JME 3700
Midterm Exam: January 3, 2017 **Name:** ______________________

Solutions

- Questions having bold–faced point values in the solution indicate the basis of how partial–credit was tallied.

1. A spherical buoy of radius R is held by a strong rope such that it rests equally between 2 non–mixing liquids, numbered 1 and 2 (Fig. 1). That is, the sphere's equator is located exactly at the liquid interface. These liquids have densities ρ and $2\,\rho$, respectively, while the buoy is hollow. Its shell is very thin (such that its volume is $4\,\pi\,R^3/3$) and it is of negligible weight. Gravitational acceleration is g.

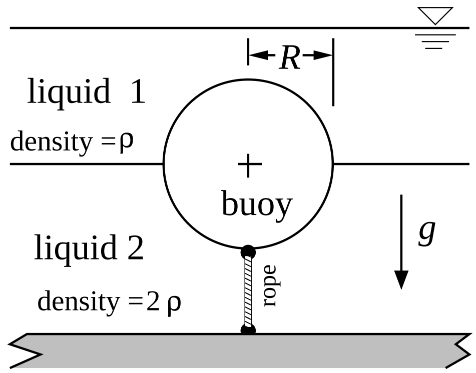

Fig. 1: *Spherical buoy*

(a) (10 pts) Determine the tension T in the rope in terms of R, ρ, and g.

Solution: Buoyancy force, call it F, has two components, one associated with half the sphere volume displacing fluid 1 and the other associated with half the sphere volume displacing fluid 2. Therefore

$$F \;=\; \frac{4\,\pi\,R^3/3}{2}\cdot\rho\,g \;+\; \frac{4\,\pi\,R^3/3}{2}\cdot 2\,\rho\,g \;=\; 2\,\pi\,R^3\,\rho\,g\,.$$

There would ordinarily be 3 relevant forces, the downward weight of the buoy, W, the upward buoyancy force, F, and the downward force of the cable tension, T. However, in this particular case, we are supposed to neglect the weight of the buoy because it is a hollow, thin shell. Therefore, static equilibrium dictates simply that $T = F$, which means

$$T \;=\; 2\,\pi\,R^3\,\rho\,g\,.$$

(b) (10 pts) Now assume that the buoy is completely filled–up with liquid from region 1, i.e. the liquid whose density is ρ. Determine the new tension in the rope.

Solution: There are two ways to approach this problem, which of course yield the same answer.

1. observe that the buoy still displaces the same configuration of the two liquids, so the buoyancy force is the same, but the buoy itself now has weight represented by its contents of liquid 1
2. observe that the top half of the buoy is "neutral", i.e. the buoyancy and weight of the top half of the sphere cancel one another, leaving only the bottom half of the sphere to consider

For the first method, static equilibrium dictates $T = F - W$, where $F = 2\,\pi\,R^3\,\rho\,g$ from above. Filled with liquid 1, the sphere now has a weight $W = 4\,\pi\,R^3\,\rho\,g/3$, whereby

$$T \;=\; 2\,\pi\,R^3\,\rho\,g \;-\; \frac{4\,\pi\,R^3\,\rho\,g}{3} \;=\; \frac{2\,\pi\,R^3\,\rho\,g}{3}\,,$$

i.e. the tension is one–third of its value of the previous question. In the second method, the top half of the sphere plays no role. The bottom half is associated with the buoyancy force of $0.5 \times 4\,\pi\,R^3/3 \times 2\,\rho\,g$ (as above), but has a weight of $0.5 \times 4\,\pi\,R^3/3 \times \rho\,g$. The difference is $2\,\pi\,R^3\,\rho\,g/3$.

2. A gate of rectangular cross section (height w and width d "into the paper") holds back a static pool of water having a density ρ (Fig. 2). The gate is hinged at point B at the bottom with a frictionless pin, while the top at point C is acted upon by a compression spring, having a spring constant k, which pushes in the counter–clockwise direction. Gravitational acceleration is g. The water level is exactly at the elevation of the top of the gate.

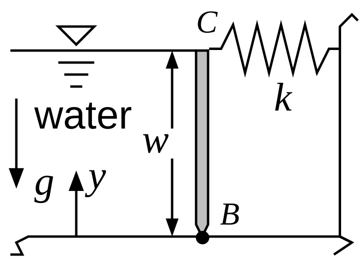

Fig. 2: *Water gate*

(a) (10 pts) Determine the magnitude of the hydrostatic force, $|F_h|$, on the gate exerted by the water as a function of ρ, g, w, and d.

