JME 3700
March 24, 2017

Homework #9 Solutions

1. (10 pts) A viscous, incompressible fluid flows steadily between two infinite parallel plates separated by a distance b (Fig. 1). Motion is driven by the plates themselves, the bottom and top plates sliding in their own planes with velocities u_o and $2u_o$, respectively. There is no pressure gradient. Determine the horizontal velocity distribution if the flow is laminar and fully–developed and the fluid has density and viscosity of ρ and μ, respectively and use this result to subsequently determine the shear stress on the bottom plate.

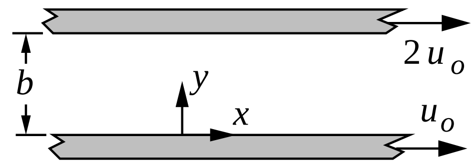

Fig. 1: *Flow between plates.*

Solution: We solve the equations of motion under the restriction of steady flow ($\partial/\partial t = 0$), fully–developed flow ($\partial/\partial x = 0$), and no pressure gradient ($\partial P/\partial x = 0$). We can write the conservation of mass equation for two–dimensions (x and y) as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \, .$$

The fully–developed assumption implies the first term drops out, leaving $dv/dy = 0$. Consequently, v is a constant, but the no–slip boundary conditions at the top and bottom imply that this constant is zero, i.e. $v = 0$.

We can now write the x momentum equation as

$$\rho \left(\frac{\partial u}{\partial t} + u \, \frac{\partial u}{\partial x} + v \, \frac{\partial u}{\partial y} \right) = - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) .$$

With the restriction above and the observation that $v = 0$, this simplifies to

$$\frac{d^2 u}{dy^2} = 0 \, ,$$

which integrates to $u = C_1 y + C_2$. Using the no–slip boundary conditions

$$u\big|_{y=b} = 2u_o \qquad \text{and} \qquad u\big|_{y=0} = u_o \, ,$$

we find $C_1 = u_o/b$ and $C_2 = u_o$, so that

$$u = u_o \left(\frac{y}{b} + 1 \right) .$$

Shear stress at the bottom can be calculated by Newton's Law of viscosity, as

$$\tau \bigg|_{y=0} = \mu \, \frac{du}{dy}\bigg|_{y=0} = \frac{\mu \, u_o}{b} \, .$$

2. (10 pts) The flow in Question 1 has no variation in the z–direction (i.e. $+z$ coming "out of the paper") and no velocity component in this direction, either. In such instances, the vorticity (which in general is a vector as is velocity) has only one component of vorticity. Determine the vorticity for the flow in Question 1 and comment on how fluid elements are "tumbling" in this flow.

Solution: Vorticity is expressed in general as

$$\nabla \times \mathbf{V} \;=\; \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} \;=\; \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} \,+\, \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} \,+\, \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}\,,$$

but clearly the only component that does not vanish in this case, including per the problem statement is the last one. Given from Question 1 that

$$u \;=\; u_o\left(\frac{y}{b} + 1\right) \qquad \text{and} \qquad v \;=\; 0\,,$$

we find that the $\hat{k}$ component of the vorticity is

$$\frac{\partial}{\partial x}\Big[0\Big] \;-\; \frac{\partial}{\partial y}\left[u_o\left(\frac{y}{b} + 1\right)\right] \;=\; -\,\frac{u_o}{b}\,.$$

The negative sign indicates the $\hat{k}$ component points "into the paper", indicating clockwise rotation, according to the right hand rule.

3. (10 pts) A particular flow is described by the potential function $\phi = C_0\,x\,y$, where C_0 is a constant. Find the corresponding stream function, ψ to within an additive constant C_1.

Solution: From the velocity potential and stream function definitions, we find

$$u \;=\; \frac{\partial \phi}{\partial x} \;=\; C_0\,y \;=\; \frac{\partial \psi}{\partial y} \qquad \rightarrow \qquad \psi \;=\; \frac{C_0\,y^2}{2} \,+\, f_1(x)$$

$$v \;=\; \frac{\partial \phi}{\partial y} \;=\; C_0\,x \;=\; -\,\frac{\partial \psi}{\partial x} \qquad \rightarrow \qquad \psi \;=\; -\,\frac{C_0\,x^2}{2} \,+\, f_2(y)\,.$$

The matching process indicates

$$f_1(x) \;=\; -\,\frac{C_0\,x^2}{2} \,+\, C_1 \qquad \text{and} \qquad f_2(y) \;=\; \frac{C_0\,y^2}{2} \,+\, C_1\,,$$

so that

$$\psi \;=\; \frac{C_0}{2}\left(y^2 - x^2\right) \,+\, C_1\,.$$

4. (10 pts) Show by either integrating the following form of the Euler equations,

$$\rho\left(u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y}\right) \;=\; -\,\frac{\partial P}{\partial x} \qquad \text{and} \qquad \rho\left(u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y}\right) \;=\; -\,\frac{\partial P}{\partial y}\,,$$

or by applying the Bernoulli equation for potential flows (neglect potential energy), that the pressure distribution in Problem 3 is

$$P \;=\; -\,\frac{\rho\, C_0^2}{2}\,\Big(x^2+y^2\Big)\;+\;C_2\,,$$

where C_2 is an additive constant.

