JME 3700
Final Exam: May 10, 2016 **Name:** ____________________

Solutions

1. A sphere of mass m and diameter D is dropped from rest off of a very tall test stand into still air (Fig. 1). The sphere's downward velocity, V, is governed by its weight, W, and its aerodynamic drag force, F_D. Assume that the coefficient of drag, C_D, is *known*. Also, the air's density is ρ and the gravitational acceleration is g. Newton's Second Law, the sum of the forces being equal to rate of change of momentum, can be written for this case as

$$\sum F = W - F_D = m \frac{dV}{dt} \,.$$

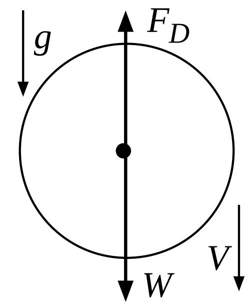

Fig. 1: *Falling sphere*

(a) (10 pts) Derive an expression for the acceleration dV/dt in terms of g, C_D, ρ, V, D, m on the basis of Newton's Second Law. *Do not solve this equation for V!*

Solution: Newton's Second Law must be fleshed–out with expressions for W and F_D. Trivially, $W = m\,g$, and the form of the drag force is

$$F_D = C_D \frac{\rho V^2 A}{2} = C_D \frac{\rho V^2 \pi D^2}{8} \,,$$

where we have calculated the visible area as $A = \pi D^2/4$. Dividing through by m, we find

$$\frac{dV}{dt} = g - \frac{C_D \rho V^2 \pi D^2}{8\,m} \,.$$

(b) (10 pts) Determine the expression for the *terminal velocity* of the sphere, assuming that the test stand is sufficiently tall for this state to be reached.

Solution: Terminal velocity is reached when the acceleration vanishes, whereby

$$\frac{dV}{dt} = g - \frac{C_D \rho V^2 \pi D^2}{8\,m} = 0 \,.$$

Solving for $V = V_T$, we find

$$V_T = \sqrt{\frac{8\,m\,g}{\rho\, C_D\, \pi D^2}} \,.$$

2. The so–called "capillary rise" of a liquid occurs due to the fluid property of surface tension, if the tube diameter is sufficiently small (Fig. 2). Here, the height h to which the liquid rises is a function of the dimensionless contact angle with the surface, θ, fluid properties of density, ρ (units of kg/m^3), and surface tension, γ (units of N/m), gravity, g (units of m/s^2), and tube diamater, D (units of m). That is, the functional dependence can be expressed as $h = h(D, g, \rho, \gamma, \theta)$.

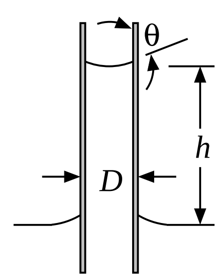

Fig. 2: *Rise.*

(a) (10 pts) Standard dimensionless analysis shows

$$\frac{h}{D} = F\left(\frac{\gamma}{\rho\, g\, D^2}\,, \theta\right)$$

meaning dimensionless rise is some (unknown) function, F, of two other independent dimensionless parameters. If $h = 0.03m$ in a reference case, use this dimensionless statement to assess what the rise would be in an experiment if D and γ are halved, ρ is doubled, and θ remains the same.

Solution: Checking consistency of the dimensionless groups, we have $\theta|_{ref} = \theta|_{exp}$ and

$$\left.\frac{\gamma}{\rho\, g\, D^2}\right|_{ref} = \left.\frac{\gamma/2}{2\,\rho\, g\,(D/2)^2}\right|_{exp} = \left.\frac{\gamma}{\rho\, g\, D^2}\right|_{exp}$$

meaning the independent dimensionless groups are the same between the reference and the experiment, i.e. we have similarity. This implies

$$\frac{h_{ref}}{D_{ref}} = \frac{h_{exp}}{D_{exp}} \qquad \therefore \quad h_{exp} = \frac{D_{exp}}{D_{ref}}\, h_{ref} = \frac{h_{ref}}{2} = \frac{0.03}{2} = 0.015\, m\,.$$

(b) (10 pts) Dimensionless analysis does not give any specific information regarding F. However, because the fluid is static, we can write a simple equation: the force due to the surface tension acting around the circumference, F_T, exactly supports the weight, W, of the fluid that rose in the tube, i.e.

$$\underbrace{\pi\, D \cdot \gamma}_{F_T} \cdot \underbrace{\cos\theta}_{\substack{\text{vertical}\\ \text{component}}} = \underbrace{\underbrace{\overbrace{\frac{\pi\, D^2}{4}}^{\text{volume}} \cdot h \cdot \rho}_{\text{mass}} \cdot g}_{W}$$

From this information, determine F.