Solution: The area of the gate is $A = w \cdot d$. The magnitude of the force is simply the area times the average pressure, i.e. the magnitude of the pressure at the centroid. Since the centroid is $h_c = w/2$ below the surface, we find

$$|F_h| \;=\; \rho\,g\,h_c\,A \;=\; \rho\,g\,\frac{w}{2}\,w\,d \;=\; \frac{\rho\,g\,w^2\,d}{2}\,.$$

(b) (10 pts) Calculate the location at which the resultant force acts as a function of w. Give your answer as a length measured vertically from the hinge point to the line–of–action of the force.

Solution: This particular configuraiton is one of the few where the line of action can be determined by inspection. However, let us apply standard theory to calculate it formally, i.e.

$$y_r \;=\; \frac{I_{xc}}{y_c\,A} \;+\; y_c\,,$$

where y_c is the location of the gate centroid measured from the surface *along an axis coincident with the gate* and I_{xc} is the area moment of inertia about the axis normal to the page. Here, $y_c = h_c = w/2$, since the gate is vertical, and $I_{xc} = d\,w^3/12$, which gives

$$y_r \;=\; \frac{\dfrac{d\,w^3}{12}}{\dfrac{w}{2}\,w\,d} \;+\; \frac{w}{2} \;=\; \frac{w}{6} \;+\; \frac{w}{2} \;=\; \frac{2\,w}{3}$$

as measured from the surface. Therefore, if measured from the hinge, the point of application is

$$w - y_r = w - \frac{2\,w}{3} = \frac{w}{3}\,.$$

(c) (10 pts) Calculate the force imparted by the spring, $|F_s|$, that is necessary to maintain this configuration in a state of static equilibrium as a function of ρ, g, w, and d.

Solution: Here we simply sum moments about the hinge. Because the pin is frictionless, static equilibrium is a balance between the hydrostatic and the spring torques, i.e.

$$|F_s| \cdot w = |F_h| \cdot \frac{w}{3}$$

$$\therefore\ |F_s| = \frac{|F_h|}{3} = \frac{\rho\, g\, w^2\, d}{6}\,.$$

3. A special fluid device takes a uniform inflow stream and converts it to an output stream having a linear flow profile (Fig. 3). The inlet is a square cross section of edge length h_0, while the outlet is a square cross section of edge length h_1. The inlet velocity profile is u_0 (a constant), while the outlet profile is $u_1 = C_1 y$, where C_1 is also a constant.

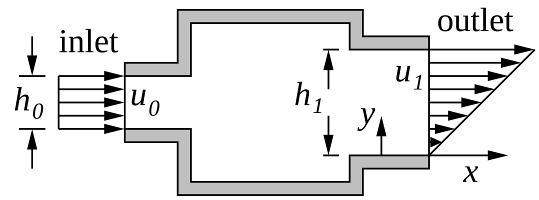

Fig. 3: *Fluid velocity conversion device*

(a) (10 pts) Given that this flow is incompressible, determine u_0 as a function of h_0, h_1, and C_1 such that mass is conserved.

Solution: We apply the conservation of mass (continuity) equation, which for this problem we can write simply as

$$\iint_{area} \mathbf{V} \cdot \mathbf{n}\, dA = 0\,.$$

Mass only crosses at the inlet and outlet. Given that the outward unit normal at the inlet is in the negative x direction, while the outward unit normal at the outlet is in the positive x direction, and given that the inlet velocity is uniform, we can write conservation of mass as

$$-\,u_0\, h_0^2 + \int_0^{h_1} C_1\, y\, \underbrace{h_1\, dy}_{dA} = 0.$$

Note that we have used a differential area for the outlet integral of $h_1\, dy$. We then find

$$u_0\, h_0^2 = C_1\, h_1 \int_0^{h_1} y\, dy = C_1\, h_1 \left(\frac{y^2}{2}\right)\Bigg|_0^{h_1} = C_1\, h_1 \left(\frac{h_1^2}{2}\right) = C_1\, \frac{h_1^3}{2}$$

$$\therefore\ u_0 = \frac{C_1\, h_1^3}{2\, h_0^2}\,.$$

(b) (10 pts) Determine the associated physical units of the constant C_1.

Solution: According to $u_1 = C_1 y$ in the opening statement, C_1 must have units of sec^{-1}, so that u_0 would be length per unit time.

4. (10 pts) You are designing a manometer system to measure the difference in level in two water tanks very accurately. The prototype is shown in Fig. 4 and specifications call for a 10–fold magnification. That is, for every unit of vertical difference h between the actual water levels of the two tanks, the manometer should show $10h$ units of displacement. Calculate the specific gravity S_g of the manometer fluid (shaded) that gives such an amplification. Take the water density to be ρ.

