JME 3700
February 18, 2017

Homework #5 Solutions

1. (10 pts) The flow in a semi–infinite domain $0 \le x < \infty$ and $0 \le y < \infty$ has the velocity distribution

$$\mathbf{V} \;=\; \frac{u_0\, x}{L}\,\hat{\imath} \;-\; \frac{u_0\, y}{L}\,\hat{\jmath} \;+\; 0\,\hat{k}\,,$$

where u_0 and L are constant velocity and length scales. A cubic "box"–type control volume of edge length L is situated as shown in Fig. 1. Demonstrate that the volumetric flow rates are the same for the top face and the right face, i.e. $|Q_t| = |Q_r|$ and comment on their sign difference.

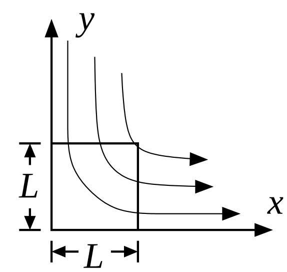

Figure 1: *Cubic CV.*

Solution: We derive Q_t and Q_r directly, according to the volumetric flow rate definition. For the top face, the outward unit normal is $\hat{n} = \hat{\jmath}$ and the y component of the velocity is $-u_0$ at $y = L$, whereby

$$\begin{aligned} Q_t &= \iint_A \mathbf{V}\cdot\hat{n}\,dA \\ &= \int_0^L\!\!\int_0^L \left(\frac{u_0\,x}{L}\,\hat{\imath} \,-\, u_0\,\hat{\jmath} \,+\, 0\,\hat{k}\right)\cdot\left(0\,\hat{\imath} \,+\, 1\,\hat{\jmath} \,+\, 0\,\hat{k}\right) dx\,dz \\ &= -\,u_0 \int_0^L dx \int_0^L dz \;=\; -\,u_0\,L^2\,. \end{aligned}$$

For the right face, the outward unit normal is $\hat{n} = \hat{\imath}$ and the x component of the velocity is u_0 at $x = L$, whereby

$$\begin{aligned} Q_r &= \iint_A \mathbf{V}\cdot\hat{n}\,dA \\ &= \int_0^L\!\!\int_0^L \left(u_0\,\hat{\imath} \,-\, \frac{u_0\,y}{L}\,\hat{\jmath} \,+\, 0\,\hat{k}\right)\cdot\left(1\,\hat{\imath} \,+\, 0\,\hat{\jmath} \,+\, 0\,\hat{k}\right) dx\,dz \\ &= u_0 \int_0^L dx \int_0^L dz \;=\; u_0\,L^2\,. \end{aligned}$$

Clearly, $|Q_t| = |Q_r|$ and the sign difference signifies flow *into* the cube at the top face and *out of* the cube at the right face.

2. (10 pts) A trough of length L having an isosceles–triangle cross–section of width and height B and H, respectively, is being filled with liquid through a submerged inflow port at a constant volumetric flow rate Q (Fig. 2). The liquid level height at the instant shown is h. Intuitively, we see the rate at which the level rises, dh/dt, will not be constant. It will be faster for small values of h and will slow down as h becomes larger. Determine dh/dt as a function of H, B, L, h, and Q, and show that, as the liquid nears the top, i.e. $h \to H$, the rate of rise approaches $dh/dt \to Q(B\,L)^{-1}$.

