JME 3700
April 3, 2017

Homework #10 Solutions

1. (10 pts) A pump increases the pressure in a certain piping system. The pressure rise across this pump, ΔP, depends upon the fluid density ρ, the volume flow rate Q, and the impeller diameter and rotation rate, D and ω, respectively. Derive a set of dimensionless parameters for this configuration, i.e. where dimensionless pressure rise is a function of other dimensionless number(s).

Solution: In functional form, the dependence is $\Delta P = \Delta P\,(D, \rho, \omega, Q)$. There are $k = 5$ variables and the dimensional components of these variables are

	ΔP	D	ρ	ω	Q
M	1	0	1	0	0
L	-1	1	-3	0	3
T	-2	0	0	-1	-1

It is clear that the determinant of the sub–matrix defined by columns ρ, ω, and Q does *not* vanish, therefore the rank of this system is 3. Consequently, there are $r = 3$ repeating variables and we can conveniently pick these three. We now write a Bridgman equation for each candidate variable. For ΔP we have

$$\Pi_1 \text{ for } \Delta P: \quad M^0\,L^0\,T^0 = \underbrace{\left(M^1L^{-1}T^{-2}\right)}_{\Delta P} \times \underbrace{\underbrace{\left(M^1L^{-3}T^0\right)^a}_{\rho}\,\underbrace{\left(M^0L^0T^{-1}\right)^b}_{\omega}\,\underbrace{\left(M^0L^3T^{-1}\right)^c}_{Q}}_{\text{repeating variables}},$$

for which we write 3 equations

$$\begin{aligned} M: \quad 0 &= 1 + 1a + 0b + 0c \\ L: \quad 0 &= -1 - 3a + 0b + 3c \\ T: \quad 0 &= -2 + 0a - 1b - 1c \end{aligned}$$

and solve as $a = -1$ and $b = -4/3$ and $c = -2/3$. The first dimensionless group is therefore $\Pi_1 = \Delta P/\left(\rho\,\omega^{4/3}\,Q^{2/3}\right)$. For D we have

$$\Pi_2 \text{ for } D: \quad M^0\,L^0\,T^0 = \underbrace{\left(M^0L^1T^0\right)}_{D} \times \underbrace{\underbrace{\left(M^1L^{-3}T^0\right)^a}_{\rho}\,\underbrace{\left(M^0L^0T^{-1}\right)^b}_{\omega}\,\underbrace{\left(M^0L^3T^{-1}\right)^c}_{Q}}_{\text{repeating variables}},$$

for which we again write 3 equations

$$\begin{aligned} M: \quad 0 &= 0 + 1a + 0b + 0c \\ L: \quad 0 &= 1 - 3a + 0b + 3c \\ T: \quad 0 &= 0 + 0a - 1b - 1c \end{aligned}$$

and solve as $a = 0$ and $b = 1/3$ and $c = -1/3$. The second dimensionless group is therefore $\Pi_2 = D\,\omega^{1/3}/Q^{1/3}$.

Strictly speaking, we could use these two expressions, Π_1 and Π_2 as is. However, they're in neither a convenient nor recognizable form. Dimensionless theory allows us to re–combine these expressions to derive new ones. Let's define $\Pi_2' = \Pi_2^3 = D^3\,\omega/Q$. Also, let

$$\Pi_1' \;=\; \frac{\Pi_1}{\Pi_2^2} \;=\; \frac{\Delta P}{\rho\,\omega^{4/3}\,Q^{2/3}} \cdot \frac{Q^{2/3}}{D^2\,\omega^{2/3}} \;=\; \frac{\Delta P}{D^2\,\rho\,\omega^2}\,.$$

In functional form, we can then write

$$\frac{\Delta P}{\rho\,(D\,\omega)^2} \;=\; F\left(\frac{D^3\,\omega}{Q}\right)\,,$$

i.e. the dimensionless variable $\Delta P/(\rho\,(D\,\omega)^2)$ *is a function of* the dimensionless variable $D^3\omega/Q$. Note how $D\,\omega$ is effectively a velocity, so that the denominator of the left hand side is a kind of dynamic pressure.

2. (10 pts) Under certain circumstances, the statement of conservation of momentum simplifies to *Prandtl's boundary layer equation*

$$u\,\frac{\partial u}{\partial x} \;+\; v\,\frac{\partial u}{\partial y} \;=\; \nu\,\frac{\partial^2 u}{\partial y^2}\,,$$

where x and y are space coordinates, $u(x,y)$ and $v(x,y)$ are velocity components, and ν is kinematic viscosity. If L and u_∞ are appropriate length and velocity scales, respectively, e.g. we can define dimensionless variables such as $x^* = x/L$ and $u^* = u/u_\infty$, re–derive this equation in dimensionless form, showing that the Reynolds number, $Re = u_\infty\,L/\nu$, arises as the natural parameter.