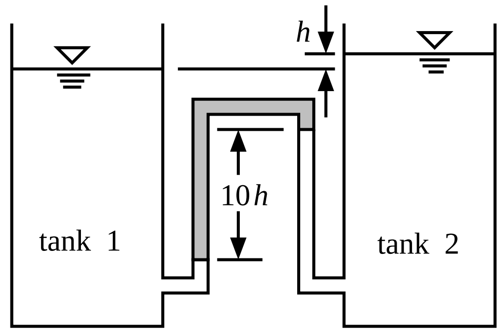

Fig. 4: *Double tank configuration*

Solution: We use the standard procedure of marching through the system, keeping track of vertical changes of pressure. The surface in each tank is at atmospheric pressure, P_{atm}. Let y be the distance from the surface in tank 1 to the left–side manometer level. Starting in tank 1, we then have

$$\begin{aligned} P_{atm} + \rho g y - S_g \rho g 10 h + \rho g 10 h - \rho g y - \rho g h &= P_{atm} \\ - S_g \rho g 10 h + \rho g 10 h - \rho g h &= 0 \\ 9 \rho g h &= S_g \rho g 10 h \\ 9 &= 10 S_g , \end{aligned}$$

from which we conclude $S_g = 0.9$.

5. Fluid of constant density ρ flows inviscidly and steadily in a horizontal pipe of diameter D_1 shown in Fig. 5 with a uniform velocity V_1 and pressure P_1. The flow meets a smoothly and gradually necked–down constriction of diameter D_2, where the uniform velocity and pressure are V_2 and P_2, respectively.

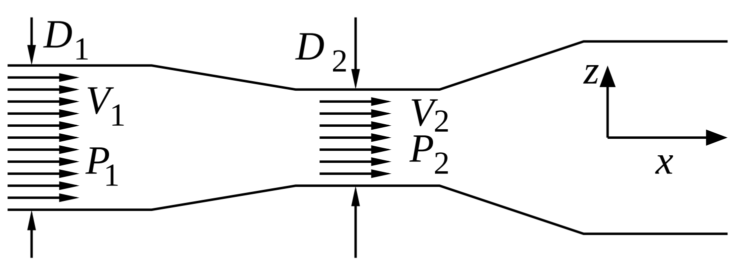

Fig. 5: *Pipe flow with a "neck–down" constriction*

(a) (10 pts) Taking the the ratio of the pipe diameters as $\beta = D_2/D_1$, determine V_1 as a function of V_2 and β, according to a suitable conservation principle.

Solution: The principle of conservation of mass between the two sections dictates

$$V_1\, A_1 \;=\; V_2\, A_2\,,$$

which can be written in terms of the diameters as

$$V_1\, \frac{\pi\, D_1^2}{4} \;=\; V_2\, \frac{\pi\, D_2^2}{4}\,.$$

Solving for V_1, we find

$$V_1 \;=\; V_2\, \left(\frac{D_2}{D_1}\right)^2 \;=\; V_2\, \beta^2\,.$$

(b) (10 pts) Suppose we can determine pressures P_1 and P_2 by some independent means, i.e. that we can directly measure the pressure difference $\Delta P = P_2 - P_1$. If the motion of fluid particles is predominantly along streamlines, calculate V_2, the throat velocity, in terms of the fluid density ρ, the pressure difference, ΔP, and the geometric factor β. That is, your answer should be a function having the form $V_2 = V_2\,(\rho, \Delta P, \beta)$.

Solution: The flow adheres to the 4 conditions that enable application of the Bernoulli equation. We apply the basic verison of the equation along the center streamline between the upstream section and the constricted section

$$\frac{P_1}{\rho} \;+\; \frac{V_1^2}{2} \;+\; g\, z_1 \;=\; \frac{P_2}{\rho} \;+\; \frac{V_2^2}{2} \;+\; g\, z_2\,.$$

Observing that the pipe is horizontal, so that $z_1 = z_2$, we solve for V_2 to obtain

$$V_2^2 \;=\; V_1^2 \;-\; \frac{2\,\Delta P}{\rho}$$

and then use $V_1 = V_2\,\beta^2$ from above to find

$$V_2^2 \;=\; \left(V_2\,\beta^2\right)^2 \;-\; \frac{2\,\Delta P}{\rho} \;=\; V_2^2\,\beta^4 \;-\; \frac{2\,\Delta P}{\rho}\,,$$

which can be simplified as

$$V_2^2\left(1 \;-\; \beta^4\right) \;=\; -\,\frac{2\,\Delta P}{\rho}\,.$$

This yields

$$V_2^2 \;=\; \frac{2\,\Delta P}{\rho\,(\beta^4 \;-\; 1)} \qquad \text{or} \qquad V_2 \;=\; \sqrt{\frac{2\,\Delta P}{\rho\,(\beta^4 \;-\; 1)}}\,.$$