JME 3700
April 8, 2017

Homework #11 Solutions

1. (10 pts) Water drains from a large tank through a system made of $D = 0.2\,m$ inner diameter pipe having a surface roughness of $\varepsilon = 0.00026\,m$ (Fig. 1). Discharge to the atmosphere is $H = 4.5\,m$ below the free surface and the volume flow rate is $Q = 0.05\,m^3/s$. If the four right–angle pipe components each have a loss coefficient of $K_a = 0.25$ and the exit itself has a loss coefficient of $K_e = 1$, calculate the total length of the straight pipe sections. (We assume that the inlet to the pipe is "well–rounded" so that its losses are negligible.) Assume a water viscosity and density of $\nu = 1.2 \times 10^{-6}\,m^2/s$ and $\rho = 1000\,kg/m^3$, respectively.

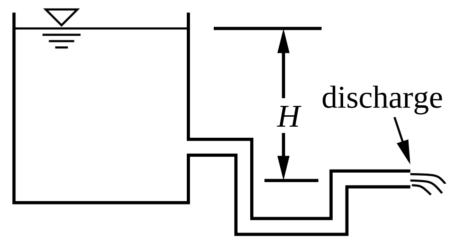

Fig. 1: *Piping system.*

Solution: We can write the extended Bernoulli equation from the free surface s to the discharge exit e in the form

$$\frac{P_s}{\rho\,g} + \frac{\overline{u}_s^2}{2\,g} + z_s - h_L = \frac{P_e}{\rho\,g} + \frac{\overline{u}_e^2}{2\,g} + z_e\,.$$

Here, $P_s = P_e$ (atmospheric pressure), $\overline{u}_s^2 = 0$, and $z_s - z_e = H$. Also, let us split the head loss h_L into 2 components, one representing the sum of all the "component losses" (the right–angle bends and the exit), h_{LC}, and one representing losses from all the straight sections of the pipe h_{LP}. Our Bernoulli equation is then

$$h_{LC} + h_{LP} = H - \frac{\overline{u}_e^2}{2\,g}\,,$$

where

$$h_{LC} = (4\,K_a + K_e)\,\frac{\overline{u}_e^2}{2\,g} \qquad \text{and} \qquad h_{LP} = f\,\frac{L}{D}\,\frac{\overline{u}_e^2}{2\,g}\,.$$

Here, L represents the total length of all the straight sections of pipe. Substituting and solving, we find

$$\begin{aligned} f\,\frac{L}{D}\,\frac{\overline{u}_e^2}{2\,g} &= H - \frac{\overline{u}_e^2}{2\,g} - (4\,K_a + K_e)\,\frac{\overline{u}_e^2}{2\,g} \\ &= H - (1 + 4\,K_a + K_e)\,\frac{\overline{u}_e^2}{2\,g} \\ L &= \frac{D}{f}\left(\frac{H}{\overline{u}_e^2/(2\,g)} - (1 + 4\,K_a + K_e)\right)\,. \end{aligned}$$

We have everything necessary to solve, except for the friction factor of the pipe flow, f. If the cross–sectional area of the pipe is A_c, then the Reynolds number is

$$Re = \frac{\overline{u}\,D}{\nu} = \frac{Q\,D}{A_c\,\nu} = \frac{0.05\,\cdot\,0.2}{\pi\,0.1^2\,\cdot\,1.2\times10^{-6}} \approx 265{,}300 \quad (\text{turbulent regime})\,.$$

Also, the roughness ratio is $\varepsilon/D = 0.00026/0.2 = 0.0013$. From these values, we find $f \approx 0.022$ from the Moody chart. Also, $\overline{u}_e = Q/A_c = \cdots = 1.59\, m/s$. Therefore,

$$\begin{aligned} L &= \frac{0.2}{0.022}\left(\frac{4.5}{1.59^2/(2\cdot 9.8)} - (1 + 4\cdot 0.25 + 1)\right) \\ &= 289.9\, m\,. \end{aligned}$$

2. (10 pts) A water company is considering replacing a long section of a very old 0.5 m inner diameter "main". This pipe has a relative roughness of $\varepsilon/D = 0.02$. One option is to dig the whole system up, but this would be very expensive. A junior engineer suggests pushing a hydrodynamically smooth plastic liner of inner diameter 0.4 m through the old pipe instead. Although this would be much cheaper, this new configuration would still need to carry the same volume flow rate $Q = 1\, m^3/s$ as the older, larger pipe. If there is to be no net increase in pressure drop, will this approach work? Assume a water viscosity of $\nu = 1.2\times 10^{-6}\, m^2/s$.

