JME 3700
April 25, 2017

Homework #13 Solutions

1. (10 pts) Water flows at $Q = 0.9\,m^3/s$ in a rectangular channel having a width of $w = 1.5\,m$. The flow encounters a smooth vertical rise of 0.06 m (Fig. 1). Calculate the downstream depth y_2 if the upstream depth is $y_1 = 0.8\,m$.

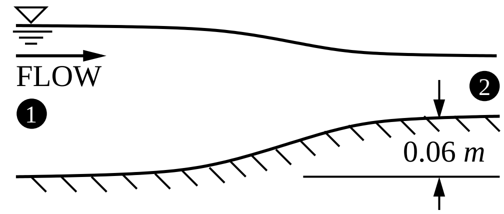

Fig. 1: *Contracting flow.*

Solution: The velocities at the two locations can be calculated from the volume flow rate as

$$\bar{u}_1 = \frac{Q}{A_1} = \frac{0.9}{1.5 \cdot 0.8} = 0.75\,m/s \qquad \text{and} \qquad \bar{u}_2 = \frac{Q}{A_2} = \frac{0.9}{1.5 \cdot y_2} = \frac{0.6}{y_2}\,.$$

Assuming the flow takes place over a small distance, we omit losses. Consequently, the Bernoulli equation is

$$\frac{P_1}{\rho\, g} + \frac{\bar{u}_1^2}{2\, g} + z_1 = \frac{P_2}{\rho\, g} + \frac{\bar{u}_2^2}{2\, g} + z_2\,,$$

where $P_1 = P_2 = 0$ (atmospheric), $z_1 = y_1 = 0.8$ (the depth of the upstream part), and $z_2 = y_2 + 0.06$. Substituting quantities, we find

$$\frac{0.75^2}{2 \cdot 9.8} + 0.8 = \frac{0.6^2}{2 \cdot 9.8\, y_2^2} + y_2 + 0.06\,,$$

which can be written in canonical form as

$$19.6\, y_2^3 - 15.0665\, y_2^2 + 0.36 = 0\,.$$

Solving[1], we find the set of roots to be $y_2 \approx \{-0.142, 0.176, 0.7347\}\,m$. Clearly, the first root is not physically valid, so we discard it. We must therefore pick the correct root from the latter two. How do we do this? Let's calculate the nature of the upstream flow, i.e. a Froude number of $Fr_1 = 0.75/\sqrt{9.8 \cdot 0.8} \approx 0.27$ indicates the flow is sub–critical. The Froude numbers for the two down–stream possibilities are

$$Fr_2 = \begin{cases} \frac{0.6/0.176}{\sqrt{9.8 \cdot 0.176}} \approx 2.6 & \text{for} \quad y_2 = 0.176 \\ \frac{0.6/0.7347}{\sqrt{9.8 \cdot 0.7347}} \approx 0.304 & \text{for} \quad y_2 = 0.7347\,. \end{cases}$$

Because there is nothing in the configuration that could transition the flow into the super–critical regime, i.e. a "bump", the downstream flow should also be sub–critical. Consequently, the downstream depth is $y_2 = 0.7347\,m$.

[1] e.g. using a root–finding program on a calculator, writing a script using the Newton–Raphson method, etc.

2. (10 pts) Determine the upper bound for the depth y of a super–critical open–channel flow if the channel is rectangular with a width of $w = 2\,m$ and has a volumetric flow rate of $Q = 40\,m^3/s$.

Solution: The velocity in the channel is

$$\bar{u} \;=\; \frac{Q}{A} \;=\; \frac{40}{2 \cdot y} \;=\; \frac{20}{y}\,.$$

If the flow must remain super–critical, then

$$Fr \;=\; \frac{\bar{u}}{\sqrt{g\,y}} \;>\; 1\,,$$

which implies

$$\begin{aligned} \sqrt{g\,y} \;&<\; \bar{u} \\ g\,y \;&<\; \bar{u}^2 \\ y \;&<\; \frac{\bar{u}^2}{g} \;=\; \frac{(20/y)^2}{g} \\ y^3 \;&<\; \frac{400}{9.8} \\ y \;&<\; 3.44\,m\,. \end{aligned}$$

3. (10 pts) A flume has a rectangular cross seciton of $2\,L$ wide by L deep, but has a very thin vertical board in its middle (Fig. 2) Assuming the Manning formula applies, find the ratio of the flow rate for this configuration, Q_1, versus the flow rate if the middle board were removed, Q_2, and comment on the cause of any observed difference. Both configurations have the same slope, S_0, and Manning coefficient, n.

