JME 3700
April 17, 2017

Homework #12 Solutions

1. (10 pts) A smooth upward air flow suspends a sphere of radius R just above the flow outlet (Fig. 1). The sphere is solid and made of a material that is twice as dense as air, the latter's density being ρ. The coefficient of drag of a sphere is $C_D \approx 0.5$ over a wide range Reynolds number, $10^4 \leq Re \leq 10^5$. Assuming this condition is satisfied, show that the flow velocity is

$$u = \sqrt{\frac{32\,R\,g}{3}}$$

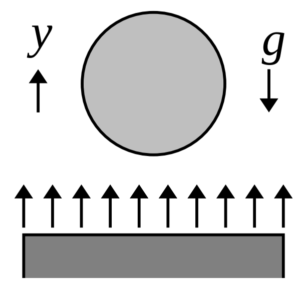

Fig. 1: *Suspended sphere.*

Solution: Given that the sphere is stationary (static), the drag force pushing up on the sphere, F, is exactly balanced by the sphere's weight W, i.e. $F = W$. We can flesh–out these terms as follows

$$W = \text{density} \cdot \text{volume} \cdot g = 2\,\rho \cdot \frac{4\,\pi\,R^3}{3} \cdot g = \frac{8\,\rho\,\pi\,R^3\,g}{3}$$

$$F = \frac{1}{2}\,C_D\,\rho\,u^2\,A = \frac{1}{2} \cdot \frac{1}{2}\,\rho\,u^2\,\pi\,R^2 = \frac{\rho\,u^2\,\pi\,R^2}{4}$$

whereby we find

$$\frac{\rho\,u^2\,\pi\,R^2}{4} = \frac{8\,\rho\,\pi\,R^3\,g}{3} \qquad \rightarrow \qquad u^2 = \frac{4 \cdot 8\,\rho\,\pi\,R^3\,g}{3\,\rho\,\pi\,R^2}$$

from which the stated result follows directly.

2. (10 pts) Fluid at an approach speed of u_0 flows around a cylinder of width b and radius R. The pressure distribution around the top half of the cylinder, $0 \leq \theta \leq \pi$, is shown in Fig. 2 in units of $\rho\,u_0^2$. (The distribution is symmetric about $\theta = \pi$, i.e. for the bottom half, $\pi \leq \theta \leq 2\,\pi$.) Determine the drag coefficient if shear stress can be neglected.

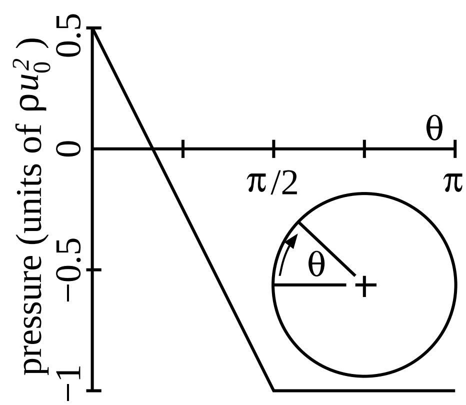

Fig. 2: *Pressure distribution.*

Solution: This problem requires integrating the pressure contribution to calculate the total pressure–based drag force. Since shear can be neglected, we have $dF_D = P \cdot dA \cdot \cos\theta$ for a differential element on the cylinder's surface. Take the differential area as $dA = R\,d\theta \cdot b$, i.e. a differential arc length times the cylinder's width. We have

$$F_D \;=\; \int_0^{2\pi} P\,(R\,d\theta\,b)\cos\theta \;=\; 2\int_0^{\pi} P\;R\,b\cos\theta\,d\theta\,.$$

The last step comes from the symmetry condition stated in the problem. The easiest way to proceed is to divide the task according to

$$P \;=\; P(\theta) \;=\; \begin{cases} \frac{1}{2}\,\rho\,u_0^2\,(1-6\,\theta/\pi) & \text{for} \quad 0 \le \theta \le \pi/2 \\ -\rho\,u_0^2 & \text{for} \quad \pi/2 \le \theta \le \pi \end{cases}$$