Solution: We can write the Bernoulli equation in the form

$$P\;+\;\frac{\rho\, V^2}{2}\;=\;C_2\,,$$

where C_2 is the aforementioned constant. Since $\mathbf{V} = C_0\, y\hat{\imath} + C_0\, x\hat{\jmath}$ from Problem 3, we have $V^2 = \mathbf{V}\cdot\mathbf{V} = C_0^2(x^2+y^2)$, from which the pressure distribution given in the questions follows directly via algebra. If integrating the Euler equations, we find

$$\frac{\partial P}{\partial x}\;=\;-\,\rho\Big(C_0\,y\cdot 0 + C_0\,x\cdot C_0\Big)\;=\;-\,\rho\,C_0^2\,x \qquad\rightarrow\qquad P\;=\;-\,\frac{\rho\,C_0^2\,x^2}{2}\;+\;f_1(y)$$

$$\frac{\partial P}{\partial y}\;=\;-\,\rho\Big(C_0\,y\cdot C_0 + C_0\,x\cdot 0\Big)\;=\;-\,\rho\,C_0^2\,y \qquad\rightarrow\qquad P\;=\;-\,\frac{\rho\,C_0^2\,y^2}{2}\;+\;f_2(x)$$

The matching process indicates

$$f_1(y)\;=\;-\,\frac{\rho\,C_0^2\,y^2}{2}\;+\;C_2 \qquad \text{and} \qquad f_2(x)\;=\;-\,\frac{\rho\,C_0^2\,x^2}{2}\;+\;C_2\,,$$

so that

$$P \;=\; -\,\frac{\rho\, C_0^2}{2}\,\Big(x^2+y^2\Big)\;+\;C_2\,,$$

as stated in the problem.

5. (10 pts) Consider fully–developed, steady laminar flow in a pipe of inner radius R (Fig. 2), whose axial profile is

$$u(r)\;=\;\frac{1}{4\,\mu}\,\frac{dP}{dx}\,\left(r^2\;-\;R^2\right)\,,$$

where pressure gradient dP/dx and viscosity μ are both constants. Here, the x–axis and the flow direction are "into the paper". Determine the shear stress, τ, realized at the inner wall (at $r = R$) in terms of the volume flow rate, Q, the radius, R, and the fluid viscosity, μ.

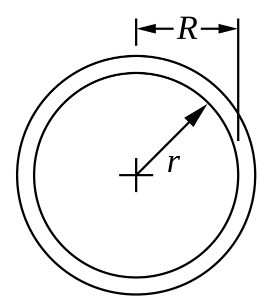

Fig. 2: *Flow in a pipe.*

Solution: As shorthand, let us define

$$C\;=\;\frac{1}{4\,\mu}\,\frac{dP}{dx}\,\cdot$$

We can evaluate both the shear stress and the volume flow rate from the velocity profile using the appropriate definitions. In particular,

$$Q \;=\; \iint\limits_{area} \mathbf{V}\cdot\vec{n}\,dA \;=\; \int_0^R C\left(r^2-R^2\right)\,2\,\pi\,r\,dr \;=\; 2\,\pi\,C\int_0^R \left(r^3 - r\,R^2\right)dr$$

$$=\; 2\,\pi\,C\left(\frac{r^4}{4}-\frac{r^2\,R^2}{2}\right)\Bigg|_0^R \;=\; 2\,\pi\,C\left(\frac{R^4}{4}-\frac{R^4}{2}\right) \;=\; -\,\frac{C\,\pi\,R^4}{2}\,.$$

Likewise,

$$\tau \;=\; \mu\,\frac{du}{dr}\bigg|_{r=R} \;=\; \mu\,C\,2\,r\big|_{r=R} \;=\; 2\,\mu\,C\,R\,.$$

We can eliminate C from these two expressions (i.e. by solving for C in the volume flow rate and substituting it into the shear stress equation) to find

$$\tau \;=\; 2\,\mu\,R\,\cdot\,\frac{-2\,Q}{\pi\,R^4} \;=\; -\frac{4\,\mu\,Q\,R}{\pi\,R^4} \;=\; -\frac{4\,\mu\,Q}{\pi\,R^3}\,.$$