicated that rapidly–varying channel flows, i.e. those where $dy/dx \not\ll 1$, are not amenable to a strictly analytical treatment. However, *some* of these phenomena can be studied with suitable approximations. One of the more interesting configurations is the *hydraulic jump* (Fig. 10.8). This phenomenon entails a very rapid depth change, i.e. a finite increase of depth over a comparatively short distance, $dy/dx \gg 1$. It can arise when there is a conflict between the upstream and downstream tendencies for the

[10.11] It is recommended by various engineering handbooks (e.g. Merritt, 1976). There are other empirically–based flow models, notably Kutter's formula (e.g. Babbitt, 1940), but they are typically more complicated and less frequently used.

[10.12] The value of 1.49 is approximately the cube root of the meter–foot conversion factor, i.e. 3.281 feet per meter. The cube root arises because of the 1/3 power dependence of the length scale for n, as mentioned in footnote 10.10.

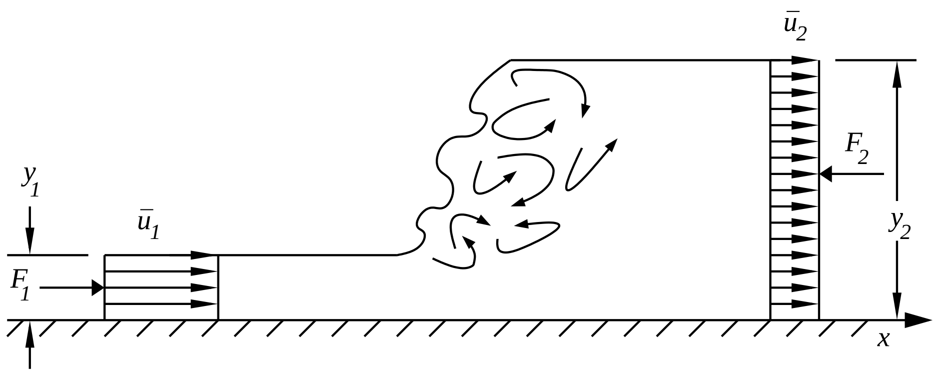

FIGURE 10.8. *Diagrammatic representation of the hydraulic jump phenomenon.*

flow regime. That is, upstream conditions may favor super–critical flow, while downstream conditions, e.g. an obstruction, requires the flow to be sub–critical.

We consider the horizontal rectangular cross–section channel shown in Fig. 10.8 having a width b. Two approximations are critical to our analysis:

- Although fluid motion in the jump region is complex, we will assume uniform steady flow in the immediate upstream and downstream areas.
- Wall shear stress τ_w between the upstream and downstream areas is neglected. Note that we did *not* make this assumption §10.4 for constant depth flow. The flow complexity in this problem completely precludes any analytical attempt to quantify τ_w, although our assumption might be interpreted in an "averaged" context. That is, there may be regions of backflow near the wall as well as flow in the streamwise direction, so that, on average, the wall shear tends to vanish.

Just as we did for constant depth flow in §10.4, we can write an integral conservation of momentum equation according to Eq. (5.14) between the upstream and downstream sections as

$$\begin{aligned} \Sigma F_x &= \frac{\partial}{\partial t}\int_{cv} u\,\rho\,dv + \int_{cs} u\,\rho\,\mathbf{V}\cdot\hat{n}\,dA \\ &= -\,\bar{u}_1\,\rho\,Q_1 + \bar{u}_2\,\rho\,Q_2 \\ &= \rho\,Q\,(\bar{u}_2 - \bar{u}_1) \\ &= \rho\,\bar{u}_1\,y_1\,b\,(\bar{u}_2 - \bar{u}_1)\ . \end{aligned} \tag{10.43}$$

Once again, the second step arises from the fact that the flow is assumed steady, so $\partial/\partial t = 0$, while the third step is a consequence of the constant volume flow rate Q that we have assumed for open–channel flows. In the last step, we have simply substituted $Q = \bar{u}_1 y_1 b$

