JME 3700
March 3, 2017

Homework #7 Solutions

1. (10 pts) Long distance pipelines can have in–line pumps periodically spaced to step–up line pressures degraded by frictional losses. Consider a single such pump with a small approach and exit length of pipe and assume this pump does not change the flow rate, Q, in the line (Fig. 1). If the local losses due to pump friction and flow dissipation are $h_L = K\,V^2/2$, where V is the average flow velocity in the pipe and K is a unitless loss coefficient characteristic of this type of pump, show that the pump's power consumption is

$$\dot{W}_s \;=\; Q\left(K\,\frac{\rho\,V^2}{2} \;+\; \Delta P\right)$$

where ρ is the liquid density and ΔP is the amount of pressure step–up.

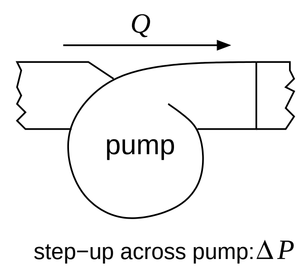

Figure 1: *Step–up pump.*

Solution: We can write the generalized Bernoulli equation across the pump as

$$\frac{P_1}{\rho} \;+\; \frac{V_1^2}{2} \;+\; g\,z_1 \;+\; h_S \;-\; h_L \;=\; \frac{P_2}{\rho} \;+\; \frac{V_2^2}{2} \;+\; g\,z_2\,.$$

We assume the line pipe diameter stays the same, whereby conservation of mass implies $V_1 = V_2 = V$, so the two kinetic energy terms drop out. Elevation changes are also evidently negligible here, and if we take $\Delta P = P_2 - P_1$ and note that $h_S = \dot{W}_s/\dot{m}$, we can re–write the equation as

$$\begin{aligned} h_S &= h_L \;+\; \frac{P_2 \;-\; P_1}{\rho} \\ \frac{\dot{W}_s}{\dot{m}} &= K\,\frac{V^2}{2} \;+\; \frac{\Delta\,P}{\rho}\,. \end{aligned}$$

The proposition follows directly when we substitute the $\dot{m} = \rho\,Q$ and solve for the power consumption, i.e.

$$\dot{W}_s \;=\; \rho\,Q\left(K\,\frac{V^2}{2} \;+\; \frac{\Delta\,P}{\rho}\right) \;=\; Q\left(K\,\frac{\rho\,V^2}{2} \;+\; \Delta\,P\right)\,.$$

2. (10 pts) Fluid of density ρ flows from a large pressurized flask through a pipe of length L that is tilted at an angle θ (Fig. 2). If the flow exits to the atmosphere and if viscous losses in the pipe have the form $CV_2^2/2$ (where C is an empirically–based constant), show that the exit velocity is

$$V_2 = \sqrt{\frac{2(P_1 - \rho\, g\, L\, \sin\theta)}{\rho\,(1+C)}}\;.$$

Gravity acts in the $-z$ direction, as shown.

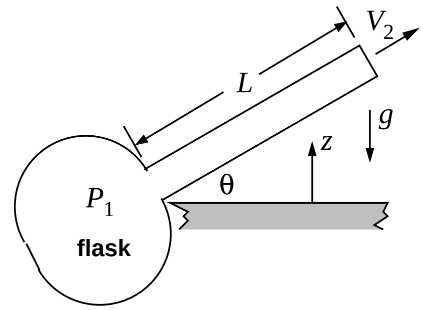

Figure 2: *Angled pipe.*

Solution: There is clearly no shaft work in this configuration, so let us write the generalized Bernoulli equation without this term in the following energy–per–unit–mass form

$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + g\, z_1 - h_L = \frac{P_2}{\rho} + \frac{V_2^2}{2} + g\, z_2\,.$$

Choosing the horizontal datum as our elevation reference, we take $z_1 = 0$. Also, $V_1 \approx 0$ because the flask is "large" and $P_2 = 0$, since the flow exits to the atmosphere. If we substitute the loss model, observe that $z_2 = L\, \sin\theta$ by trigonometry, and then simplify, we find

$$\frac{P_1}{\rho} - C\,\frac{V_2^2}{2} = \frac{V_2^2}{2} + g\, L\, \sin\theta\,.$$

We can solve algebraically for V_2 to find the result stated in the question.

3. (10 pts) Water of depth D flows at a uniform velocity of V in a uniform–width concrete drainage channel, as shown in Fig. 3. The flow encounters pipe, partially embedded in the channel–bed, that spans the channel at a right angle to the flow. The depth "necks down" to $D/2$ downstream of the pipe. Calculate the head loss h_L (in units of length) associated with this scenario. Hint: reference your calculation to the water's surface at the "upstream" u and "downstream" d points. Comment on any constraints your solutions suggests between D and V.

