JME 3710
October 26, 2018

Homework #4 Solutions

1. (10 pts) Spherical metallic shot having $k = 40\,W/(m \cdot K)$ and a radius of $R = 0.002\,m$ is heat–treated in a 2–step process. It first free–falls a distance H for the purpose of cooling from $425\,^oC$ to $225\,^oC$, upon which it splashes into a water bath for quenching (Fig. 1). The free–fall starts from zero initial velocity and accelerates according to $g = 9.8\,m/s^2$, during which the convection coefficient can be taken as a constant value of $h = 60\,W/(m^2 \cdot K)$. Neglecting considerations of aerodynamic resistance, determine the necessary H if the ambient air temperature is $T_\infty = 25\,^oC$ and the shot has a cooling time–constant of $\tau = 4\,s$.

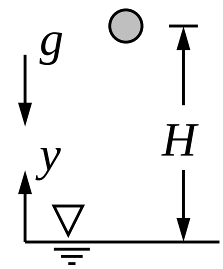

Figure 1: *Falling shot.*

Solution: The wording of the problem suggests that the capacitance model is appropriate, but we must confirm this assumption by first checking the Biot number. Given that the length scale of a sphere is $R/3$ (on the basis of volume to surface area ratio), we find

$$Bi \;=\; \frac{h\,L}{K} \;=\; \frac{h\,R}{3\,K} \;=\; \frac{60 \cdot 0.002}{3 \cdot 40} \;=\; 0.001 \ll 0.1\,,$$

which satisfies the requirement. Therefore, the temperature response is given by the solution $\theta = \theta_0\,\exp(-t/\tau)$, where $\theta(t) = T(t) - T_\infty$ is our transformed temperature. We can solve for the required drop time (time of flight) as $t = \tau\ln(\theta_0/\theta)$, from which we find

$$t \;=\; \tau\,\ln\left[\frac{T(0) - T_\infty}{T(t) - T_\infty}\right] \;=\; 4 \cdot \ln\left[\frac{425 - 25}{225 - 25}\right] \;\approx\; 2.773\,s\,.$$

We can readily determine the height H necessary to travel for a given time t in free–fall (e.g. by considering the integration for a particle under uniform acceleration) as $H = g\,t^2/2$, whereby we find

$$H \;=\; \frac{9.8 \cdot 2.773^2}{2} \;\approx\; 37.7\,m\,.$$

2. (10 pts) If the water bath is at the same ambient temperature as the air in Question 1, i.e. $T_\infty = 25\,^oC$, determine what fractions of the total initial heat energy content of the shot are dissipated during the free–fall and in the water bath, respectively.

Solution: Total heat energy transferred over some period of time in the capacitance model is

$$Q \;=\; \rho\,V\,c\,\theta_0\left(1 - e^{-t/\tau}\right)\,,$$

meaning the *fraction* is $f = 1 - e^{-t/\tau}$. From Question 1,

$$\frac{t}{\tau} \;=\; \frac{2.773}{4} \;\approx\; 0.693$$

meaning that the fraction of the total heat transfer that occurred during the free–fall step is $f \;=\; 1 - e^{-0.693} \;\approx\; 0.5$. (Note that this could have been solved *exactly* as 0.5 using the laws of logarithms.) Consequently, the other 50% of the heat is dissipated in the water bath. At the end of the process, the water, air, and shot are all in thermal equilibrium at $T_\infty = 25\,^oC$.

3. (10 pts) A boiler (partial cross–section shown in Fig. 2) has a pressure wall of thickness $L = 0.02\,m$ (shaded area) and is very well insulated on the outside (as marked). The wall is made of a material having $k = 55\,W/(m\,K)$ and $\alpha = 1.28 \times 10^{-5}\,m^2/s$ and the x–coordinate is as shown. The wall is initially at a uniform temperature of $T_o = 300\,K$, after which flow from the combustion process commences, creating conditions of $T_\infty = 1200\,K$ and $h = 1375\,W/(m^2\,K)$ within the chamber. Estimate the time (in seconds) required for every point in the wall to be at a temperature of at least $600\,K$.

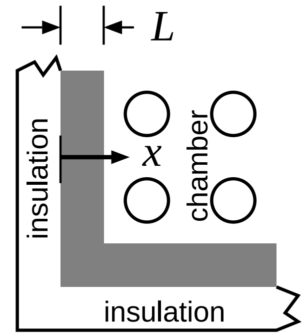

Figure 2: *Boiler.*

Solution: Intuitively, we know that, because the heating comes from the "inside", the temperature at the outer surface of the wall will lag, i.e. will be the lowest. Therefore, the problem is really to calculate the time required for the location $x^* = 0$ to reach $600\,K$. Calculating the Biot number first, here using $L = 0.02\,m$ as the characteristic length scale,