Consequently, we write the integral in two parts as[1]

$$\begin{aligned} F_D \;&=\; 2\,b\,R\int_0^{\pi/2}\frac{1}{2}\,\rho\,u_0^2\,(1-6\,\theta/\pi)\,\cos\theta\,d\theta \;-\; 2\,b\,R\int_{\pi/2}^{\pi}\rho\,u_0^2\,\cos\theta\,d\theta \\ &=\; 2\,b\,R\left[\frac{1}{2}\,\rho\,u_0^2\Big(\sin\theta \;-\; 6\,(\cos\theta+\theta\,\sin\theta)\,/\pi\Big)\Big|_0^{\pi/2} \;-\; \rho\,u_0^2\,\sin\theta\Big|_{\pi/2}^{\pi}\right] \\ &=\; 2\,b\,R\left[\frac{1}{2}\,\rho\,u_0^2\left(-2+\frac{6}{\pi}\right) \;-\; \rho\,u_0^2\,(0-1)\right] \\ &=\; 2\,b\,R\cdot\frac{3}{\pi}\,\rho\,u_0^2 \\ &=\; \frac{6\,b\,R}{\pi}\,\rho\,u_0^2\,. \end{aligned}$$

The coefficient of drag is now determined directly from its definition. Taking the frontal "visible" area as $A = b \cdot 2\,R$, we have

$$C_D \;=\; \frac{F_D}{0.5\,\rho\,u_0^2\,A} \;=\; \frac{1}{0.5\,\rho\,u_0^2\,\cdot\,b\cdot 2\,R}\;\cdot\;\frac{6\,b\,R}{\pi}\,\rho\,u_0^2 \;=\; \frac{6}{\pi} \;\approx\; 1.91\;.$$

[1] The non–trivial integral involving $\theta\cos\theta$ can be worked out, or simply looked up in a handbook, e.g. integral #393 in CRC Standard Mathematical Tables, 26–th Ed. (1981).

3. (10 pts) A barrier is in the form of a 1–1–$\sqrt{2}$ right triangle having a leg length of L and a width of W "into the paper" (Fig. 3). A flow creates a positive linear pressure distribution on the front vertical face, i.e. $P(y) = P_0 y/L$, and a similar linear negative distribution on the back side (the hypotenuse) having the same maximum magnitude of $|P_0|$. Here, the coefficient of drag is defined as

$$C_D = \frac{F_D}{\frac{1}{2}\rho u_\infty^2 A_F},$$

where ρ and u_∞ are fluid density and approach velocity, F_D is the total drag force, and $A_F = L \cdot W$. If the flow structure is such that $P_0 = \rho u_\infty^2/4$, determine C_D.

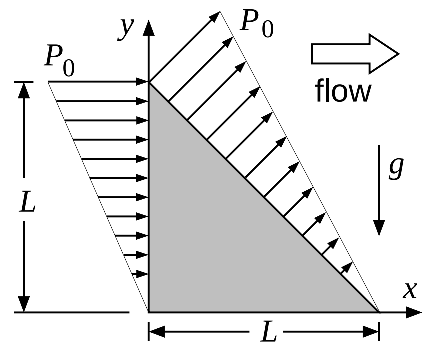

Fig. 3: *Triangle barrier.*

Solution: To determine the total drag force, we can calculate the resultant forces from the 2 pressure distributions, say F_1 on the front face and F_2 on the hypotenuse. While these can be cast by integrating, we might note that these distributions are linear and might further recall from hydrostatics that the magnitude is the average pressure times the area. That is

$$F_1 = \frac{P_0 W L}{2} \quad \text{and} \quad F_2 = \frac{\sqrt{2}\, P_0 W L}{2},$$

where each of these forces acts perpendicularly to its surface. For F_2, we must also take the component acting in the horizontal direction, meaning multiplying by $\sqrt{2}/2$. The total drag force is then

$$F_D = F_1 + \frac{\sqrt{2}}{2} F_2 = \frac{P_0 W L}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}\, P_0 W L}{2} = P_0 W L .$$

We can substitute this expression into the drag coefficient and subsequently substitute $P_0 = \rho u_\infty^2/4$ to obtain

$$C_D = \frac{P_0 W L}{\frac{1}{2}\rho u_\infty^2 A_F} = \frac{\frac{1}{4}\rho u_\infty^2 L W}{\frac{1}{2}\rho u_\infty^2 L W} = \frac{1}{2}.$$