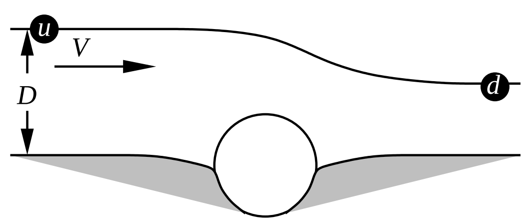

Figure 3: *Flow over a pipe obstacle.*

Solution: Head loss should be calculated in units of length, so we start with the extended Bernoulli equation in the form

$$\frac{P_u}{\rho\, g} + \frac{V_u^2}{2\, g} + z_u - h_L = \frac{P_d}{\rho\, g} + \frac{V_d^2}{2\, g} + z_d\,,$$

where we have dropped the term associated with shaft work, since that does not apply in this problem. According to the problem statement, we have $V_u = V$, $z_u = D$, and $z_d = D/2$.

Atmospheric conditions imply $P_u = P_d = 0$. Conservation of mass implies $V_u A_u = V_d A_d$, so that $V_d = 2\,V$. Substituting these quantities and solving, we find

$$\begin{aligned} h_L &= \frac{V_u^2 - V_d^2}{2\,g} + z_u - z_d \\ &= \frac{V^2 - (2\,V)^2}{2\,g} + D - \frac{D}{2} \\ &= \frac{D}{2} - \frac{3\,V^2}{2\,g}\,. \end{aligned}$$

We know that h_L cannot be negative, i.e. $h_L \geq 0$, which suggests the physical constraint $D \geq 3\,V^2/g$.

4. (10 pts) A pump at an elevation of $h_1 = 5m$ above the bottom of a very large reservoir of depth $d = 2\,m$ lifts water of density $\rho = 1000\,kg/m^3$ over a dam, pumping it into a piping system at an elevation of $h_2 = 3\;m$ (Fig. 4). The pipe diameter is $D = 0.1\;m$ and the pipe flow velocity is $V = 1\;m/s$. A pressure gauge in the pipe reads $P = 15,000\;Pa$ above atmospheric. Calculate the amount of work added to the water in units of J/kg that produces a useful mechanical effect, i.e. the difference between the "shaft work" input by the pump and the overall losses associated with this flow.

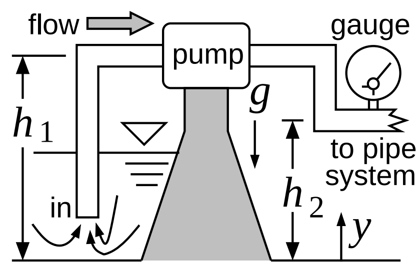

Figure 4: *Reservoir pumping.*

Solution: We can imagine a streamline connecting a point on the reservoir surface "R" to a point in the pipe system "S", for which we can write the generalized Bernoulli equation in the following energy–per–unit–mass form

$$\frac{P_R}{\rho} + \frac{V_R^2}{2} + g\,z_R - h_L + h_S = \frac{P_S}{\rho} + \frac{V_S^2}{2} + g\,z_S\,.$$

In the suggested coordinate system in the figure, $z_R = d = 2$ and $z_S = h_2 = 3$. Also, since the reservoir is said to be very large, we assume that the associated velocity is small enough to be neglected, i.e. $V_R = 0$, and we also observe that the surface is at atmospheric pressure so that $P_R = 0$. Static pressure and velocity in the pipe are as stated in the problem, i.e. $P_S = 15,000$ and $V_S = 1$. These substitutions, along with a little rearangement yields

$$\begin{aligned} \text{work that produces useful effect} &= h_S - h_L = \frac{P_S}{\rho} + \frac{V_S^2}{2} + g\left(z_S - z_R\right) \\ &= \frac{15000}{1000} + \frac{1^2}{2} + 9.8 \cdot \left(3 - 2\right) = 15 + 0.5 + 9.8 \\ &= 25.3\;\frac{J}{kg}\,. \end{aligned}$$

Evidently, the elevation of the pump, h_1, does not play a role in this issue.

5. (10 pts) If the overall losses due to entrance effects, flow friction, pump losses, etc. in the flow in Problem 4 are $h_L = 5\ J/kg$, calculate the power consumption of the pump $\dot{W}_s$ in units of Watts.

Solution: We concluded in Problem 4 that $h_S - h_L = 25.3\ J/kg$, whereby

$$h_S \;=\; 25.3 \;+\; h_L \;=\; 25.3 \;+\; 5 \;=\; 30.3\ J/kg\,.$$

The form of the mechanical "shaft work" term is $h_S = \dot{W}_s/\dot{m}$, whereby $\dot{W}_s = \dot{m} \cdot h_S$. The mass flow rate is calculated as

$$\dot{m} \;=\; \rho\, V_S\, A_S \;=\; \rho\, V_S\, \frac{\pi \cdot D^2}{4} \;=\; 1000 \cdot 1 \cdot \frac{\pi \cdot 0.1^2}{4} \;\approx\; 7.85\ kg/s\,,$$

so that $\dot{W}_s = 7.85 \cdot 30.3 \approx 238\, W$.