Solution: Let us first analyze the old pipe system. The average velocity is

$$\overline{u}_{old} = \frac{Q}{A_{old}} = \frac{1}{\pi\, 0.25^2} \approx 5.093\, m/s\,.$$

Consequently, the Reynolds number for the old system is

$$Re_{old} = \frac{\overline{u}_{old}\, D_{old}}{\nu} = \frac{5.093\cdot 0.5}{1.2\times 10^{-6}} \approx 2.1\times 10^6\,.$$

A quick glance at the Moody chart for $Re_{old} = 2.1\times 10^6$ and $\varepsilon/D_{old} = 0.02$ indicates $f_{old} \approx 0.048$, so that the pressure drop from the *Darcy* equation is

$$\Delta P_{old} = f_{old}\,\frac{L}{D_{old}}\,\frac{\rho\,\overline{u}_{old}^2}{2} = 0.048\,\frac{L}{0.5}\,\frac{\rho\, 5.093^2}{2} \approx 1.245\,\rho\, L\,.$$

Now we analyze the new system, which has a smaller cross section, but must still carry the same volume flow rate. The average velocity is

$$\overline{u}_{new} = \frac{Q}{A_{new}} = \frac{1}{\pi\, 0.2^2} \approx 7.958\, m/s\,.$$

Consequently, the Reynolds number for the new system is slightly higher at

$$Re_{new} = \frac{\overline{u}_{new}\, D_{new}}{\nu} = \frac{7.958\cdot 0.4}{1.2\times 10^{-6}} \approx 2.65\times 10^6\,.$$

However, the inside is now hydrodynamically smooth, i.e. $\varepsilon/D_{old} \to 0$. A quick glance at the Moody chart indicates $f_{new} \approx 0.01$, so that the pressure drop from the *Darcy* equation is

$$\Delta P_{new} = f_{new}\,\frac{L}{D_{new}}\,\frac{\rho\,\overline{u}_{new}^2}{2} = 0.01\,\frac{L}{0.4}\,\frac{\rho\, 7.958^2}{2} \approx 0.792\,\rho\, L\,.$$

The new configuration actually has substantially better performance (less pressure drop), but can still carry the same volume flow rate! Evidently, the gain in using a "smoother" material outweighed the liability of using a significantly smaller cross section.

3. (10 pts) A pump moves water into a very large tank at a flow rate of $Q = 0.01\ m^3/s$ through $L = 100\ m$ of hydrodynamically smooth pipe (Fig. 2, not shown to scale). The pipe is horizontal and has an inner diameter of $D = 0.075m$. At the instant shown, there is an $h = 10\ m$–deep column of water already in the tank. What pressure, P_p, must the pump step the flow up to at its outlet? Assume $\nu = 1 \times 10^{-6}\, m^2/s$ and $\rho = 1000\, kg/m^3$ and that the viscous losses due to the flow entering the tank are characterized by the loss coefficient $K = 1$.

