JME 3700
February 10, 2017

Homework #4 Solutions

1. (10 pts) Viscous effects alter a uniform flow within a pipe roughly according to that shown in Fig. 1, i.e. from uniform at the inlet ($x = 0$) to a fully–developed profile downstream ($x \gg L$). Suppose that, near the wall of the pipe, the velocity component along the axis, i.e. in the x–direction, evolves approximately as

$$u \;=\; u_0 \, e^{-x/L} \;,$$

where u_0 and L are some constant velocity and length references. Assuming here that other velocity components are negligible, determine the acceleration of a fluid element along this axis and comment whether acceleration is positive or negative and where (i.e. value of x) its magnitude is maximum.

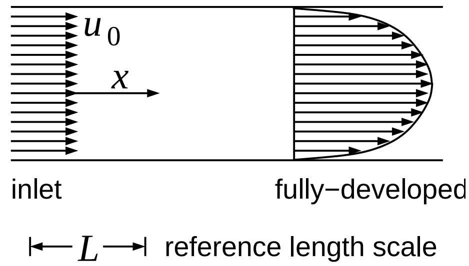

Figure 1: *developing pipe flow*

Solution: We would typically start with the so–called total derivative for u:

$$a_x \;=\; \frac{\partial u}{\partial t} \;+\; u\,\frac{\partial u}{\partial x} \;+\; v\,\frac{\partial u}{\partial y} \;,$$

but since u in the problem statement is not a function of t and since v is said to be negligible, we find

$$a_x \;=\; u\,\frac{\partial u}{\partial x} \;=\; u_0\,e^{-x/L}\cdot\left(-\,\frac{1}{L}\right)u_0\,e^{-x/L} \;=\; -\,\frac{u_0^2}{L}\,e^{-2x/L} \;,$$

with the negative sign indicating negative acceleration, i.e. "deceleration", and the maximum magnitude being u_0^2/L at the inlet, i.e. at $x = 0$.

2. (10 pts) In a certain case, the steady–state two–dimensional velocity field is given by

$$\mathbf{V} \;=\; u\,(x,y)\,\hat{i} \;+\; v\,(x,y)\,\hat{j} \;,$$

where $u\,(x,y) = x - y$ and $v\,(x,y) = xy^2 - 27$. Find the location of any stagnation points, i.e. points where the velocity vanishes.

Solution: Stagnation points imply $u = v = 0$, so we simply need to determine where these relations are satisfied for the specific u and v given in the problem. From $u = x - y = 0$, we immediately see that any stagnation points, if they exist, must lie along the line $x = y$. Starting with $v = xy^2 - 27 = 0$, we can substitute $x = y$, which gives $x^3 - 27 = 0$. Solving, we find $x = 3$. Consequently, given the requirement of $x = y$, there is a single stagnation point at $(x, y) = (3, 3)$.

3. (10 pts) We have a velocity field whose form is $\mathbf{V} = u(x,y)\,\hat{\imath} + v(x,y)\,\hat{\jmath}$, where $u(x,y) = C\,x^2$ and $v(x,y) = C\,y^2$ and where $C \neq 0$ is a constant. Determine the two components of the acceleration, a_x and a_y, and any point(s) where the acceleration vanishes.

   Solution: This is a matter of applying the definition of the acceleration according to the Eulerian description of motion in two dimensions, i.e.

$$a_x \;=\; \frac{\partial u}{\partial t} \,+\, u\,\frac{\partial u}{\partial x} \,+\, v\,\frac{\partial u}{\partial y} \qquad \text{and} \qquad a_y \;=\; \frac{\partial v}{\partial t} \,+\, u\,\frac{\partial v}{\partial x} \,+\, v\,\frac{\partial v}{\partial y}\,.$$

   Taking appropriate derivatives, we find

$$a_x \;=\; 0 \,+\, C\,x^2\,(2\,C\,x) \,+\, C\,y^2\,(0) \;=\; 2\,C^2\,x^3$$

   and

$$a_y \;=\; 0 \,+\, C\,x^2\,(0) \,+\, C\,y^2\,(2\,C\,y) \;=\; 2\,C^2\,y^3\,.$$

   Since $C \neq 0$, the only point at which $a_x = a_y = 0$ is at $(x,y) = (0,0)$.

4. (10 pts) The steady–state two–dimensional velocity field for a flow is given by

$$\mathbf{V} \;=\; u(x,y)\,\hat{\imath} \,+\, v(x,y)\,\hat{\jmath}\,,$$

   where $u(x,y) = 2\,x^2\,y$ and $v(x,y) = -2\,x\,y^2$. Show that streamlines can be expressed in the form $x\,y = C_1$, where C_1 is some constant.

   Solution: This can be done by using the fact that the slope of the streamline dy/dx is the same as the slope of the velocity vector v/u, i.e. that velocity is *tangent* to the streamlines. Substituting the velocities, we find

$$\frac{dy}{dx} \;=\; \frac{v}{u} \;=\; -\,\frac{2\,x\,y^2}{2\,x^2\,y} \;=\; -\,\frac{y}{x} \qquad \text{which separates as} \qquad \frac{dy}{y} \;=\; -\,\frac{dx}{x}\,.$$

   We can integrate as

$$\int \frac{dy}{y} \;=\; -\int \frac{dx}{x} \qquad \text{to find} \qquad \ln y \;=\; -\,\ln x \,+\, C_0\,,$$

   where C_0 is a constant. Exponentiating, we can write this expression as

$$\begin{aligned} e^{\ln y} &= e^{-\ln x + C_0} \\ &= e^{-\ln x}\,e^{C_0} \\ y &= \frac{C_1}{x}\,, \end{aligned}$$

   where $C_1 = e^{C_0}$ is still a constant. Clearly then, we have $x\,y = C_1$, as the problem asked us to show.

5. (10 pts) The temperature distribution in a certain fluid is $T = C_0 t + C_1 x + C_2 y$, where C_0, C_1, and C_2 are constants. Find the expression for the time rate of change of temperature for a differential fluid element moving in a flow whose velocity is $\mathbf{V} = C_1 y\,\hat{\imath} + C_2 x\,\hat{\jmath}$.

Solution: The total derivative of temperature is

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y}\,.$$

Taking appropriate derivatives, we find

$$\frac{DT}{Dt} = C_0 + C_1\,y\,(C_1) + C_2\,x\,(C_2) = C_0 + C_1^2\,y + C_2^2\,x\,.$$