Unlike the case for constant depth flow, $\bar{u}_1 \neq \bar{u}_2$, so the momentum terms do not cancel. As for the applied forces, there are no shear or body (gravitational) components, as in the case of constant depth channel flow in Eq. (10.31). Moreover, the surface forces due to pressure at the inlet and outlet boundaries do *not* cancel one another (as they did in Eq. 10.31) because the surface elevations are not the same. Therefore, the sum of the forces is simply made up of the pressure terms

$$\Sigma F_x = F_1 - F_2 . \tag{10.44}$$

At both of these locations the flow is uniform and steady, so that we can once again assume hydrostatic pressure

$$F_1 = P_1 A_1 = \left(\frac{\rho\, g\, y_1}{2}\right)(y_1\, b) = \frac{\rho\, g\, y_1^2\, b}{2} ,$$

with a similar expression for F_2.

Substituting Eq. (10.44) with the appropriate expressions for the forces into Eq. (10.43), we find the momentum equation takes the form

$$\begin{aligned} \frac{\rho\, g\, y_1^2\, b}{2} - \frac{\rho\, g\, y_2^2\, b}{2} &= \rho\, \bar{u}_1\, y_1\, b\, (\bar{u}_2 - \bar{u}_1) \\ \frac{g\, y_1^2}{2} - \frac{g\, y_2^2}{2} &= \bar{u}_1\, y_1\, (\bar{u}_2 - \bar{u}_1) \\ \frac{y_1^2}{2} - \frac{y_2^2}{2} &= \frac{\bar{u}_1\, y_1}{g}\, (\bar{u}_2 - \bar{u}_1) \qquad (10.45) \end{aligned}$$

We also can still invoke conservation of mass as an independent equation

$$\bar{u}_1\, y_1\, b = \bar{u}_2\, y_2\, b = Q . \tag{10.46}$$

Lastly, we can invoke energy considerations in the form of the extended Bernoulli equation introduced in Eq. (5.34) on page 55

$$\frac{P_{1,bot}}{\rho\, g} + \frac{\bar{u}_1^2}{2\, g} + z_1 = \frac{P_{2,bot}}{\rho\, g} + \frac{\bar{u}_2^2}{2\, g} + z_2 + h_L , \tag{10.47}$$

where we use the "bottom" subscript on pressure, so as not to confuse this with the expressions for pressure in Eq. (10.44). Specifically, we are writing the equation along the straight streamline that coincides with the channel bed. Thus,

$$P_{1,bot} = \rho\, g\, y_1 \qquad \text{and} \qquad P_{2,bot} = \rho\, g\, y_2 .$$

Also, there is no change in z over the horizontal channel bed, so that we find

$$y_1 + \frac{\bar{u}_1^2}{2\, g} = y_2 + \frac{\bar{u}_2^2}{2\, g} + h_L . \tag{10.48}$$

Here, the head loss term h_L accounts for viscous losses that occur in the jump itself, but does not account for losses arising from wall shear stress. Recall, τ_w was neglected under one of our original assumptions.

We now have as governing equations Eq. (10.45), (10.46), and (10.48). They are a set of non–linear algebraic equations, so we suspect that there might be more than one admissible solution. In fact, the trivial solution is rather obvious: $y_1 = y_2$, $\bar{u}_1 = \bar{u}_2$, and $h_L = 0$. To obtain a non–trivial solution, we perform the following algebraic manipulations, which begin with using Eq. (10.46) in the form of $\bar{u}_2 = \bar{u}_1 y_1 / y_2$ to eliminate $\bar{u}_2$ in Eq. (10.45):

$$\begin{aligned} \frac{y_1^2}{2} - \frac{y_2^2}{2} &= \frac{\bar{u}_1\, y_1}{g}\left(\frac{\bar{u}_1\, y_1}{y_2} - \bar{u}_1\right) \\ &= \frac{\bar{u}_1^2\, y_1}{g}\left(\frac{y_1}{y_2} - 1\right) \\ &= \frac{\bar{u}_1^2\, y_1}{g\, y_2}\Big(y_1 - y_2\Big)\,. \end{aligned}$$

