JME 3700
February 25, 2017

Homework #6 Solutions

1. (10 pts) A vane is mounted on a frictionless pin and smoothly redirects a steady horizontal jet of water flowing in the $+x$ direction *downward* to flow in the $-y$ direction (Fig. 1, vane is cross–hatched). Reaction forces at the pin are F_x and F_y and the velocity and cross–sectional area of the water stream are u and A, respectively. Assume potential energy changes do not play a role in this case. If the control volume is taken as the dashed surface and the combined weight of the vane and the water inside the control volume at the instant shown is W, determine u such that $F_y = 0$.

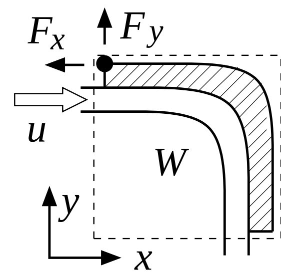

Figure 1: *Flow redirector*

Solution: Since the flow is steady–state, the unsteady $\partial/\partial t$ term vanishes from the momentum equation. While mass only crosses at the inlet and outlet, there is not *vertical* component of momentum at the inlet, so we can ignore that part. Consequently, the relevant form of the momentum equation in this instance is

$$\sum F_y = \iint_{\text{outlet}} u\,\rho\,\mathbf{V}\cdot\mathbf{n}\,dA\,,$$

where the forces are F_y and W. Note that we do not have any sort of velocity distribution, so we do not even formally do the integral, but rather do a straight multiply. Properly attending to the signs of the forces and of the dot product and noting that the velocity at the outlet is $-u$, we find

$$F_y - W = (-u)\cdot\rho\cdot(+u)\cdot A = -\rho\,u^2\,A\,,$$

so that, if $F_y = 0$, we find

$$W = \rho\,u^2\,A\,,$$

$$u = \sqrt{\frac{W}{\rho\,A}}$$

2. (10 pts) The device in Fig. 2 has an inlet at $x = 2\,m$ that is $0.1m$ tall ("surface 1") and an outlet at $y = 2m$ that is $0.1\,m$ wide ("surface 2"). It extends uniformly $1m$ "into the paper". A fluid of density ρ has a motion within the device and on its boundaries given by the steady–state velocity distribution

$$\mathbf{V} \;=\; -\,C\,x\,\hat{i} \;+\; C\,y\,\hat{j}\,,$$

where C is a constant having units of sec^{-1} and x and y are in units of m. Determine the reaction force, F_x, in the x–direction (magnitude and direction) needed to hold the device in place. *Hint:* Motion is tangent along the shaded surfaces, i.e. no fluid crosses those parts of the boundary.

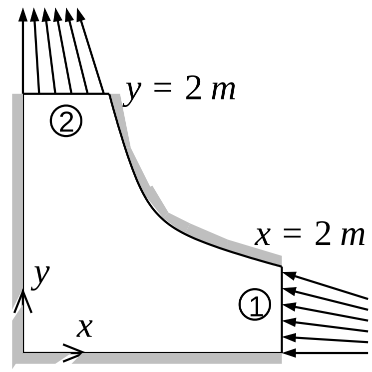

Figure 2: *Flow redirector*

Solution: Take the device itself as the control volume and assume F_x points to the right (positive x). Since the flow is steady–state, the unsteady $\partial/\partial t$ term vanishes from the momentum equation. Moreover, mass only crosses at surfaces 1 and 2, so the relevant form of the momentum equation in this instance is

$$F_x \;=\; \iint_1 u\,\rho\,\mathbf{V}\cdot\mathbf{n}\,dA \;+\; \iint_2 u\,\rho\,\mathbf{V}\cdot\mathbf{n}\,dA\,.$$