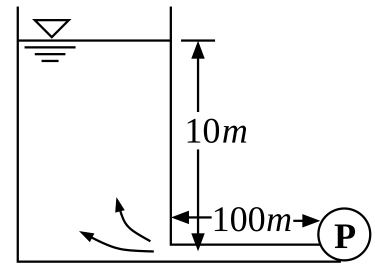

Fig. 2: *Pumping water.*

Solution: Assume flow in the pipe is fully–developed. We can write the extended Bernoulli equation from the pump p to the the free surface of the tank s in the form

$$\frac{P_p}{\rho} + \frac{\bar{u}_p^2}{2} + z_p\, g - h_L = \frac{P_s}{\rho} + \frac{\bar{u}_s^2}{2} + z_s\, g\,.$$

Here, $P_s = 0$ (atmospheric pressure), $\bar{u}_s = 0$ (large tank), and $z_s - z_p = h$. The head loss, h_L, has 2 contributions: losses in the $L = 100\, m$ straight section and losses in the tank entrance, i.e.

$$h_L = f\,\frac{L}{D}\,\frac{\bar{u}_p^2}{2} + K\,\frac{\bar{u}_p^2}{2}\,,$$

where we know $K = 1$. Our Bernoulli equation is then

$$\frac{P_p}{\rho} + \frac{\bar{u}_p^2}{2} = g\,h + f\,\frac{L}{D}\,\frac{\bar{u}_p^2}{2} + K\,\frac{\bar{u}_p^2}{2}\,,$$

which can be solved for the pressure as

$$P_p = \rho\, g\, h + \frac{\rho\, \bar{u}_p^2}{2}\left(f\,\frac{L}{D} + K - 1\right)\cdot$$

We have everything necessary to evaluate, except for the friction factor of the pipe flow, f. If the cross–sectional area of the pipe is A_c, then the Reynolds number is

$$Re = \frac{\bar{u}_p\, D}{\nu} = \frac{Q\, D}{A_c\, \nu} = \frac{0.01 \cdot 0.075}{\pi\,(0.075/2)^2 \cdot 1 \times 10^{-6}} \approx 170,000 \quad \text{(turbulent regime)}\,.$$

The pipe is "smooth", i.e. $\varepsilon/D = 0$ and from these values, we find $f \approx 0.016$ from the Moody chart. Also, $\bar{u}_p = Q/A_c = \cdots \approx 2.26\, m/s$. Therefore,

$$\begin{aligned} P_p &= 1000 \cdot 9.8 \cdot 10 + \frac{1000 \cdot 2.26^2}{2}\left(0.016\,\frac{100}{0.075} + 1 - 1\right) \\ &\approx 152,700\, N/m^2\,. \end{aligned}$$

4. (10 pts) Oil flows steadily in a horizontal pipe of constant diameter $D_1 = 0.2\,m$ at an average velocity of $\overline{u}_1 = 1\ m/s$ (Fig. 3). Oil viscosity is $\nu = 0.001\ m^2/s$ and the flow is fully–developed. The flow eventually reaches a smooth contraction and enters into a smaller pipe of diameter $D_2 = 0.1\ m$, quickly becoming fully developed. Determine the respective friction factors f_1 and f_2 in the larger and smaller pipes.

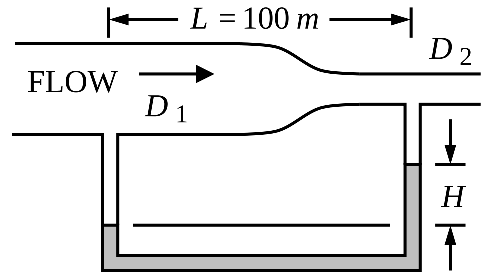

Fig. 3: *Oil flow in a pipe.*

Solution: For the large pipe, calculate the Reynolds number as

$$Re_1 \;=\; \frac{\overline{u}_1\, D_1}{\nu} \;=\; \frac{1 \cdot 0.2}{0.001} \;=\; 200\,,$$

whereby we see the flow is laminar, since $Re_1 < 2100$, so that the friction factor follows the formula $f = 64/Re$, whereby

$$f_1 \;=\; \frac{64}{Re} \;=\; \frac{64}{200} \;=\; 0.32\,.$$

For the smaller section of pipe, the average velocity can be calculated via conservation of mass as

$$\frac{\overline{u}_1\, D_1^2}{4} \;=\; \frac{\overline{u}_2\, D_2^2}{4} \qquad \therefore \quad \overline{u}_2 \;=\; \frac{\overline{u}_1\, D_1^2}{D_2^2} \;=\; \frac{1 \cdot 0.2^2}{0.1^2} \;=\; 4\,m/s\,.$$