Solution: This is a question of non–dimensionalizing each of the components of the expression and then determining how they can be manipulated into the required form. First, list the dimensionless variables.

$$x^* \;=\; \frac{x}{L} \qquad y^* \;=\; \frac{y}{L} \qquad u^* \;=\; \frac{u}{u_\infty} \qquad v^* \;=\; \frac{v}{u_\infty}\,.$$

Next, non–dimensionalize the first–derivative terms using the Chain Rule:

$$\frac{\partial u}{\partial x} \;=\; \frac{\partial\,(u^* u_\infty)}{\partial x^*}\,\frac{\partial x^*}{\partial x} \;=\; \left(u_\infty\,\frac{\partial u^*}{\partial x^*}\right)\frac{1}{L} \;=\; \frac{u_\infty}{L}\,\frac{\partial u^*}{\partial x^*}\,,$$

$$\frac{\partial u}{\partial y} \;=\; \frac{\partial\,(u^* u_\infty)}{\partial y^*}\,\frac{\partial y^*}{\partial y} \;=\; \left(u_\infty\,\frac{\partial u^*}{\partial y^*}\right)\frac{1}{L} \;=\; \frac{u_\infty}{L}\,\frac{\partial u^*}{\partial y^*}\,.$$

The second derivative term is simply an additional application of Chain Rule, resulting in

$$\frac{\partial^2 u}{\partial y^2} \;=\; \frac{\partial}{\partial y}\,\frac{\partial u}{\partial y} \;=\; \cdots \;=\; \frac{u_\infty}{L^2}\,\frac{\partial^2 u^*}{\partial y^{*2}}\,.$$

Putting these pieces together, we find

$$\frac{u_\infty^2}{L}\, u^* \,\frac{\partial u^*}{\partial x^*} \;+\; \frac{u_\infty^2}{L}\, v^* \,\frac{\partial u^*}{\partial y^*} \;=\; \nu\, \frac{u_\infty}{L^2}\, \frac{\partial^2 u^*}{\partial y^{*2}} \;,$$

from which it is easy to see that

$$u^* \,\frac{\partial u^*}{\partial x^*} \;+\; v^* \,\frac{\partial u^*}{\partial y^*} \;=\; \frac{1}{Re}\, \frac{\partial^2 u^*}{\partial y^{*2}} \;,$$

where $Re = u_\infty\, L/\nu$ is the Reynolds number.

3. (10 pts) The equation in Question 2 can be generalized somewhat to describe the flow arising from buoyancy–induced convection by adding a term, as

$$u\, \frac{\partial u}{\partial x} \;+\; v\, \frac{\partial u}{\partial y} \;=\; \nu\, \frac{\partial^2 u}{\partial y^2} \;+\; g\, \beta\, T \;,$$

where $T = T(x, y)$ is a temperature distribution (units of K), g is gravitational acceleration, and β is the thermal expansion coefficient of the fluid (in units of K^{-1}). Show by similar operations as used in Question 2 and by the rules of "dimensionless algebra", that if temperature is non–dimensionalized as $T^* = T/\Delta T_R$, where ΔT_R is a reference temperature difference in units of K, a new dimensionless group called the Grashof number (the relative importance of buoyancy effects to viscous effects) appears, as defined by

$$Gr \;=\; \frac{g\, \beta\, \Delta T_R\, L^3}{\nu^2} \;.$$

Solution: Referring back to the solution of Question 2, we can start at the point where the differential terms had been non–dimensionalized and we can add the new buoyancy term, as written in terms of T^* as

$$\frac{u_\infty^2}{L}\, u^* \,\frac{\partial u^*}{\partial x^*} \;+\; \frac{u_\infty^2}{L}\, v^* \,\frac{\partial u^*}{\partial y^*} \;=\; \nu\, \frac{u_\infty}{L^2}\, \frac{\partial^2 u^*}{\partial y^{*2}} \;+\; g\, \beta\, \Delta T_R\, T^* \;,$$

from which we can divide through by u_∞^2/L to obtain

$$u^* \,\frac{\partial u^*}{\partial x^*} \;+\; v^* \,\frac{\partial u^*}{\partial y^*} \;=\; \frac{1}{Re}\, \frac{\partial^2 u^*}{\partial y^{*2}} \;+\; \frac{g\, \beta\, \Delta T_R\, L}{u_\infty^2}\, T^*$$

where $Re = u_\infty\, L/\nu$ is again the Reynolds number, as obtained in Question 2. The coefficient of T^* is dimensionless, but it is not the Grashof number. Recalling the rules of dimensionless algebra, we can further develop this group by multiplying by the square of the Reynolds number, as

$$Gr \;=\; \frac{g\, \beta\, \Delta T_R\, L}{u_\infty^2} \cdot \left(\frac{u_\infty\, L}{\nu}\right)^2 \;=\; \frac{g\, \beta\, \Delta T_R\, L^3}{\nu^2} \;.$$

4. (10 pts) Your engineering group is charged with characterizing the performance of a new aircraft carrier hull design. The design is only a proposal and will not actually be built unless it

will give an appreciable increase in performance over current configurations. Assume the flow is governed primarily by the effects described by the Reynolds and the Froude numbers,

$$Re = \frac{U_s L_s}{\nu} \qquad \text{and} \qquad Fr = \frac{U_s}{\sqrt{g L_s}},$$

respectively. Here, U_s is the ship's cruising speed and L_s is its hull length. If the proposed design specifies $U_s = 10\ m/s$ and $L_s = 350\ m$, comment on whether you could study the performance experimentally using a 1/100 scale model ship towed in a water channel. Assume $\nu = 1 \times 10^{-6}\ m^2/s$ for water.