Noting that the left hand side can be factored into $(y_1 + y_2)(y_1 - y_2)/2$, we can cancel a factor of $(y_1 - y_2)$ to obtain the equation[10.13]

$$\frac{y_1 + y_2}{2} = \frac{\bar{u}_1^2\, y_1}{g\, y_2}\,.$$

Now multiply by a factor of $2y_2/y_1^2$ to obtain

$$\frac{y_2}{y_1} + \left(\frac{y_2}{y_1}\right)^2 = \frac{2\,\bar{u}_1^2}{g\, y_1}\,. \tag{10.49}$$

Let us make a few observations to assist us in solving Eq. (10.49). First, we see the right hand side contains the square of the upstream Froude number, which we can write as

$$Fr_1 = \frac{\bar{u}_1}{\sqrt{g\, y_1}}\,.$$

Second, if we make the change of variables $\xi = y_2/y_1$, it is clear that Eq. (10.49) can be put into the standard form

$$\xi^2 + \xi - 2\,Fr = 0$$

and solved by the quadratic formula[10.14] to obtain

$$\xi = \frac{-1 \pm \sqrt{1 + 8\,Fr_1^2}}{2}\,. \tag{10.50}$$

[10.13] Notice that we are cognizant of the fact that we are pursuing a non–trivial solution, so that $y_1 \neq y_2$. Thus, we are not removing "a factor of zero" from the equation.

[10.14] When an equation is written in the standard form of

$$a\,\xi^2 + b\,\xi + c = 0\,,$$

the quadratic formula gives the solution

$$\xi = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\,.$$

Let us examine Eq. (10.50) carefully. The Froude number is always positive, based on a simple observation of its definition. Therefore, of the two solutions in Eq. (10.50), the one with the minus sign will always make the right hand side, as a whole, negative. This violates the concept of the hydraulic jump, i.e. the downstream surface must be *higher* than the upstream one. Therefore, the only physically plausible solution is the one with the plus sign. Restoring our original variables, we thus find

$$\frac{y_2}{y_1} = \frac{-1+\sqrt{1+8\,Fr_1^2}}{2}\,. \tag{10.51}$$

Eq. (10.51) gives the change in height as a function of the upstream Froude number.

THEOREM 10.1 (Existence of Hydraulic Jumps). *Hydraulic jumps can only exist if the upstream Froude number exceeds one.*

PROOF. The condition $y_2 > y_1$ allows us to write Eq. (10.51) in the form

$$\frac{-1+\sqrt{1+8\,Fr_1^2}}{2} > 1\,.$$

Solving this equation yields $Fr > 1$. □

Therefore, it is a condition that the upstream flow must be super–critical for a hydraulic jump to exist. We can also re–arrange Eq. (10.48) to obtain the head loss in dimensionless form by dividing by y_1 to obtain:

$$\begin{aligned} 1 + \frac{\bar{u}_1^2}{2\,g\,y_1} &= \frac{y_2}{y_1} + \frac{\bar{u}_2^2}{2\,g\,y_1} + \frac{h_L}{y_1} \\ \frac{h_L}{y_1} &= 1 + \frac{Fr_1^2}{2} - \frac{y_2}{y_1} - \frac{\bar{u}_2^2}{2\,g\,y_1}\,. \end{aligned}$$

From the conservation of mass statement of Eq. (10.46), we find

$$\bar{u}_2 = \bar{u}_1^2\left(\frac{y_1}{y_2}\right)^2\,,$$

which we substitute into the above expression to obtain

$$\begin{aligned} \frac{h_L}{y_1} &= 1 - \frac{y_2}{y_1} + \frac{Fr_1^2}{2} - \frac{\bar{u}_1^2}{2\,g\,y_1}\left(\frac{y_1}{y_2}\right)^2 \\ &= 1 - \frac{y_2}{y_1} + \frac{Fr_1^2}{2} - \frac{Fr_1^2}{2}\left(\frac{y_1}{y_2}\right)^2 \end{aligned}$$

$$\frac{h_L}{y_1} = 1 - \frac{y_2}{y_1} + \frac{Fr_1^2}{2}\left[1 - \left(\frac{y_1}{y_2}\right)^2\right]\,. \tag{10.52}$$