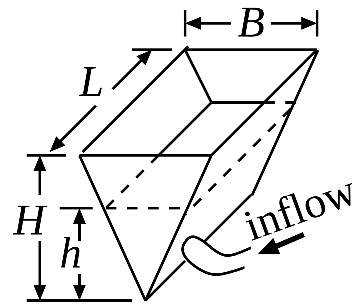

Figure 2: *Filling trough.*

Solution: Take the control volume as the entire trough and start with the general form of the continuity (conservation of mass) equation *for the liquid* (we ignore the air, assuming that the two fluids do not mix) as

$$\frac{\partial}{\partial t}\iiint_V \rho\, dV \;+\; \iint_A \rho\, \mathbf{V}\cdot\mathbf{n}\, dA \;=\; 0\,.$$

We can factor the density, such that

$$\frac{\partial}{\partial t}\iiint_V dV \;+\; \iint_A \mathbf{V}\cdot\mathbf{n}\, dA \;=\; 0\,,$$

where the first term is now the rate of change of the volume of liquid within the tank (control volume) and the second is the volumetric rate crossing the control volume boundary. Since the latter only occurs at one place (the submerged inflow port), we can write the latter intergral directly in terms of Q, i.e.

$$\frac{dV}{dt} \;-\; Q \;=\; 0\,.$$

Say we define a new variable b as the width of the liquid level, whereby the volume of liquid in the tank at the instant shown is readily calculated as $V = b\,h\,L/2$. However, because of *similar triangles*, we can write

$$\frac{H}{B} \;=\; \frac{h}{b}\,,$$

so that $b = h\,B/H$ and we can eliminate b so that

$$V \;=\; \frac{b\,h\,L}{2} \;=\; \frac{B\,L}{2\,H}\,h^2\,.$$

We can now use this expression in the conservation law above, which gives

$$\frac{d}{dt}\left(\frac{B\,L}{2\,H}\,h^2\right) \;=\; Q\,,$$

or, carrying through the derivative operation,

$$\frac{h\,B\,L}{H}\cdot\frac{dh}{dt} \;=\; Q \qquad \to \qquad \frac{dh}{dt} \;=\; \frac{Q\,H}{h\,B\,L}\,.$$

It is immediately clear from the form of the expression that

$$\frac{dh}{dt} \to \frac{Q}{B\,L} \quad \text{as} \quad h \to H$$

thus confirming the problem statement.

3. (10 pts) Using the conservation of mass equation, demonstrate that the total time required to fill the trough described in Problem 2, if starting from a completely empty condition, is

$$t_f \;=\; \frac{B\,H\,L}{2\,Q}\,.$$

Solution: Let us start with the conservation of mass equation in the form

$$\frac{dV}{dt} \;-\; Q \;=\; 0$$

that we developed in Problem 2. This is a *separable* differential equation that can be written as $dV = Q\,dt$ and integrated from the lower bound of an "empty state" at $t = 0$ to a "full state" at $t = t_f$ as

$$\int_0^V dV' \;=\; \int_0^{t_f} Q\,dt \qquad \text{which yields} \qquad t_f \;=\; \frac{V}{Q}\,.$$

The volume of the full trough is $V = B\,H\,L/2$, from which we see the problem statement is satisfied.

4. (10 pts) Liquid flowing in a pipe of radius R has an unusual axial velocity profile shown in Fig. 3,

$$u = \left[1 - \left(\frac{r}{R}\right)^3\right]\,V_{max}$$

where the maximum velocity V_{max} occurs at the axis, i.e. at $r = 0$. The velocity at the inner wall of the pipe, $r = R$, vanishes. Find an expression for the average velocity V_{avg} in terms of the maximum velocity.

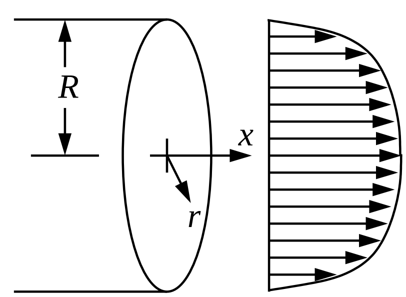

Figure 3: *Pipe flow profile.*

Solution: The velocity profile stated in the problem can be written formally as $\mathbf{V} = u\,\hat{i} + 0\hat{j}$, i.e. it is parallel to the x–direction. The average velocity is defined as $V_{avg} = Q/A$, where A is the cross–sectional area, i.e. oriented as $\mathbf{n} = 1\,\hat{i} + 0\,\hat{j}$ (perpendicular to the flow). Invoking the definition of the volume flow rate, we can write

