JME 3710
Midterm Exam: November 8, 2016 **Name:** ____________________

Solutions

1. A thin–walled steel tube of length $L = 10\,m$ carries a low–pressure flow of fluid at a mean temperature of $T_f = 390\,K$ "into the paper" (Fig. 1). The outer and inner surface radii are $R_o = 1.05\,m$ and $R_i = 1\,m$, respectively, and the temperature of the outer surface, i.e. at R_o, is measured as $T_o = 310\,K$. (Temperature at R_i is not presently known.) The convection coefficient on the flow side is known to be $h = 200 W/(m^2\,K)$ and the thermal conductivity of the pipe wall can be taken as $k = 20\,W/(m\,K)$. Presume that this configuration can be analyzed using the "circuit analogy" theory of 1–D steady heat transfer.

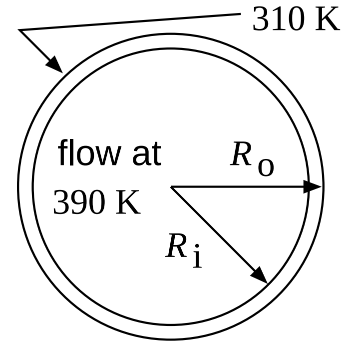

Fig. 1: Pipe flow.

(a) (10 pts) Since the wall is thin, one might assert that the formally cylindrical geometry, could (for engineering purposes) be analyzed in rectangular coordinates, i.e. as a flat sheet having the same area for heat transfer (length times circumference). Confirm that this simplification would be acceptable (say within 2% error) based on comparing the total thermal resistances of the two configurations. (For rectangular, you can use a diameter equaling twice the average of the radii in calculating circumference.)

Solution: The appropriate physical model is 2 resistors in series, one representing conduction in the wall and one representing convection in the interior of the pipe. No exterior conduction resistor is necessary here, since we know the temperature of the outside surface of the pipe. Using the appropriate formulas for resistances, we can calculate the total (equivalent) resistor for the formally cylindrical configuration as

$$R_{total,C} = \frac{\ln(R_o/R_i)}{2\,\pi\,L\,k} + \frac{1}{2\,\pi\,R_i\,L\,h} = \frac{\ln(1.05/1)}{2\cdot\pi\cdot 10\cdot 20} + \frac{1}{2\cdot\pi\cdot 1\cdot 10\cdot 200}$$
$$= 3.883\times 10^{-5} + 7.958\times 10^{-5} = 1.184\times 10^{-4}\ K/W\ ,$$

while for the rectangular simplification, with thickness $\tau = R_o - R_i = 0.05$ and diameter (for the area calculation) of $D = 2\cdot 1.025 = 2.05$, the value would be

$$R_{total,R} = \frac{\tau}{k\,\pi\,D\,L} + \frac{1}{\pi\,D\,L\,h} = \frac{0.05}{20\cdot\pi\cdot 2.05\cdot 10} + \frac{1}{\pi\cdot 2.05\cdot 10\cdot 200}$$
$$= 3.882\times 10^{-5} + 7.764\times 10^{-5} = 1.165\times 10^{-4}\ K/W\ ,$$

There is less than a 2% difference.

(b) (10 pts) Using the rectangular approximation from the previous question, calculate the total rate of heat loss of the pipe flow in units of W and comment whether thermal insulation should be used.

Solution: The circuit analog indicates $\Delta T = q\cdot R_{total}$, meaning we can write

$$q = \frac{T_f - T_o}{R_{total}} = \frac{390 - 310}{1.165\times 10^{-4}} = 686,695 \approx 686,700\,W\ .$$

This is a large rate of heat loss which could be drastically reduced using properly designed insulation.

(c) (10 pts) Calculate the temperature at the inner surface of the pipe, T_i to the nearest whole unit of K.

Solution: The inner surface temperature can be calculated by determining the interior node of the circuit. Either end can be used, i.e.

$$390 - T_i = 686,695 \cdot 7.764 \times 10^{-5} \quad \text{or} \quad T_i - 310 = 686,695 \cdot 3.882 \times 10^{-5} \,,$$

both of which can be solved algebraically to give $T_i = 337\ K$.

2. The design of an autoclave calls for a viewing window made of a special double–pane glass enclosure. Design considerations dictate that the distance between the two vertical panes is $L = 0.01\ m$ (this is the length scale of the problem) and preliminary estimations indicate the temperature difference between the inner and outer glass surfaces temperatures will be roughly $\Delta T = 200\ K$. An unusual gas having a density dependence of

$$\rho = \rho_c \, e^{-T/T_c}$$

is proposed to fill the enclosure, where ρ_c and T_c are constants. Various additional fluid properties are in Table 1.

