JME 3700
February 3, 2017

Homework #3 Solutions

1. (10 pts) A smooth hose of inner diameter D drains a large–surface–area pool by *siphoning* (Fig. 1). The liquid density is ρ and its depth in the pool is d_l. The inlet of the hose is placed at d_i above the bottom of the pool, while the exit is at d_e below the bottom of the pool. If inviscid flow is assumed, determine the mass flow rate of this process in terms of the relevant d–values and gravity, g.

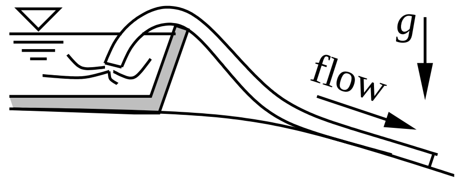

Figure 1: *Siphoning a pool.*

Solution: We can imagine a streamline starting at the liquid surface, going into the hose, and coming out of the exit. Choosing points 1 and 2 to be the liquid surface and hose exit, respectively, we write the Bernoulli equation

$$P_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = P_2 + \frac{\rho V_2^2}{2} + \rho g z_2 .$$

Here, both pressures are atmospheric, whereby P_1 and P_2 vanish. Given that the pool is "large", we assume $V_1 \approx 0$. Choosing our reference as $z_2 = 0$, we find

$$g z_1 = \frac{V_2^2}{2} \qquad \rightarrow \qquad V_2 = \sqrt{2 g z_1} .$$

Now, z_1 is the elevation of point 1 above point 2 (since we chose point 2 as our reference), i.e. $z_1 = d_e + d_l$. The mass flow rate is $\dot{m} = \rho V_2 A_2$, where $A_2 = \pi D^2/4$ is the cross–sectional area of the hose. Putting this together, we find

$$\dot{m} = \frac{\rho \sqrt{2 g (d_e + d_l)} \, \pi D^2}{4} .$$

Evidently, the process is independent of how deep the inlet of the hose is placed into the pool.

2. (10 pts) Fig. 2 shows a simple velocity measuring device for liquids. The device has a circular cross–section of diameter D. The liquid whose velocity is to be determined has a density ρ, while the manometer fluid (the shaded portion) has a density of ρ_m, where $\rho_m < \rho$. Calculate the velocity V, given the manometer fluid displacement shown in the figure. Reference points 1 and 2 may help to cast the problem.

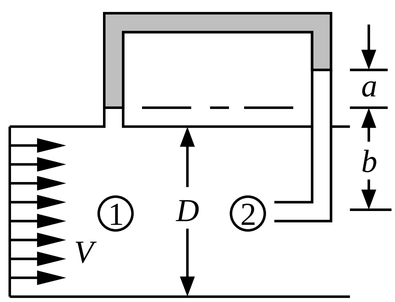

Figure 2: *Simple flowmeter.*

Solution: We imagine the streamlines in this device to be coincident with the flow direction, i.e. straight. Consequently, we can write two equations relating the pressure at points 1 and 2:

one related to the manometer readings $P_1 - \rho\, g\, b - \rho_m\, g\, a + \rho\, g\,(a+b) = P_2$, and one from the Bernoulli equation $P_1 + \rho\, V^2/2 = P_2$. From the former, we find $P_2 - P_1 = g\, a\,(\rho - \rho_m)$, while solving for V from the latter gives $V = \sqrt{2\,(P_2 - P_1)\,/\rho}$. Substituting, we find

$$V = \sqrt{\frac{2\, g\, a\,(\rho - \rho_m)}{\rho}} \; .$$

Note that the solution confirms the information in the problem statement that $\rho_m < \rho$, otherwise the square–root contains a negative argument. This is compatible with the physical observation that if the manometer fluid were heavier than the liquid, it would fall out of the manometer tube.

3. (10 pts) Liquid of density ρ is in a large diameter pressurized tank and flows steadily and inviscidly into a small–diameter pipe of cross–sectional area A_P (Fig. 3). The free surface in the tank is at a height $h/2$ above a reference plane. Static pressure within the pipe is indicated by a manometer having a reading h. The flow exits to atmosphere through a nozzle of area A_N, where $A_N < A_P$. Show that the flow velocity in the pipe can be expressed as

$$V_P = \sqrt{\frac{2\, g\, h\, A_N^2}{A_P^2 - A_N^2}} \; .$$

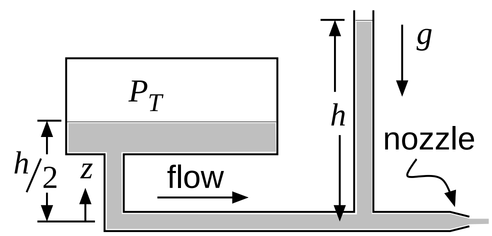

Figure 3: *Tank & pipe system.*

Solution: The Bernoulli equation written between a point directly under the static pressure tap and another point at the outlet just past the nozzle is

$$P_P + \frac{\rho\, V_P^2}{2} + \rho\, g\, z_P = P_N + \frac{\rho\, V_N^2}{2} + \rho\, g\, z_N \, .$$

Since $z_P = z_N = 0$, according to our coordinate system, the potential energy terms drop out. Also, $P_N = 0$, because the flow exists to atmospheric pressure, and $P_P = \rho\, g\, h$, according to hydrostatics, whereby

$$\rho\, g\, h + \frac{\rho\, V_P^2}{2} = \frac{\rho\, V_N^2}{2} \, .$$

Both V_P and V_N are unknown, but we can invoke conservation of mass in the form of the volumetric flow rates, i.e. $V_P\, A_P = V_N\, A_N$, whereby $V_N = V_P\, A_P/A_N$. Substitution and a little algebra show

$$\begin{aligned}
\rho\, g\, h + \frac{\rho\, V_P^2}{2} &= \frac{\rho\, V_P^2\, A_P^2}{2\, A_N^2} \\
2\, g\, h &= \frac{V_P^2\, A_P^2}{A_N^2} - V_P^2 = V_P^2 \left(\frac{A_P^2}{A_N^2} - 1 \right) = V_P^2 \left(\frac{A_P^2 - A_N^2}{A_N^2} \right) \\
V_P^2 &= \frac{2\, g\, h\, A_N^2}{A_P^2 - A_N^2} \, ,
\end{aligned}$$

from which the proposition follows directly by taking the square root of both sides.