$$Bi \;=\; \frac{h\,L}{k} \;=\; \frac{1375 \cdot 0.02}{55} \;=\; 0.5 \;>\; 0.1\,,$$

meaning we cannot use the lumped capacitance analysis method. The next available method is the so–called one–term approximation to the exact solution

$$\frac{T(x,t) \;-\; T_\infty}{T_o \;-\; T_\infty} \;=\; \theta^* \;=\; C_1\, e^{-\zeta_1^2 t^*} (\cos\zeta_1 x^*)\,,$$

but this is valid only for $t^* > 0.2$. We do not know t^* *a priori*, so it would seem reasonable to solve for t^* using this method and then check to see if our solution is valid afterwords. Note that $\theta^* = (600 - 1200)/(300 - 1200) \approx 0.667$ for the temperature values stated in the problem. Since $x^* = 0$, we are actually working with the simplified solution $\theta^* = C_1\, e^{-\zeta_1^2 t^*}$, which can be solved for t^* as

$$t^* \;=\; -\,\frac{\ln(\theta^*/C_1)}{\zeta_1^2}\,,$$

where ζ_1 and C_1 are the eigenvalue and mode coefficient, respectively. Using the Biot number value of 0.5, we find $\zeta_1 = 0.6533$ and $C_1 = 1.0701$ (from the appropriate table), so that

$$t^* \;=\; -\,\frac{\ln(0.667/1.0701)}{0.6533^2} \;\approx\; 1.109\,.$$

This value is well above the threshold of 0.2 for the solution to be reasonable, so we'll accept it. It simply remains to calculate the actual time from the dimensionless expression

$$t \;=\; \frac{t^*\,L^2}{\alpha} \;=\; \frac{1.109 \cdot 0.02^2}{1.28 \times 10^{-5}} \;\approx\; 35\,sec\,.$$

4. (10 pts) Estimate the temperature of the inner surface of the wall for the configuration in Question 3.

   Solution: The inner surface is at $x^* = 1$ and, since the parameters of the so–called one–term approximation to the exact solution were already established in Question 3, we find

$$\frac{T - T_\infty}{T_o - T_\infty} = \theta^* = C_1\, e^{-\zeta_1^2 t^*} \cos(\zeta_1 x^*)$$

$$\frac{T - 1200}{300 - 1200} = 1.0701\, e^{-0.6533^2 \cdot 1.109} \cos\Big(0.6533 \cdot 1\Big) \approx 0.5293$$

$$T \approx 724\, K\,.$$

5. (10 pts) A thermocouple of main length dimension D is to be designed and constructed. In this particular problem, there are 2 choices: a spherical shape, where the diameter is D, or a cube shape, where the "superdiagonal" (distance between opposing corners) is D (Fig. 3). The main design criterion in this problem is that responsiveness of the thermocouple should be maximized (i.e. response time minimized). Either design will be fabricated from the same material (same thermal conductivity k, same specific heat c, same density ρ) and operate in the same environment, i.e. in a fluid of freestream temperature T_∞. If, as a first approximation, we assume both designs have the same convection coefficient h and both have a low Biot number, determine which is the better design and why.

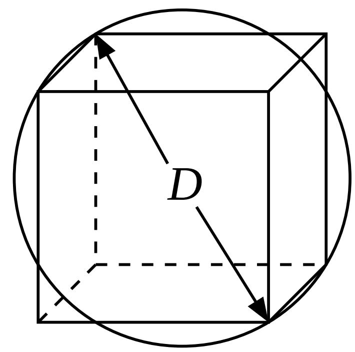

Figure 3: *Thermocouple.*

   Solution: Given a low Biot number, we presume that the capacitance model applies, whereby the response time is completely determined by the time–constant of the system, τ, i.e.

$$\tau = \frac{\rho\, V\, c}{h\, A_s}$$

and the better design is the one which minimizes this parameter. For the cube, the edge length is $D/\sqrt{3}$, meaning $V = (D/\sqrt{3}\,)^3$ and $A_s = 6(D/\sqrt{3}\,)^2$, whereby we find a time–constant of

$$\tau_{cube} = \frac{3\,\rho\, D^3\, c}{3\sqrt{3}\; h\,(6\, D^2)} \approx 0.0962\,\frac{\rho\, D\, c}{h}\,,$$

while for the sphere, $V = \pi D^3/6$ and $A_s = \pi D^2$, whereby we find a time–constant of

$$\tau_{sphere} = \frac{\rho\,\pi\, D^3\; c}{6\, h\,\pi\, D^2} \approx 0.1667\,\frac{\rho\, D\, c}{h}\,.$$

The cube has significantly smaller time constant (by a factor of exactly $3^{-1/2}$), so it responds more quickly and is the better design for this particular problem. As a point of interest, note further that the time constant in both cases continues to decrease as D is reduced, because the surface area to volume ratio becomes progressively larger. In practical terms, we are not free to make D arbitrarily small because other factors are relevant.