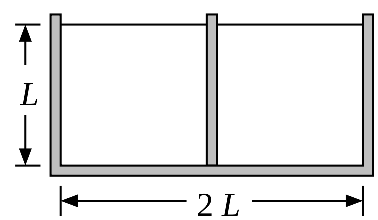

Fig. 2: *Rectangular flume.*

Solution: The flow rate of configuration 1 is simply twice the flow rate in the individual $L \times L$ channels, i.e.

$$Q_1 \;=\; 2\,\frac{A\,R_h^{2/3}\sqrt{S_0}}{n}\,,$$

where $A = L^2$ and $R_h = A/P = L^2/(3\,L) = L/3$. Consequently,

$$Q_1 \;=\; 2\,\frac{L^2\,(L/3)^{2/3}\sqrt{S_0}}{n} \;=\; \frac{2}{3^{2/3}} \cdot \frac{(L)^{8/3}\sqrt{S_0}}{n} \;\approx\; 0.961 \cdot \frac{(L)^{8/3}\sqrt{S_0}}{n}\,.$$

For configuration 2, $A = 2\,L^2$ and $R_h = 2\,L^2/(4\,L) = L/2$, so that the Manning formula gives

$$Q_2 \;=\; \frac{2\,L^2\,(L/2)^{2/3}\sqrt{S_0}}{n} \;=\; \frac{2}{2^{2/3}} \cdot \frac{(L)^{8/3}\sqrt{S_0}}{n} \;\approx\; 1.26 \cdot \frac{(L)^{8/3}\sqrt{S_0}}{n}\,.$$

The ratio is around 0.76, or, equivalently, Q_2 is around 30% more than Q_1, because the vertical board adds considerably more shear stress by virtue of the no–slip boundary condition on its surface.

4. (10 pts) Roman engineer Flöwous Maximus is tasked with determining the maximum distance, D, that an outpost can be built outside the main city, given that its only water supply will be via aquaduct from the city's water aquafier. The maximum vertical fall between the aquafier and the surrounding countryside is $h = 2\ m$ and the duct itself would be a right–triangle section of finished brick, having $n = 0.015\ s/m^{1/3}$ (Fig. 3). Find D in units of km if the aquaduct's design size is $w = 2\ m$ and the outpost's anticipated peak consumption is $Q = 2\ m^3/s$.

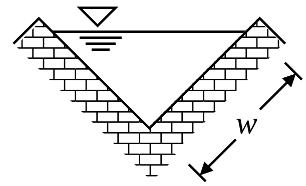

Fig. 3: *Aquaduct channel.*

Solution: The Manning equation is

$$Q = \frac{A\, R_h^{2/3} \sqrt{S_0}}{n} \,,$$

where for this configuration we have

$$A = \frac{w^2}{2} \,, \qquad R_h = \frac{A}{P} = \frac{w^2/2}{2\,w} = \frac{w}{4} \,, \qquad S_0 = \frac{h}{D} \,,$$

the last relation being what would give the maximum distance, D, i.e. it presumes the maximum "fall". Squaring the Manning equation and substituting the above, we find

$$Q^2 = \frac{(w^2/2)^2\, (w/4)^{4/3}\, (h/D)}{n^2} = \frac{w^{16/3}\, h}{4^{7/3}\, D\, n^2} \,.$$

This equation is readily solved for D as

$$D = \frac{w^{16/3}\, h}{4^{7/3}\, Q^2\, n^2} = \frac{2^{16/3} \cdot 2}{4^{7/3} \cdot 2^2 \cdot 0.015^2} = 3527.6\, m \approx 3.5\, km \,.$$

5. (10 pts) Suppose that, at certain times, the aquaduct in Problem 4 has to be shut off for maintenance. However, the outpost has be notified in advance so they can store water ahead of time. Flöwous has an idea that he can simply pass a message in a bottle via the aquaduct informing his deputies at the outpost of the shutoff. To the nearest hour, how long of a time, t, does it take Flöwous' message to reach his deputies?

Solution: The water in the aquaduct moves with a velocity given roughly by

$$V = \frac{R_h^{2/3} \sqrt{S_0}}{n} = \frac{(w/4)^{2/3} \sqrt{h/D}}{n} = \frac{(2/4)^{2/3} \sqrt{2/3528}}{0.015} = 0.999 \approx 1\, m/s \,,$$

meaning that, for a message moving at this speed, it will take around

$$t = \frac{3528\, m}{1\, m/s} = 3528\, s \approx 1\, h \,.$$