We see that the energy dissipated is a function of the upstream Froude number and the jump ratio, as given by Eq. (10.51). The form of this expression makes it somewhat difficult to visualize. It is therefore helpful to plot both Eqs. (10.51) and (10.52), as has been done in Fig. 10.9. While the

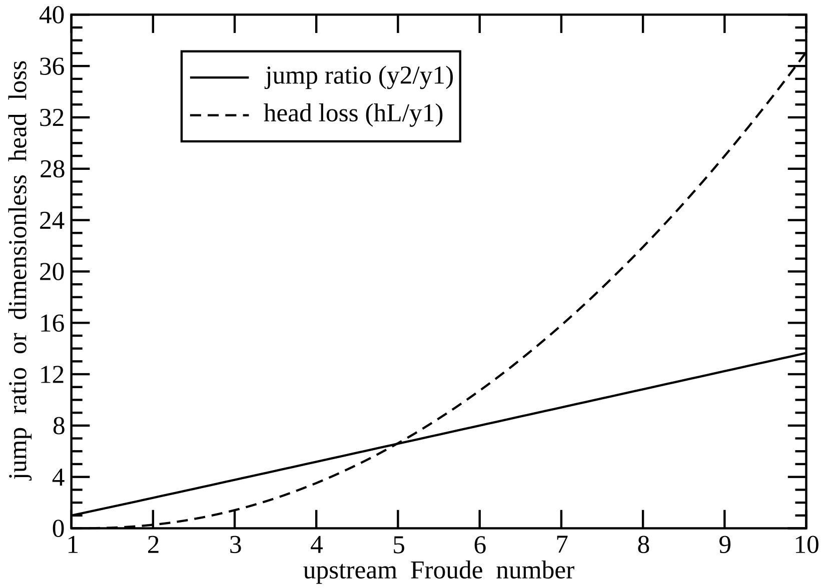

FIGURE 10.9. *Hydraulic jump characteristics.*

jump ratio increases essentially linearly with the upstream Froude number, the head loss increases much more dramatically. Hydraulic jumps are clearly very dissipative in nature.

APPENDIX A

Mohr's Circle Analog to Pascal's Law

A.1. Transformation Equations of Plane Stress

In solid mechanics, the problem of stress transformation is typically cast according to a differential element under general plane stress (Popov, 1976), e.g. as shown in Figure A.1. We wish to determine the stresses realized

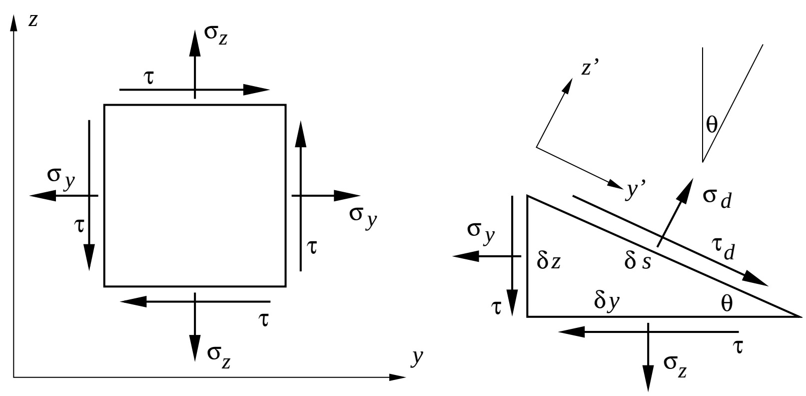

FIGURE A.1. *General plane stress on a differential element (left), and transformation to an orientation θ (right)*

along a diagonal face, which is oriented at an arbitrary angle θ, i.e. the normal stress σ_d and the shear stress τ_d.