Now, calculate the Reynolds number as

$$Re_2 \;=\; \frac{\overline{u}_2\, D_2}{\nu} \;=\; \frac{4 \cdot 0.1}{0.001} \;=\; 400\,,$$

whereby we see the flow is still laminar, so that the friction factor is

$$f_2 \;=\; \frac{64}{Re} \;=\; \frac{64}{400} \;=\; 0.16\,.$$

5. (10 pts) The configuration in Question 4 is exactly horizontal and two static ports are separated by $L = 100\ m$, as measured along the pipe. These ports are connected by a liquid mercury manometer (also shown in Fig. 3), whose fluid density is $\rho_m = 13500\ kg/m^3$. Here, the contraction is halfway between the static ports and is very short compared to the overall length of $L = 100\,m$. Therefore, taking as a first approximation that f_1 applies over the first $50m$ of pipe length and f_2 applies over the subsequent $50m$ of length, determine the manometer reading, H. Gravitational acceleration is $g = 9.8\,m/s^2$ and oil density is $\rho_o = 900\,kg/m^3$. From a design perspective, comment whether this manometer configuration would be a good method for measuring pressure drop.

Solution: The extended Bernoulli equation for this system can be written as

$$\frac{P_1}{\rho_o\, g} \;+\; \frac{\bar{u}_1^2}{2\,g} \;+\; y_1 \;-\; h_L \;=\; \frac{P_2}{\rho_o\, g} \;+\; \frac{\bar{u}_2^2}{2\,g} \;+\; y_2\,,$$

where we take stations "1" and "2" as the pipe centerline points at the upstream and downstream static ports, respectively. Noting that $y_1 = y_2$ because the pipe is horizontal, we can write this equation in the form

$$\frac{P_1 - P_2}{\rho_o\, g} = h_L + \frac{\overline{u}_2^2 - \overline{u}_1^2}{2\, g} .$$

Substituting the Darcy–Weissbach relation in 2 parts, one for the first $L/2 = 50\, m$ and one for the second $L/2 = 50\, m$, we find

$$\begin{aligned}
P_1 - P_2 &= f_1 \frac{L}{2\, D_1} \frac{\rho_o\, \overline{u}_1^2}{2} + f_2 \frac{L}{2\, D_2} \frac{\rho_o\, \overline{u}_2^2}{2} + \frac{\rho_o\, \overline{u}_2^2}{2} - \frac{\rho_o\, \overline{u}_1^2}{2} \\
&= \frac{\rho_o\, \overline{u}_1^2}{2} \left(f_1 \frac{L}{2\, D_1} - 1 \right) + \frac{\rho_o\, \overline{u}_2^2}{2} \left(f_2 \frac{L}{2\, D_2} + 1 \right) \\
&= \frac{900 \cdot 1^2}{2} \left(0.32 \cdot \frac{100}{2 \cdot 0.2} - 1 \right) + \frac{900 \cdot 4^2}{2} \left(0.16 \cdot \frac{100}{2 \cdot 0.1} + 1 \right) \\
&= 35,500 + 583,200 \\
&= 618,700 \, \frac{N}{m^2} .
\end{aligned}$$

Calling W the vertical distance between the pipe centerline and the top of the downstream mercury column, we can write a manometer relationship as

$$P_1 + \rho_o\, g\, W + \rho_o\, g\, H - \rho_m\, g\, H - \rho_o\, g\, W = P_2 ,$$

which can be re–arranged as

$$P_1 - P_2 = g\, H\, (\rho_m - \rho_o) ,$$

where we would like to solve for H. Using $P_1 - P_2$ found above, we calculate

$$H = \frac{P_1 - P_2}{g(\rho_m - \rho_o)} = \frac{618,700}{9.8 \cdot (13,500 - 900)} \approx 5\, m.$$

The fact that the vertical displacement of manometer fluid is so large suggests that this would not actually be a very good design choice for measuring pressure drop.