Solution: The force balance given in the question is easily rearranged as

$$\frac{h}{D} = \frac{4\,\gamma\,\cos\theta}{\rho\, g\, D^2}\,,$$

the right hand side clearly being F.

3. Constant–depth, gravity–driven flow occurs in a rectangularly–shaped cross–section channel at a design depth of $h = 1\, m$. However, the channel width, x, is yet to be determined (Fig. 3). Assume the Manning equation is a good description of this flow and that the duct is made of corrugated metal having a Manning factor of $n = 0.022\ s/m^{1/3}$. The channel's slope is a constant $S_0 = 0.0014$.

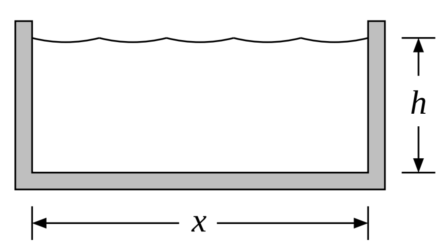

Fig. 3: *Rectangle channel*

(a) (10 pts) Assuming that the channel is to be designed, in this particular case, to minimize flow resistance, show that the width should be chosen as $x = 2h$. *Hint:* Flow rate depends upon hydraulic radius, $R_h = A/P$, meaning that for a given area, the radius is maximized by *minimizing* P. Write P as a function of A and x and minimize it by a standard method from differential calculus involving the first derivative.

Solution: By inspection, we have $A = x \cdot h$ and $P = 2 \cdot h + x$, whereby

$$P = 2 \cdot \frac{A}{x} + x \qquad \text{and} \qquad \frac{dP}{dx} = -\frac{2A}{x^2} + 1 = 0$$

is the condition by which we can find the value of x that results in an extremum. Solving, we find

$$x = \frac{2A}{x} = 2h .$$

The second derivative is $4A/x^3$, which is clearly positive at $x = 2h$, indicating this extremum is a minimum, not a maximum, thus proving the proposition.

(b) (10 pts) Estimate the volumetric flow rate of water for this optimal–flow channel.

Solution: The depth is $h = 1\,m$ and the previous section indicates $x = 2\,m$, whereby the hydraulic radius is

$$R_h = \frac{A}{P} = \frac{x\,h}{2h + x} = \frac{2 \cdot 1}{2 \cdot 1 + 2} = \frac{1}{2} .$$

The Manning formula then gives the estimated flow rate as

$$Q = \frac{A\,R_h^{2/3}\,S_0^{1/2}}{n} = \frac{2 \cdot 0.5^{2/3} \cdot \sqrt{0.0014}}{0.022} \approx 2.14\,m^3/s .$$

4. Steady, fully–developed flow of water occurs in a square cross–section duct of side length $w = 0.01\,m$ at a design average velocity of $\overline{u} = 0.05\,m/s$ (relevant views shown in Fig. 4). The cross–section is constant over the total duct length of $L = 200\,m$ and there is a smooth rise from inlet to outlet of $d = 1\,m$. Gravitational acceleration is $g = 9.8\,m/s^2$ downward.

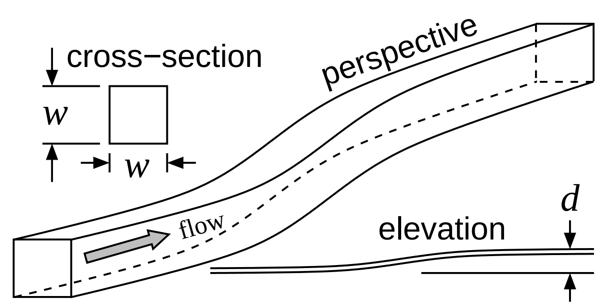

Fig. 4: *Flow in a square duct*

(a) (5 pts) The length scale in this type of problem for all calculations is the so–called hydraulic diameter, D_h. Calculate D_h in units of m.