4. (10 pts) Using the what is known from Question 3, show that the static pressure within the tank, P_T, can be expressed as

$$P_T = \frac{\rho g h}{2} \left(\frac{A_P^2 + A_N^2}{A_P^2 - A_N^2} \right) .$$

Solution: The Bernoulli equation written between the liquid surface within the pressurized tank and the same point directly under the static pressure tap is

$$P_T + \frac{\rho V_T^2}{2} + \rho g z_T = P_P + \frac{\rho V_P^2}{2} + \rho g z_P .$$

Since the tank is large, we neglect the kinetic energy term containing V_T. Again, $z_P = 0$, but here $z_T = h/2$. We already know the pressure and kinetic energy terms in the pipe from Question 3. Consequently,

$$P_T + \frac{\rho g h}{2} = \rho g h + \frac{\rho}{2} \cdot \frac{2 g h A_N^2}{A_P^2 - A_N^2} .$$

Additional algebra shows

$$\begin{aligned} P_T &= \frac{\rho g h}{2} + \frac{\rho g h A_N^2}{A_P^2 - A_N^2} = \rho g h \left(\frac{1}{2} + \frac{A_N^2}{A_P^2 - A_N^2} \right) \\ &= \rho g h \left(\frac{A_P^2 - A_N^2}{2(A_P^2 - A_N^2)} + \frac{2 A_N^2}{2(A_P^2 - A_N^2)} \right) \\ &= \frac{\rho g h}{2} \cdot \frac{A_P^2 - A_N^2 + 2 A_N^2}{A_P^2 - A_N^2} , \end{aligned}$$

from which the proposition follows directly by simplifying.

5. (10 pts) Fig. 4 shows a so–called venturi flow meter. Here, the main tube has a diameter D and the flow constricts to a smaller tube of diameter $D/2$. A manometer having fluid of density ρ_m indicates pressure difference between locations 1 and 2. Given lengths b and h, determine the volume flow rate Q for a fluid of density ρ if the flow can be assumed to be inviscid.

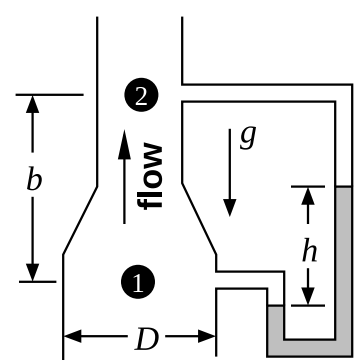

Figure 4: *Venturi flow.*

Solution: There are 3 relevant equations relating positions 1 and 2. First is the Bernoulli equation

$$P_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = P_2 + \frac{\rho V_2^2}{2} + \rho g z_2 ,$$

which we can simplify by choosing the reference elevation as $z_1 = 0$, whereby $z_2 = b$. Then

$$P_1 + \frac{\rho V_1^2}{2} = P_2 + \frac{\rho V_2^2}{2} + \rho g b .$$

Next is conservation of mass, $V_1 A_1 = V_2 A_2$, written more exactly as

$$V_1 \frac{\pi D^2}{4} = V_2 \frac{\pi (D/2)^2}{4} , \quad \text{which simplifies to} \quad V_2 = 4 V_1 .$$

Finally, we have the manometer equation. For convenience, let's take x_1 as the vertical distance between location 1 and the lower manometer fluid level and x_2 as the vertical distance between location 2 and the upper manometer fluid level. Then

$$P_1 + \rho g x_1 - \rho_m g h - \rho g x_2 = P_2 , \text{ which simplifies to } P_1 + \rho g(x_1 - x_2) - \rho_m g h = P_2 .$$

Note that $b + x_1 = h + x_2$, whereby $x_1 - x_2 = h - b$. The manometer equation can then be further simplified to

$$P_1 + \rho g(h - b) - \rho_m g h = P_2 ,$$

doing away with x_1 and x_2, which turn out not to be relevant to the problem. Now we can substitute V_2 from conservation of mass and P_2 from the manometer equation into the Bernoulli equation, finding

$$P_1 + \frac{\rho V_1^2}{2} = P_1 + \rho g(h - b) - \rho_m g h + \frac{\rho (4 V_1)^2}{2} + \rho g b .$$

This simplifies in several steps

$$\frac{\rho V_1^2}{2} = (\rho - \rho_m) g h + \frac{\rho 16 V_1^2}{2}$$

$$\frac{15 \rho V_1^2}{2} = (\rho_m - \rho) g h \quad \rightarrow \quad V_1 = \sqrt{\frac{2 (\rho_m - \rho) g h}{15 \rho}}$$

from which we can calculate $Q = V_1 A_1$ as

$$Q = \frac{\pi D^2}{4} \sqrt{\frac{2 (\rho_m - \rho) g h}{15 \rho}} .$$