**JME 3700**
**March 15, 2017**

## Homework #8 Solutions

1. (10 pts) A certain flow is described by the potential function $\phi = x^3y - xy^3 + x - y + 6$. Determine the stream function $\psi$ to within some additive constant $C$ for this flow.

   *Solution:* First, let us determine what the velocity components are from the definition of the potential function

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left(x^3\,y - x\,y^3 + x - y + 6\right) = 3\,x^2\,y - y^3 + 1$$

$$v = \frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}\left(x^3\,y - x\,y^3 + x - y + 6\right) = x^3 - 3\,x\,y^2 - 1\,.$$

   We can now deduce the stream function from its definition

$$u = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v = -\frac{\partial \psi}{\partial x}\,.$$

   Substituting from the expression for $u$ above, we find

$$\frac{\partial \psi}{\partial y} = 3\,x^2\,y - y^3 + 1 \qquad \rightarrow \qquad \psi = \frac{3\,x^2\,y^2}{2} - \frac{y^4}{4} + y + f_1(x)\,,$$

   where $f_1(x)$ is some function *only of* $x$. Likewise, substituting from the expression for $v$ above, we find

$$\frac{\partial \psi}{\partial x} = -\,x^3 + 3\,x\,y^2 + 1 \qquad \rightarrow \qquad \psi = \frac{3\,x^2\,y^2}{2} - \frac{x^4}{4} + x + f_2(y)\,,$$

   where $f_2(y)$ is some function *only of* $y$. By comparing and matching the two expressions and also allowing for additive constants, we find $f_1(x) = x - x^4/4 + C_1$ and $f_2(y) = y - y^4/4 + C_2$, where $C_1$ and $C_2$ are the constants. Again by matching, we must have $C = C_1 = C_2$, whereby

$$\psi = \frac{3\,x^2\,y^2}{2} - \frac{x^4 + y^4}{4} + x + y + C\,.$$

2. (10 pts) It has been claimed that a necessary condition for describing fluid motion using a potential function is that the vorticity, or rotation, must be zero. Determine whether the motion described in Question 1 is consistent with this claim.

   *Solution:* In Question 1, we found the velocity components characterizing that flow to be

$$u = 3\,x^2\,y - y^3 + 1 \qquad \text{and} \qquad v = x^3 - 3\,x\,y^2 - 1\,.$$

   Rotation, which is the curl of the velocity vector, has only a "z" component in a 2–dimensiontal flow, i.e.

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,.$$

Evaluating, we find

$$\begin{aligned}\zeta_z &= \frac{\partial}{\partial x}\left[x^3 - 3\,x\,y^2 - 1\right] - \frac{\partial}{\partial y}\left[3\,x^2\,y - y^3 + 1\right] \\ &= \left[3\,x^2 - 3\,y^2\right] - \left[3\,x^2 - 3\,y^2\right] = 0\,,\end{aligned}$$

whereby we see that there is no rotation associated with this flow. Finally, we note that a factor of 1/2 is optional in this calculation for some definitions of vorticity, but that the final answer is the same.

3. (10 pts) A certain incompressible flow is described by the stream function $\psi = 4x^2y^2 - y^4$. Find the equation(s) of the streamline(s) passing through the origin.

*Solution:* The value of the stream function passing through the origin is $\psi = 4\cdot 0^2\cdot 0^2 - 0^4 = 0$. Consequently, any streamlines passing through $(0,0)$ are governed by the equation $4x^2y^2 - y^4 = 0$. We can determine what these curves are as

$$\begin{aligned} 4x^2y^2 - y^4 &= 0 \\ 4x^2 - y^2 &= 0 \\ y^2 &= 4x^2 \\ y &= \pm 2x\,, \end{aligned}$$

which actually turn out to be straight lines.

4. (10 pts) A particular two–dimensional velocity field is given by

$$\mathbf{V} = u_0 \sin\left(\frac{x}{L}\right)\hat{\imath} - \frac{u_0\,y}{L}\cos\left(\frac{x}{L}\right)\hat{\jmath}\,,$$

where $u_0$ and $L$ are constants having units of speed $(m/s)$ and length $(m)$, respectively. Determine the volumetric dilation rate and give a physical interpretation of the result. Also, determine the vorticity and state whether the flow is irrotational or not.

*Solution:* The dilation is simply $\nabla\cdot\mathbf{V}$, which is calculated as

$$\frac{\partial}{\partial x}\left[u_0\sin\left(\frac{x}{L}\right)\right] + \frac{\partial}{\partial y}\left[-\frac{u_0\,y}{L}\cos\left(\frac{x}{L}\right)\right] =$$

$$\frac{u_0}{L}\cos\left(\frac{x}{L}\right) - \frac{u_0}{L}\cos\left(\frac{x}{L}\right) = 0\,.$$

The vanishing dilation indicates that amount of mass in a fixed volumetric element is constant, i.e. the flow is incompressible. The vorticity in two dimensions only has a "z" component, i.e.

$$\zeta_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\,,$$

which evaluates in this case to

$$\zeta_z = \frac{u_0\,y}{L^2}\sin\left(\frac{x}{L}\right) - 0 = \frac{u_0\,y}{L^2}\sin\left(\frac{x}{L}\right)\,,$$

whereby we see that the vorticity does *not* vanish, so the flow is not irrotational.

5. (10 pts) You are trying to determine the velocity field $\mathbf{V} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$ for a particular 3–dimensional incompressible flow. A special 2–dimensional flow measurement device indicates that two of the components are $u = x^2y + xz$ and $v = -xy^2 + yz$ and then tells you that you're on your own for the third component, $w$. Determine $w$ to within an additive function $f(\,)$. Make sure to note the dependent variables of this function.

*Solution:* The flow is incompressible, so it must satisfy

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,.$$

Evaluating terms from the device's readout, we find

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(x^2y + xz\right) = 2xy + z \qquad \text{and} \qquad \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}\left(-xy^2 + yz\right) = -2xy + z\,.$$

According to the conservation statement, the remaining term must be

$$\frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = -\,(2xy + z) - (-2xy + z) = -\,2z\,.$$

The term clearly integrates to $-z^2$, however, we note that, since this is an indefinite integration, we can have an additive function $f$ that can depend on all other variables, except $z$. Therefore, we find

$$w = -z^2 + f(x, y)\,.$$