For static equilibrium, the forces must sum to zero in each coordinate direction. Normally, we would write an equation for each of the y and z directions, i.e. $\Sigma F_y = 0$ and $\Sigma F_z = 0$. However, in that case, σ_d and τ_d would appear in each equation and would not be readily separable. Rather, we write static equilibrium with respect to the transformed coordinate system (y', z'), so that σ_d and τ_d are identically separate. Each stress must be multiplied by the area over which it acts to obtain the resulting force. Assuming a unit depth, we can write in the y' direction

$$\begin{aligned}\tau_d \cdot \delta s \cdot 1 \quad &= \quad \tau \cdot \delta y \cdot 1 \cdot \cos\theta \; - \\ &\quad \sigma_z \cdot \delta y \cdot 1 \cdot \sin\theta \; - \; \tau \cdot \delta z \cdot 1 \cdot \sin\theta \; + \; \sigma_y \cdot \delta z \cdot 1 \cdot \cos\theta \,.\end{aligned}$$

From geometry, we have $\delta y = \delta s \cos\theta$ and $\delta z = \delta s \sin\theta$. Using these results and simplifying, we obtain

$$\begin{aligned}
\tau_d\,\delta s &= \tau\,\delta s\,\cos^2\theta - \sigma_z\,\delta s\,\cos\theta\,\sin\theta - \tau\,\delta s\,\sin^2\theta + \sigma_y\,\delta s\,\cos\theta\,\sin\theta \\
&= \tau\,\delta s\left(\cos^2\theta - \sin^2\theta\right) + \delta s\,\cos\theta\,\sin\theta\,(\sigma_y - \sigma_z) \\
&= \tau\,\delta s\,\cos 2\theta + \delta s\,\sin 2\theta\,\frac{\sigma_y - \sigma_z}{2}\,.
\end{aligned}$$

The last line results from trigonometric identities (Beyer, 1984). Canceling δs, we finally write

$$\tau_d = \tau\,\cos 2\theta - \sin 2\theta\,\frac{\sigma_z - \sigma_y}{2}\,. \tag{A.1}$$

For the z' direction, we have

$$\begin{aligned}
\sigma_d\cdot\delta s\cdot 1 &= \tau\cdot\delta y\cdot 1\cdot\sin\theta + \\
&\quad \sigma_z\cdot\delta y\cdot 1\cdot\cos\theta + \tau\cdot\delta z\cdot 1\cdot\cos\theta + \sigma_y\cdot\delta z\cdot 1\cdot\sin\theta\,.
\end{aligned}$$

Again, using geometric relations and simplifying, we find

$$\begin{aligned}
\sigma_d\,\delta s &= \tau\,\delta s\,\cos\theta\,\sin\theta + \sigma_z\,\delta s\,\cos^2\theta + \tau\,\delta s\,\cos\theta\,\sin\theta + \sigma_y\,\delta s\,\sin^2\theta \\
&= 2\,\tau\,\delta s\,\cos\theta\,\sin\theta + \sigma_z\,\delta s\,\cos^2\theta + \sigma_y\,\delta s\,\sin^2\theta \\
&= \tau\,\delta s\,\sin 2\theta + \sigma_z\,\delta s\,\frac{1 + \cos 2\theta}{2} + \sigma_y\,\delta s\,\frac{1 - \cos 2\theta}{2}\,.
\end{aligned}$$

Similar to the derivation for τ_d, the last line results from trigonometric identities (Beyer, 1984). Canceling δs and rearranging terms, the result is

$$\sigma_d = \tau\,\sin 2\theta + \frac{\sigma_z + \sigma_y}{2} + \cos 2\theta\,\frac{\sigma_z - \sigma_y}{2}\,. \tag{A.2}$$

A.2. Mohr's Circle

Eqs. (A.1) and (A.2) represent a circle written in parametric form. This can be shown as follows. First, cast the equations as

$$\tau_d = \tau\,\cos 2\theta - \sin 2\theta\,\frac{\sigma_z - \sigma_y}{2}$$

and

$$\sigma_d - \frac{\sigma_z + \sigma_y}{2} = \tau\,\sin 2\theta + \cos 2\theta\,\frac{\sigma_z - \sigma_y}{2}\,.$$