4. (10 pts) The triangular barrier in Question 3 has a mass of m and its center of gravity (centroid) is at $(x, y) = (L/3, L/3)$. Noting that gravitational acceleration g acts in the negative y–direction, show that the minimum value for m such that barrier does not begin to lift at its front, i.e. at $(x, y) = (0, 0)$ and rotate about $(x, y) = (L, 0)$ is

$$m > \frac{3\rho u_\infty^2 W L}{8 g}.$$

Solution: For rotation about $(x, y) = (L, 0)$, we can sum the moments about that point, where the counter–clockwise moment of the weight of the barrier is exactly offset by the clockwise moments of the two pressure loads. This would be the condition under which the resulting m is the mass such that larger values would not allow lifting. Because the pressure loads have linear distributions, the lines–of–action of their resultant forces both follow the "two–thirds rule", e.g. the resultant of the horizontal distribution on the front face acts at $y = 2\,L/3$. Given the location of the centroid from the problem statement, we can then write

$$\begin{aligned} m\,g \cdot \frac{2\,L}{3} &= \frac{P_0\,W\,L}{2} \cdot \frac{2\,L}{3} + \frac{\sqrt{2}\;P_0\,W\,L}{2} \cdot \frac{2\,\sqrt{2}\;L}{3} \\ m\,g &= \frac{P_0\,W\,L}{2} + \frac{\sqrt{2}\;P_0\,W\,L}{2} \cdot \sqrt{2} = \frac{3\,P_0\,W\,L}{2} \\ m &= \frac{3\,P_0\,W\,L}{2\,g}\,, \end{aligned}$$

so that, in substituting $P_0 = \rho\,u_\infty^2/4$, we find

$$m = \frac{3\,W\,L}{2\,g} \cdot \frac{\rho\,u_\infty^2}{4} = \frac{3\,\rho\,u_\infty^2\,W\,L}{8\,g}\,.$$

Therefore, the actual mass, m, would have to be greater than this value.

5. (10 pts) A dragster of mass $m = 730\;kg$ crosses the finish line at $u_0 = 120\;m/sec$, after which the driver deploys a chute to slow the vehicle. The chute has an effective area of $2.3\,m^2$ and exerts a drag force, F_D (Fig. 4). It can be taken as a hemisphere having a coefficient of drag of $C_D = 1.4$. If the aerodynamic drag and the rolling resistance of the car itself are neglected (as being presumably small compared to the chute drag), determine the required time for the car to slow to a speed of $u_1 = 45\;m/sec$. Take local air density as $\rho = 1.2\;kg/m^3$.

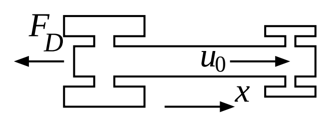

Fig. 4: *Dragster.*

Solution: Using the free–body diagram in Fig. 4, we can write Newton's Second Law as

$$-\,F_D = m\,\frac{du}{dt}\,.$$

Substituting the definition of the drag in terms of the coefficient of drag, i.e. $F_D = C_D \rho u^2 A/2$ and re–arranging terms, we can write

$$\frac{du}{dt} = -\,\frac{F_D}{m} = -\,\frac{C_D\,\rho\,A\,u^2}{2\,m}\,.$$

This is a *separable* differential equation that we can solve as follows:

$$\int_{u_0}^{u_1} \frac{du}{u^2} = -\,\frac{C_D\,\rho\,A}{2\,m} \int_0^t dt \qquad \rightarrow \qquad -\,\frac{1}{u}\bigg|_{u_0}^{u_1} = -\,\frac{C_D\,\rho\,A}{2\,m}\,t\bigg|_0^t$$

which can be simplified as

$$t = \frac{2\,m}{C_D\,\rho\,A}\left(\frac{1}{u_1} - \frac{1}{u_0}\right) = \frac{2 \cdot 730}{1.4 \cdot 1.2 \cdot 2.3}\left(\frac{1}{45} - \frac{1}{120}\right) \approx 5.3\;sec\,.$$