Solution: First, let us calculate the Reynolds and Froude numbers of the full–scale ship:

$$Re = \frac{10 \cdot 350}{1 \times 10^{-6}} = 3.5 \times 10^9 \qquad \text{and} \qquad Fr = \frac{10}{\sqrt{9.8 \cdot 350}} \approx 0.17\,.$$

To study the problem experimentally, we must be able to construct a model system that has the same values for these parameters, i.e. $Re_{model} = 3.5 \times 10^9$ and $Fr_{model} = 0.17$. Let us take the Froude number first. A 1/100 scale–down implies $L_{model} = 3.5\,m$, which gives

$$Fr_{model} = 0.17 = \frac{U_{model}}{\sqrt{9.8 \cdot 3.5}}\,.$$

Solving, we find that the model must be towed at a speed of $U_{model} = 1\ m/s$. This seems reasonable — so far, so good. Now, we must also satisfy the requirement that the Reynolds numbers match. If we are towing in a water channel, then the above parameters give us a Reynolds number of

$$Re = \frac{1 \cdot 3.5}{1 \times 10^{-6}} = 3.5 \times 10^6\,.$$

This is too small by 3 orders of magnitude! It is clear that L_{model} is fixed according to the problem statement and U_{model} is then fixed by the requirement of matching the Froude number. The only way to get the Reynolds number to match appropriately is to use a liquid having a viscosity 1000 times less than water. Consequently, a water channel could not be used for this problem.

5. (10 pts) A cylinder of diameter D bobs up and down in a pool (Fig. 1). The bobbing frequency ω is assumed to be a function of D, as well as the cylinder's mass, m, and the specific weight of the liquid, γ, i.e. $\omega = \omega\,(D, m, \gamma)$. Cast this relationship in terms of relevant dimensionless variables. Also, determine whether the bobbing frequency increases or decreases if the mass of the cylinder is decreased. *Hint.* If a physical system has only a single dimensionless variable, that variable is a constant.

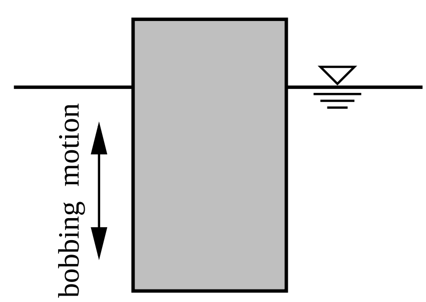

Fig. 1: *Bobbing cylinder.*

Solution: There are $k = 4$ variables and the dimensional components of these variables are

	ω	D	m	γ
M	0	0	1	1
L	0	1	0	-2
T	-1	0	0	-2

It is clear that the determinant of the sub–matrix defined by columns ω, D, and m does *not* vanish, therefore the rank of this system is 3. Consequently, there are $r = 3$ repeating variables and we can conveniently pick these three. There is only $k - r = 4 - 3 = 1$ candidate variable, γ, for which we can write the following Bridgman equation

$$\Pi \text{ for } \gamma \;: \quad M^0\, L^0\, T^0 \;=\; \underbrace{\left(M^1 L^{-2} T^{-2}\right)}_{\gamma} \times \underbrace{\underbrace{\left(M^0 L^0 T^{-1}\right)^a}_{\omega}\, \underbrace{\left(M^0 L^1 T^0\right)^b}_{D}\, \underbrace{\left(M^1 L^0 T^0\right)^c}_{m}}_{\text{repeating variables}} \,.$$

We can now write 3 equations in M, L, and T as

$$\begin{aligned} M \;:\quad 0 &= 1 + 0a + 0b + 1c \\ L \;:\quad 0 &= -2 + 0a + 1b + 0c \\ T \;:\quad 0 &= -2 - 1a + 0b + 0c \end{aligned}$$

and solve as $a = -2$ and $b = 2$ and $c = -1$. The sole dimensionless group that characterizes this problem is then

$$\Pi \;=\; \frac{\gamma\, D^2}{m\, \omega^2} \cdot$$

The key observation here is that, because the system is only governed by one group, the value of this group must be a constant. In other words, Π *is not a function of any other dimensionless variables — therefore its value must be constant.* By inspection, we see that if m decreases, ω must increase.