At surfaces 1 and 2, we have respective velocity, outward unit normal vectors, and differential areas of

$$\begin{array}{llll} \text{surface 1:} & \mathbf{V} \;=\; -\,2\,C\,\hat{i} \;+\; C\,y\,\hat{j} & \mathbf{n} \;=\; 1\,\hat{i} \;+\; 0\,\hat{j} & dA \;=\; 1\cdot dy \\ \text{surface 2:} & \mathbf{V} \;=\; -\,C\,x\,\hat{i} \;+\; 2\,C\,\hat{j} & \mathbf{n} \;=\; 0\,\hat{i} \;+\; 1\,\hat{j} & dA \;=\; 1\cdot dx\,, \end{array}$$

whereby the momentum equation is solved as

$$\begin{aligned} F_x &= \int_0^{0.1} (-\,2\,C)\,\rho\,(-\,2\,C)\cdot 1\,dy \;+\; \int_0^{0.1} (-\,C\,x)\,\rho\,(+\,2\,C)\cdot 1\,dx \\ &= 4\,\rho\,C^2\int_0^{0.1} dy \;-\; 2\,\rho\,C^2\int_0^{0.1} x\,dx \\ &= 4\,\rho\,C^2\,y\,\Big|_0^{0.1} \;-\; 2\,\rho\,C^2\;\frac{x^2}{2}\,\Big|_0^{0.1} \\ &= 0.4\,\rho\,C^2 \;-\; 0.01\,\rho\,C^2 \qquad = \; 0.39\,\rho\,C^2\,. \end{aligned}$$

The result is positive, meaning our original assumption of the force being in the positive x direction was correct.

3. (10 pts) Liquid of density ρ hits a cart on frictionless wheels at speed v and is deflected, as shown in Fig. 3. Specifically, the stream splits evenly, where the cross–sectional area of the downward flow and that of the flow at angle θ are each exactly half of the area of the incoming stream, A. Velocities of the two split streams each remain v. If the cart's motion is resisted by a linear–elastic (Hookean) spring having a spring constant k, determine the length h by which the spring compresses, relative to its natural length. Comment on whether there exists a θ for which $h = 0$. Assume steady flow.

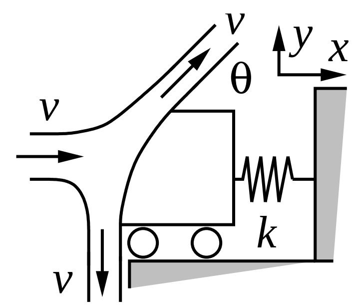

Figure 3: *Cart and spring.*

Solution: This problem depends upon the horizontal component of momentum

$$\Sigma F_x = \iint u \, \rho \, \mathbf{V} \cdot \hat{n} \, dA \, ,$$

where u is the velocity component in the x direction. Note that we have already omitted the $\partial/\partial t$ term, which vanishes because the problem is steady. The sole horizontal force on the control volume is the spring compression in the $-x$ direction. Given that the spring is Hookean, its magnitude is simply $h\,k$, so that the momentum equation takes the form

$$-h\,k = \underbrace{\Big(+v\Big)\,\rho\,\Big(-v\Big)\,A}_{\text{inlet stream}} + \underbrace{\Big(0\Big)\,\rho\,\Big(+v\Big)\,\frac{A}{2}}_{\text{downward stream}} + \underbrace{\Big(+v\,\cos\theta\Big)\,\rho\,\Big(+v\Big)\,\frac{A}{2}}_{\text{stream at angle }\theta} \, .$$

Note there is no horizontal momentum component for the downward stream. Solving for the spring deflection, h, we find

$$h = \frac{\rho\,v^2\,A}{k}\left(1 - \frac{\cos\theta}{2}\right) \, .$$

The case of no deflection, $h = 0$, implies

$$\left(1 - \frac{\cos\theta}{2}\right) = 0 \, ,$$

since the other variables are all positive, finite constants. However, this condition further implies $\cos\theta = 2$, which obviously has no solution. The scenario of no deflection cannot be made to exist for this case.

4. (10 pts) A barge dredges a river channel as shown in Fig. 4. The sand/water mixture (specific gravity $S_G = 1.3$) from the riverbed enters the barge bottom vertically and is discharged onto the shore from a nozzle at an angle $\theta = 30^o$ to the horizontal at a steady velocity of $V = 10\, m/s$. Calculate the tension T in the horizontal cable anchored to shore if the nozzle has a cross–sectional area of $A = 1\, m^2$.