$$V_{avg} \;=\; \frac{1}{A}\iint\limits_A \mathbf{V}\cdot\mathbf{n}\,dA\,.$$

We must simply integrate this profile over the circular pipe cross section. Taking a differential "ring" element of $dA = 2\,\pi\,r\,dr$, we find

$$\begin{aligned} V_{avg} &= \frac{1}{A}\int_0^R \left[1-\left(\frac{r}{R}\right)^3\right] V_{max}\cdot\,2\,\pi\,r\,dr \\ &= \frac{1}{\pi\,R^2}\,2\,\pi\,V_{max}\int_0^R \left[1-\left(\frac{r}{R}\right)^3\right] r\,dr \\ &= \frac{2\,V_{max}}{R^2}\left(\frac{r^2}{2} - \left.\frac{r^5}{5\,R^3}\right|_0^R\right) \\ &= \frac{2\,V_{max}}{R^2}\left(\frac{R^2}{2} - \frac{R^2}{5}\right) \\ &= \frac{2\,V_{max}}{R^2}\cdot\frac{3\,R^2}{10} \\ V_{avg} &= \frac{3\,V_{max}}{5}\,. \end{aligned}$$

5. (10 pts) A 2–D flow emanates obliquely at an angle θ from the top of a fluidic device, as shown in Fig. 4 with a velocity distribution of

$$\mathbf{V} \;=\; U_0\left(1\;-\;\frac{x}{L}\right)\cdot\left(\cos\theta\;\hat{\imath}\;+\;\sin\theta\;\hat{\jmath}\right)\;,$$

where U_0 is a velocity reference and L is the width of the outlet. The outlet area is horizontal, being aligned with the x axis, and there is no variation in the depth dimension "into the paper". If this outlet has a depth dimension of w and the fluid has a density ρ, determine the mass flow rate, $\dot{m}$, out of the device.

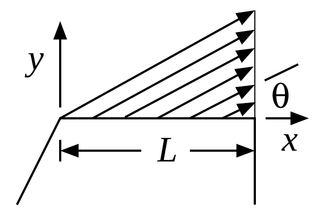

Figure 4: *Outlet*

Solution: The mass flow rate through the inlet is given by the general expression

$$\dot{m} \;=\; \iint\limits_A \rho\,\mathbf{V}\cdot\mathbf{n}\,dA\,.$$

The outward–facing unit normal is perpendicular to the outlet, i.e. $\mathbf{n} \;=\; 0\,\hat{\imath}\;+\;1\,\hat{\jmath}$, and there is no need to integrate in the depth dimension, since the problem statement indicates there is no variation. Consequently, we take $dA \;=\; w\cdot dx$ and integrate over $0 \le x \le L$ to find

$$\begin{aligned} \dot{m} &= \int_0^L \rho\;U_0\left(1\;-\;\frac{x}{L}\right)\left(\cos\theta\;\hat{\imath}\;+\;\sin\theta\;\hat{\jmath}\right)\;\cdot\;\left(0\,\hat{\imath}\;+\;1\,\hat{\jmath}\right)\;w\,dx \\ &= \rho\,U_0\,w\int_0^L\left(1\;-\;\frac{x}{L}\right)(0\,\cos\theta\;+\;1\,\sin\theta)\;dx \;=\; \rho\,U_0\,w\,\sin\theta\int_0^L\left(1\;-\;\frac{x}{L}\right)\;dx \\ &= \rho\,U_0\,w\,\sin\theta\left.\left(x\;-\;\frac{x^2}{2\,L}\right)\right|_0^L \;=\; \rho\,U_0\,w\,\sin\theta\left(L\;-\;\frac{L^2}{2\,L}\right) \\ &= \frac{\rho\,U_0\,w\,L\,\sin\theta}{2}\,. \end{aligned}$$