Table 1: fluid properties

property
$T_c = 400\ K$
$\rho_c = 1\ kg/m^3$
$c = 400\ J/(kg\ K)$
$k = 50\ W/(m\ K)$
$\nu = 2 \times 10^{-5}\ m^2/s$
$\alpha = 2.5 \times 10^{-5}\ m^2/s$

(a) (10 pts) Determine the volumetric thermal expansion coefficient, β, for this fluid and comment on how it differs from that of an ideal gas.

Solution: We evaluate the definition of β directly using the fluid's density response equation

$$\beta = -\frac{1}{\rho}\frac{\partial \rho}{\partial T} = -\frac{1}{\rho}\left[-\rho\,\frac{1}{T_c}\right] = \frac{1}{T_c}\,.$$

The expansion coefficient is evidently constant for this fluid, whereas for an ideal gas it is the inverse of the gas temperature, and therefore, varies according to specific problem.

(b) (10 pts) The design calls for the window size (height) to be $H = 0.1\ m$. The nature of the double–pane enclosure means that natural convection between the panes will tend to increase the heat transfer over and above pure conduction of heat through the window. If the overall convection Nusselt number follows

$$\overline{Nu_L} = 0.22\left(\frac{Pr}{Pr + 0.2}\cdot Ra\right)^{0.28}\cdot\left(\frac{L}{H}\right)^{0.25}$$

and if pure conduction is represented as a Nusselt number of 1, calculate the increase in heat transfer in the form of the ratio of the two Nusselt numbers.

Solution: Using values from the table, the Prandtl number is

$$Pr = \frac{\nu}{\alpha} = \frac{2 \times 10^{-5}}{2.5 \times 10^{-5}} = 0.8$$

and the Rayleigh number is

$$Ra = \frac{g\,\beta\,\Delta T\,L^3}{\nu\,\alpha} = \frac{9.8\,\cdot\,0.0025\,\cdot\,200\,\cdot\,0.01^3}{2\times10^{-5}\,\cdot\,2.5\times10^{-5}} = 9800\,.$$

Incidentally, since this value is exceeds 1000, we would expect natural convection to occur. Using the given correlation, the convection Nusselt number is found to be

$$\overline{Nu_L} = 0.22\left(\frac{0.8}{0.8\,+\,0.2}\,\cdot\,9800\right)^{0.28}\cdot\left(\frac{0.01}{0.1}\right)^{0.25} \approx 1.52\,.$$

Therefore, given the ratio 1.5 : 1, the actual (convection) increases the heat transfer by about 50% as compared to pure conduction alone.

3. Investigators have examined laminar flow forced convection over a flat plate subject to a certain heat flux and have found a correlation for the *local* Nusselt number of

$$Nu_x = \frac{h_x\,x}{k} = 0.453\,Re_x^{1/2}\,Pr^{1/3}\,,$$

where $Re_x = u_\infty x/\nu$ and Pr are the Reynolds and Prandtl numbers, respectively.

(a) (10 pts) Show that the overall convection coefficient is

$$\overline{h_x} = \left(\frac{k}{x}\right)0.906\,Re_x^{1/2}\,Pr^{1/3}$$

Solution: The local convection coefficient is

$$h_x = \frac{k}{x}\,0.453\,\left(\frac{u_\infty\,x}{\nu}\right)^{1/2}Pr^{1/3} = 0.453\,k\,\left(\frac{u_\infty}{\nu}\right)^{1/2}Pr^{1/3}\,x^{-1/2}\,,$$

which can be substituted directly into the definition for the overall convection coefficient $\overline{h_x}$ to obtain

$$\begin{aligned}
\overline{h_x} &= \frac{1}{x}\int_0^x h_x\,dx && \text{definition}\\
&= \frac{1}{x}\int_0^x 0.453\,k\,\left(\frac{u_\infty}{\nu}\right)^{1/2}Pr^{1/3}\,x^{-1/2}\,dx && \text{substitute correlation}\\
&= \frac{1}{x}\,0.453\,k\,\left(\frac{u_\infty}{\nu}\right)^{1/2}Pr^{1/3}\int_0^x x^{-1/2}\,dx && \text{move constants outside}\\
&= \left(\frac{k}{x}\right)0.453\,\left(\frac{u_\infty}{\nu}\right)^{1/2}Pr^{1/3}\left(2\cdot x^{1/2}\right)\Big|_0^x && \text{evaluate integral}\\
&= \left(\frac{k}{x}\right)2\cdot0.453\,\left(\frac{u_\infty\,x}{\nu}\right)^{1/2}Pr^{1/3} && \text{simplify}\\
&= \left(\frac{k}{x}\right)0.906\,Re_x^{1/2}\,Pr^{1/3} && \text{ans.}
\end{aligned}$$

(b) (10 pts) Using the result in the previous question, determine the expression for the overall Nusselt number, $\overline{Nu_x}$.

