JME 3700
Midterm Exam: March 20, 2017 **Name:** ____________

Solutions

1. Flow in a pipe reaches a flow splitter, which for this problem can be considered as a control volume, with flow directions indicated by non–shaded arrows (Fig. 1). The flow is steady and the velocity distribution at each of the 3 cross sections (dashed segments labeled "1" through "3") can be taken as uniform. The cross sections of areas 1, 2, and 3 are A, $A/3$, and $A/2$, respectively, and the velocities at sections 1 and 3 are V and $1.5\,V$, respectively. The fluid has a density of ρ and, at the instant of time shown, the total weight of the control volume (fluid in the splitter, plus the splitter itself) is W. Dimensions are small enough to neglect any changes of potential energy.

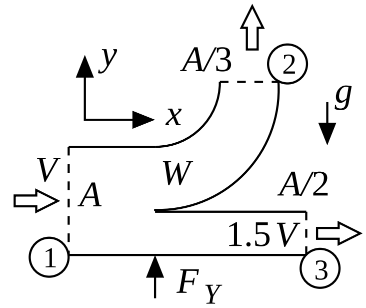

Fig. 1: *Flow splitter*

(a) (10 pts) Determine the outlet velocity, V_2, at section 2 in terms of V.

Solution: By conservation of mass, the total rates of inflow must equal the total rates of outflow. Consequently

$$\begin{aligned} A\,V &= \frac{3\,V}{2}\cdot\frac{A}{2} + V_2\cdot\frac{A}{3} \\ \frac{12\,V}{12} &= \frac{9\,V}{12} + \frac{4\,V_2}{12} \\ \frac{4\,V_2}{12} &= \frac{3\,V}{12} \qquad\qquad \therefore \quad V_2 = \frac{3\,V}{4}\,. \end{aligned}$$

(b) (10 pts) Determine the vertical component of the force, F_Y, that is required to hold the splitter in place. Include the weight of the control volume, W, in this calculation.

Solution: We can write the equation for conservation of vertical momentum in a steady flow as

$$\text{sum of forces in the } y \text{ direction} = \iint v\,\rho\,\mathbf{V}\cdot\mathbf{n}\,dA\,,$$

where v is the vertical component of velocity. Forces are F_Y and W in the positive and negative y directions, respectively. Only section 2 has vertical momentum fluxing at the control volume boundary, with the $\mathbf{V}\cdot\mathbf{n}$ product being positive, whereby

$$\begin{aligned} F_Y - W &= \left(+\frac{3\,V}{4}\right)\rho\left(+\frac{3\,V}{4}\right)\left(\frac{A}{3}\right) \\ F_Y &= \frac{3\,\rho\,V^2\,A}{16} + W\,. \end{aligned}$$

2. A contraption consists of a barrel of diameter D, partially submerged to a depth of L in a pool of liquid of density ρ (Fig. 2). There is also a horizontal gate of area $D \times D$ on a hinge that is at a depth H below the liquid free surface. Both of these elements are tied with vertical strings to a horizontal bar under the pool hinged with a fulcrum at its center. Gravity, g, points in the negative y–direction. This contraption is at static equilibrium. For the purposes of this problem, the masses of the barrel, gate, bar, and strings, as well as the friction for all hinges and pivot points are small enough to be neglected.

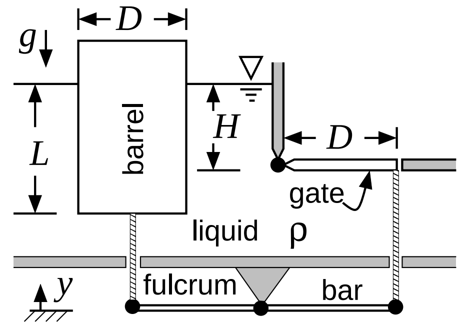

Fig. 2: *Hydrostatic contraption*

(a) (10 pts) Determine the tension in the left string, T_B, holding the barrel in place.

Solution: Since the mass of the barrel is negligible, static equilibrium dictates that the upward buoyancy force is equal to the downward tensile force. Since buoyancy is equal to the weight of the displaced liquid (Archimedes' Principle), we have

$$T_B \;=\; \rho \cdot g \cdot \text{volume} \;=\; \frac{\rho\, g\, L\, D^2}{4}\,.$$

(b) (10 pts) Determine the tension in the right string, T_G, holding the gate in place.

Solution: The hydrostatic pressure at the depth of the gate is $\rho\, g\, H$ and this pressure is uniformly distributed over the gate because the gate is horizontal. Consequently the hydrostatic force, directed upward, has a magnitude of simply the pressure times the area, i.e.

$$F_G \;=\; \rho \cdot g \cdot H \cdot \text{area} \;=\; \rho\, g\, H\, D^2\,.$$

This resultant force is applied at the middle of the gate, since the pressure distribution is uniform. Summing moments about the frictionless hinge and neglecting the weight of the gate, we have

$$T_G \cdot D \;=\; F_G \cdot \frac{D}{2}\,,$$

from which we see that

$$T_G \;=\; \frac{F_G}{2} \;=\; \frac{\rho\, g\, H\, D^2}{2}\,.$$

(c) (10 pts) There is a certain relationship between L and H that must exist for the entire contraption to be in static equilibrium. Determine that relationship.