Solution: From the definition of D_h, we write directly

$$D_h = \frac{4A}{P} = \frac{4w^2}{4w} = w ,$$

meaning that the hydraulic diameter is $D_h = w = 0.01\,m$.

(b) (5 pts) Given the material properties of water: $\nu = 1\times10^{-6} m^2/s$ (kinematic viscosity) and $\rho = 10^3 kg/m^3$ (density), show that the flow is laminar using the appropriate dimensionless argument.

Solution: Here, we use the Reynolds number based on D_h and $\overline{u}$ using consistent physical units, whereby

$$Re \;=\; \frac{\overline{u}\, D_h}{\nu} \;=\; \frac{0.05 \cdot 0.01}{1 \times 10^{-6}} \;=\; 500\,.$$

Since this is well below the critical value of around 2100, the flow is indeed observed to be laminar.

(c) (10 pts) Suppose the flow is sustained by a pump (not depicted in Fig. 4) and that we examine this flow using the generalized Bernoulli equation

$$\frac{P_1}{\rho\, g} \;+\; \frac{\overline{u}_1^2}{2\, g} \;+\; z_1 \;+\; h_S \;-\; h_L \;=\; \frac{P_2}{\rho\, g} \;+\; \frac{\overline{u}_2^2}{2\, g} \;+\; z_2\,,$$

where all terms have conventional meaning and where the labels "1" and "2" indicate inlet and outlet, respectively. Assuming conditions are such that the static pressures at the inlet and outlet are identical, simplify this equation to obtain the basic symbolic expression for the shaft work, h_S. (You will solve for its numerical value below.)

Solution: Pressure terms cancel one another because $P_1 = P_2$ (stated in the question) and kinetic energy terms cancel because the constant cross–section implies (by conservation of mass) that $\overline{u}_1 = \overline{u}_2$. This leaves

$$h_S \;=\; h_L \;+\; z_2 \;-\; z_1 \;=\; h_L \;+\; d\,,$$

since $z_2 - z_1$ is the height d to which the flow is elevated at the outlet.

(d) (10 pts) As stated above, the duct's rise is very gentle, implying the viscous losses are due entirely to shear stress between the fluid and the duct (i.e. losses are of the "major" type). Given a dimensionless parameter called the *aspect ratio*, ϕ, of the cross–section (i.e. the ratio of height to width), the Darcy friction factor for laminar flows in non–circular ducts of roughly this type can be approximated as

$$f \;=\; \frac{26.4\,\phi^2 \;-\; 50.2\,\phi \;+\; 80.7}{Re}\,,$$

where Re is the Reynolds number. Combine this information with appropriate theory to quantify your expression in the previous question, ultimately reporting the value of h_S in units of m.

Solution: Given that $\phi = 1$ for this case, we can calculate the friction factor directly as

$$f \;=\; \frac{26.4 \cdot 1^2 \;-\; 50.2 \cdot 1 \;+\; 80.7}{500} \;\approx\; 0.114\,.$$

Then, using the Darcy–Weissbach equation, we find the loss term to be

$$h_L \;=\; f\,\frac{L}{D_h}\,\frac{\overline{u}^2}{2\,g} \;=\; 0.114 \cdot \frac{200}{0.01} \cdot \frac{0.05^2}{2 \cdot 9.8} \;\approx\; 0.291\, m\,.$$

Based on the previous expression, we immediately see that

$$h_S \;=\; h_L \,+\, d \;=\; 0.291 \,+\, 1 \;=\; 1.291\, m\,.$$

(e) (10 pts) In the form of the Bernoulli equation used here, h_S is the energy per unit weight of the fluid required to sustain the flow, i.e. in units of $N \cdot m$ per N. Use this observation to calculate the power consumption, $\dot{W}$, of the pump in units of J/s, i.e. Watts.

Solution: The power is h_S multiplied by the weight–based flow rate, i.e.

$$\dot{W} \;=\; g\, \dot{m} \cdot h_S \;=\; g\, \rho\, \overline{u}\, A \cdot h_S \;=\; 9.8 \cdot 1000 \cdot 0.05 \cdot 0.01^2 \cdot 1.291 \;\approx\; 0.063\, W\,.$$