Now, square both equations, so that

$$\begin{aligned}
\tau_d^2 &= \left(\tau\,\cos 2\theta - \sin 2\theta\,\frac{\sigma_z - \sigma_y}{2}\right)^2 \\
&= \tau^2\,\cos^2 2\theta - 2\,\tau\,\cos 2\theta\,\sin 2\theta\,\frac{\sigma_z - \sigma_y}{2} \\
&\quad + \sin^2 2\theta\,\left(\frac{\sigma_z - \sigma_y}{2}\right)^2
\end{aligned} \tag{A.3}$$

and

$$\left(\sigma_d - \frac{\sigma_z + \sigma_y}{2}\right)^2 = \left(\tau \sin 2\theta + \cos 2\theta \, \frac{\sigma_z - \sigma_y}{2}\right)^2$$

$$= \tau^2 \sin^2 2\theta + 2\,\tau \sin 2\theta \cos 2\theta \, \frac{\sigma_z - \sigma_y}{2}$$

$$+ \cos^2 2\theta \left(\frac{\sigma_z - \sigma_y}{2}\right)^2 . \tag{A.4}$$

Lastly, add Eqs. (A.3) and (A.4) to obtain

$$\tau_d^2 + \left(\sigma_d - \frac{\sigma_z + \sigma_y}{2}\right)^2 = \tau^2 + \left(\frac{\sigma_z - \sigma_y}{2}\right)^2 . \tag{A.5}$$

Eq. (A.5) is Mohr's Circle of stress. In a given configuration, σ_y, σ_z, and τ are known. Eq. (A.5) permits straightforward calculations to determine the normal and shear stresses, σ_d and τ_d respectively, realized on any arbitrarily oriented plane. We can recast Eq. (A.5) into a more familiar geometric form as

$$(x - a)^2 + y^2 = b^2 ,$$

where $\sigma_d \to x$ and $\tau_d \to y$ define points on the circle, which is centered at $(x, y) = (a, 0)$ and has a radius b. This equation can be plotted as a circle whose properties describe the state of stress for any orientation (Popov, 1976).

A.3. Analog to Pascal's Law

From our discussion of fluid statics in Chapter 2, we recall that there are no shear stresses. Not only is $\tau = 0$, but the shear stress on any arbitrary orientation must also vanish, i.e. $\tau_d = 0$. Because τ_d is one of the two independent variables describing Mohr's Circle, $\tau_d \to 0$ implies the circle collapses to a point having a radius of zero. The radius for $\tau = 0$ reduces to

$$b = \sqrt{\tau^2 + \left(\frac{\sigma_z - \sigma_y}{2}\right)^2} = \sqrt{\left(\frac{\sigma_z - \sigma_y}{2}\right)^2} = \frac{\sigma_z - \sigma_y}{2} = 0 .$$

From this, we immediately deduce $\sigma_z = \sigma_y$.

Furthermore, invoking the vanishing shear requirement for Eq. (A.5), we find

$$0 + \left(\sigma_d - \frac{\sigma_z + \sigma_y}{2}\right)^2 = 0 + \left(\frac{\sigma_z - \sigma_y}{2}\right)^2 .$$

According to $\sigma_z = \sigma_y$, we can further deduce $\sigma_d = \sigma_z = \sigma_y$. Therefore, all normal stresses will be equal, regardless of the orientation of the surface on which the stress is applied. For the case of fluid statics, normal stress represents the fluid static pressure, so pressure at a point is uniform, regardless of direction. This is an independent proof of Pascal's Theorem given in Chapter 2 because we did not have to invoke the concept shrinking our elemental volume to zero.