Solution: The form of the Nusselt number (e.g. see question statement) implies the trivial algebraic solution

$$\overline{Nu_x} \;=\; \frac{\overline{h_x}\,x}{k} \;=\; 0.906\,Re_x^{1/2}\,Pr^{1/3}\,.$$

4. Heat conduction in a hypothetical electro–thermal widget is governed by the standard conduction equation, albeit one having an unusual source term

$$\underbrace{\frac{\partial T}{\partial t} \;=\; \alpha\,\frac{\partial^2 T}{\partial x^2}}_{\text{standard equation}} \;-\; \underbrace{\left(\frac{x^2}{L^2} \,+\, 2\right)\cdot\frac{\alpha\,T_0}{L^2}\cdot e^{-\frac{\alpha t}{L^2}}}_{\text{unusual source term}}\,,$$

where L and T_0 are reference length and temperature, respectively, α is thermal diffusivity, and temperature varies as $T(x,t)$. A team of engineers led by Dr. Matt Hamatishon reports the correct temperature distribution, i.e. the solution to this equation, to be

$$T(x,t) \;=\; T_0\,\frac{x^2}{L^2}\,e^{-\frac{\alpha t}{L^2}}\,.$$

(a) (10 pts) As part of the team, you are assigned to determine the general expression for the x–direction heat flux, q'', in the widget, given the thermal conductivity of its material is k.

Solution: We already have the temperature solution, so the conduction can be determined simply by applying Fourier's Law of heat conduction

$$q'' \;=\; -\,k\,\frac{\partial T}{\partial x}\,\cdot$$

Applying standard differentiation, we find

$$\frac{\partial T}{\partial x} \;=\; 2\,T_0\,\frac{x}{L^2}\,e^{-\frac{\alpha t}{L^2}}\,,$$

which means the actual heat flux is

$$q'' \;=\; -\,2\,k\,T_0\,\frac{x}{L^2}\,e^{-\frac{\alpha t}{L^2}}\,,$$

(b) (10 pts) Using your previous answer, evaluate the flux at the surface $x = 0$.

Solution: Direct substitution shows

$$q''\Big|_{x=0} \;=\; -\,2\,k\,T_0\,\frac{0}{L^2}\,e^{-\frac{\alpha t}{L^2}} \;=\; 0\,.$$

(c) (10 pts) You are a little skeptical of Dr. Hamatishon's solution and insist on checking it. Show that it does indeed satisfy the somewhat unusual 1–D time–dependent conduction equation given above.[1]

Solution: Here, it is sufficient to take the designated derivatives and confirm that they have the relationship shown in the equation. We already have $\partial T/\partial x$, so taking another derivative, we get

$$\frac{\partial^2 T}{\partial x^2} = \frac{2\,T_0}{L^2}\,e^{-\frac{\alpha t}{L^2}}\,.$$

We also get the partial derivative with respect to t as

$$\frac{\partial T}{\partial t} = -\frac{\alpha}{L^2}\,T_0\,\frac{x^2}{L^2}\,e^{-\frac{\alpha t}{L^2}}\,.$$

The solution can be proved by re–arranging the equation to show

$$\begin{aligned}\frac{\partial T}{\partial t} - \alpha\,\frac{\partial^2 T}{\partial x^2} &= -\frac{\alpha}{L^2}\,T_0\,\frac{x^2}{L^2}\,e^{-\frac{\alpha t}{L^2}} - \alpha\,\frac{2\,T_0}{L^2}\,e^{-\frac{\alpha t}{L^2}} \\ &= -\left(\frac{x^2}{L^2} + 2\right)\cdot\frac{\alpha\,T_0}{L^2}\cdot e^{-\frac{\alpha t}{L^2}}\,,\end{aligned}$$

which is indeed the source term. The equation is clearly satisfied.

[1] A relationship that may be helpful for this case is

$$\frac{d}{dt}\left(e^{f(t)}\right) = \frac{df}{dt}\cdot e^{f(t)}$$