Solution: Summing moments about the frictionless fulcrum of the bar, it is clear by inspection that $T_G = T_B$, since the fulcrum is in the center. Therefore

$$\frac{\rho\, g\, H\, D^2}{2} \;=\; \frac{\rho\, g\, L\, D^2}{4}\,,$$

implying $H = L/2$, or equivalently $L = 2\,H$.

3. An incompressible liquid of density ρ flows inviscidly and steadily in a pipe, with several reference positions along a streamline labeled (Fig. 3). At position 1, pipe diameter is D and velocity and static pressure are V_1 and P_1, respectively. At positions 2 and 3, diameter has decreased to $D/2$ and a manometer filled with a fluid of specific gravity $S_G > 1$ takes static pressure measurements at positions 1 and 2 with a differential reading of H. The flow between positions 2 and 3 drops a total elevation of L. Gravity, g, points in the negative z–direction.

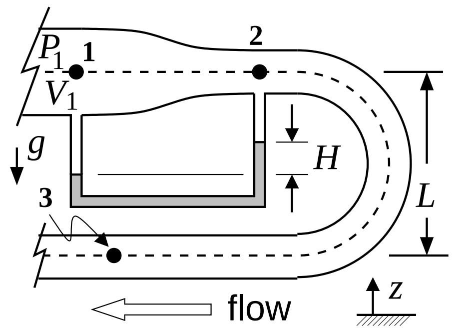

Fig. 3: *Curved pipe*

(a) (10 pts) Determine the velocity at position 2, i.e. V_2, in terms of V_1.

Solution: According to conservation of mass, the volumetric flow rate must be constant, whereby

$$\frac{\pi D^2}{4} V_1 = \frac{\pi (D/2)^2}{4} V_2 \qquad \therefore \quad V_2 = 4 V_1 .$$

(b) (10 pts) Determine the static pressure drop $\Delta P = P_1 - P_2$ between positions 1 and 2 in terms of V_1 and ρ.

Solution: Because the flow meets the 4 requirements for the Bernoulli equation, i.e. incompressible, inviscid, steady flow along a streamline, we can write this equation between positions 1 and 2 as

$$P_1 + \frac{\rho V_1^2}{2} + \rho g z_1 = P_2 + \frac{\rho V_2^2}{2} + \rho g z_2 ,$$

but the horizontal orientation means the potential energy terms vanish. Substituting $V_2 = 4 V_1$ and simplifying, we find

$$P_1 + \frac{\rho V_1^2}{2} = P_2 + \frac{\rho (4 V_1)^2}{2}$$

$$\Delta P = P_1 - P_2 = \frac{15 \rho V_1^2}{2} .$$

(c) (10 pts) Determine the reading H on the manometer in terms of V_1, and gravity, g, and the manometer fluid specific gravity, S_G.

Solution: Here, we write a manometer equation connecting positions 1 and 2. Letting y be the height from the left–side manometer fluid surface to position 1, we have

$$P_1 + \rho g y - S_G \rho g H - \rho g (y - H) = P_2 ,$$

from which we can find

$$P_1 - P_2 = (S_G - 1) \rho g H .$$

Setting this expression equivalent to ΔP from the previous question, we find

$$(S_G - 1)\,\rho\,g\,H = \frac{15\,\rho\,V_1^2}{2} \qquad \therefore \quad H = \frac{15\,V_1^2}{2\,g\,(S_G - 1)}\,.$$

(d) (10 pts) In this particular flow, we observe that the static pressure decreases from position 1 to 2 because of the contraction, but it increases from position 2 to 3 because of the elevation change. Determine the elevation change, L, in terms of V_1 and g such that the static pressures at positions 1 and 3 are equal, i.e. such that $P_1 = P_3$.

Solution: Continuing with the same streamline and writing the Bernoulli equation between positions 2 and 3, we have

$$P_2 + \frac{\rho\,V_2^2}{2} + \rho\,g\,z_2 = P_3 + \frac{\rho\,V_3^2}{2} + \rho\,g\,z_3\,.$$

We note that $V_2 = V_3$ because the pipe is of a constant diameter, meaning

$$P_3 = P_2 + \rho\,g\,(z_2 - z_3) = P_2 + \rho\,g\,L\,.$$

From above, we found that $P_2 = P_1 - 15\,\rho\,V_1^2/2$, which can be substituted to obtain

$$P_3 = P_1 - \frac{15\,\rho\,V_1^2}{2} + \rho\,g\,L\,.$$

In order for $P_1 = P_3$, the second and third terms on the right hand side must cancel each other, meaning

$$\begin{aligned} \rho\,g\,L &= \frac{15\,\rho\,V_1^2}{2} \\ L &= \frac{15\,V_1^2}{2\,g}\,. \end{aligned}$$

4. (10 pts) The standard Bernoulli equation is based on four main assumptions. List these.

Solution: The Bernoulli equation requires

1. incompressible flow
2. inviscid flow
3. steady–state flow
4. flow along a streamline