APPENDIX B

Flow Meter Relationships

B.1. Venturi Meter

Here, we show the derivation of Eq. (3.40) on pp. 34. First, solve the volume flow rate equation, Eq. (3.39), for v_1 as $v_1 = v_2 A_2 / A_1$, so that

$$v_1^2 \;=\; \frac{A_2^2}{A_1^2}\, v_2^2 \,.$$

Now substitute into the Bernoulli equation, Eq. (3.38), to obtain

$$P_1 \;+\; \frac{1}{2}\,\rho\,\frac{A_2^2}{A_1^2}\,v_2^2 \;=\; P_2 \;+\; \frac{1}{2}\,\rho\,v_2^2 \,.$$

Grouping terms, we find

$$P_1 \;-\; P_2 \;=\; \frac{1}{2}\,\rho\,v_2^2\,\left(1 \;-\; \frac{A_2^2}{A_1^2}\right)\,,$$

from which we may solve for v_2 as

$$v_2 \;=\; \sqrt{\frac{2\,(P_1 \;-\; P_2)}{\rho(1 - A_2^2/A_1^2)}}\,.$$

Substituting this directly into Eq. (3.39) yields Eq. (3.40).

B.2. Sluice Gate

Deriving Eq. (3.42) on pp. 35 proceeds along similar lines. First, solve the volume flow rate equation, Eq. (3.41), for v_1 as $v_1 = v_2 z_2 / z_1$, so that

$$v_1^2 \;=\; \frac{z_2^2}{z_1^2}\, v_2^2 \,.$$

In the Bernoulli equation, Eq. (3.28), we find that the pressures cancel, so that we can write

$$\frac{1}{2}\,\rho\,v_1^2 \;+\; \gamma\,z_1 \;=\; \frac{1}{2}\,\rho\,v_2^2 \;+\; \gamma\,z_2 \,.$$

Substituting the expression for v_1^2, we obtain

$$\frac{1}{2}\,\rho\,\frac{z_2^2}{z_1^2}\,v_2^2 \;+\; \gamma\,z_1 \;=\; \frac{1}{2}\,\rho\,v_2^2 \;+\; \gamma\,z_2 \,.$$

Grouping terms yields

$$\gamma\,(z_1 \;-\; z_2) \;=\; \frac{1}{2}\,\rho\,v_2^2\,\left(1 \;-\; \frac{z_2^2}{z_1^2}\right)\,,$$

from which we may solve for v_2 as

$$v_2 \;=\; \sqrt{\frac{2\,\gamma\,(z_1 \;-\; z_2)}{\rho(1 - z_2^2/z_1^2)}}\,.$$

Since density is embedded in the specific weight, we factor and cancel ρ to finally obtain

$$v_2 \;=\; \sqrt{\frac{2\,g\,(z_1 \;-\; z_2)}{1 - z_2^2/z_1^2}}\,.$$

Substituting this directly into Eq. (3.41) yields Eq. (3.42).

APPENDIX C

Document History

Version	Year	Remarks
1.0	2002	initial implementation
1.1	2003	added figures; improved treatment of Reynolds' Transport Theorem; corrected a few typographic mistakes
1.2	2004	changed from GPL public license to Creative Commons license; added open channel flow chapter, subject index, Mohr's Circle analog for Pascal's Law, pressure nomenclature, example suggestions in margins, Archimedes' Principle; better treatment of relative velocity for moving control volumes; physical interpretation of the curl of a vector; added a few figures in potential flow discussion; discussion of physical aspects of viscous fluids
1.21	2006	updated page and problem references to the 5-th edition of the Munson, Young, and Okiishi text; tidied most figures; added footnotes 1.5, 2.9, 2.10, 6.18; a few typographical and typesetting corrections; added some "internal" page references; changed notation of freestream velocity from u_0 to u_∞ in Ch. 9

Epilogue

Although a considerable amount of material has been covered here, we have actually only "scratched the surface". The study of fluid mechanics is not only an indispensable engineering tool, but an active and evolving research area as well. Developments continue to be made in experimental techniques for measurements (e.g. pressure–sensitive paint and laser–based measurement methods), numerical approaches, and fundamental theory. Many sub–topics in fluid mechanics have evolved to become enormous independent research areas on their own, for example transition and turbulence. Moreover, the quantitative application of fluid mechanics results continues to be broadened, for example in civil engineering (structural aerodynamics), medicine & physiology (vascular and pulmonary mechanics), and nanotechnology (small–scale flows). This subject is one of the most challenging areas for engineering experimentalists and theorists, with numerous unsolved problems remaining. Hopefully, this course has stimulated your interest to learn more about fluid mechanics.

About the Author

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