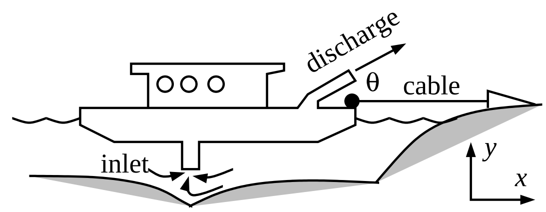

Figure 4: *River dredging.*

Solution: This problem depends upon the horizontal component of momentum

$$\Sigma F_x \;=\; \iint u\, \rho\, \mathbf{V} \cdot \hat{n}\, dA\,,$$

where u is the velocity component in the x direction. Note that we have already omitted the $\partial/\partial t$ term, which vanishes because the problem is steady. We intuitively see that the inlet plays no role here because there is no component of horizontal momentum transfer. Taking the outline of the barge as the control volume, we must evaluate the flux term for horizontal momentum only for the nozzle discharge. Assuming the cable tension is a force in the $+x$ direction and neglecting self–canceling forces due to hydrostatic pressure on the barge below the water–line, the momentum equation becomes

$$T \;=\; (+V\cos\theta)\, \rho_{\text{mixture}}\, (+V)\;\, A \;=\; V^2\, S_G\, \rho\, A\, \cos\theta$$

Substituting values, we find

$$T \;=\; 10^2 \cdot 1.3 \cdot 1000 \cdot 1 \cdot \cos 30 \;\approx\; 1.126 \times 10^5\, N\,.$$

5. (10 pts) A control volume is defined by a cube of edge length $1\, m$, i.e. a “unit cube”, situated as shown in Fig. 5. For a flow having a velocity field $\mathbf{V}(x, y, t) = C_1\, x\, t\, \hat{\imath} - C_2\, y\, \hat{\jmath}$, find the instantaneous rate of change of momentum in the x direction *within* the cube at any instant in time, i.e.

$$\sum F_x \;=\; \underbrace{\frac{\partial}{\partial t} \iiint_V u\, \rho\, dv}_{\text{this term.}} + \frac{\partial}{\partial t} \iint_A u\, \rho\, \mathbf{V} \cdot \mathbf{n}\, dA$$

Here, C_1 and C_2 are constants having units of sec^{-2} and sec^{-1}, respectively.

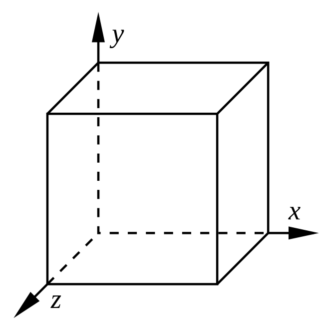

Figure 5: *Unit cube.*

Solution: The problem is asking us to evaluate the specific term

$$\frac{\partial}{\partial t} \iiint_V u\, \rho\, dv\,,$$

where u is the velocity component in the x direction, in the x momentum equation over the unit cube control volume. Note that the velocity field $u = C_1\, x\, t$ is *not* constant, however, density and C_1 are (they can be moved outside both the integral and differential). Evaluation is then accomplished in the following, rather mechanical way

$$\begin{aligned}
\frac{\partial}{\partial t}\iiint_V u\,\rho\,dv &= C_1\,\rho\,\frac{\partial}{\partial t}\int_0^1\int_0^1\int_0^1 (x\,t)\;dx\,dy\,dz \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\int_0^1\int_0^1 \left(\frac{x^2\,t}{2}\right)\Big|_0^1\,dy\,dz \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\int_0^1\int_0^1 \left(\frac{t}{2}\right)\,dy\,dz \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\int_0^1 \left(\frac{t\,y}{2}\right)\Big|_0^1\,dz \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\int_0^1 \left(\frac{t}{2}\right)\,dz \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\left(\frac{t\,z}{2}\right)\Big|_0^1 \\
&= C_1\,\rho\,\frac{\partial}{\partial t}\left(\frac{t}{2}\right) \\
&= \frac{C_1\,\rho}{2}\,.
\end{aligned}$$

Note that the units of the answer would be in Newtons, since the rest of the problem is given in terms of primary SI units.