MEMS 5403
Final Exam: May 10, 2016 **Name:** ____________________

Solutions

1. A fluid flows in the x–direction through a porous medium (cross–hatched section in Fig. 1). The fluid is at uniform temperature, $T = T_\infty$, as it enters at $x = 0$, at which point it is exposed to a constant temperature heated wall, $T = T_S$, where $T_S > T_\infty$. The fluid has kinematic viscosity ν and thermal diffusivity α. No matter how far the flow proceeds in the x–direction, the fluid always retains its freestream characteristics very far from the wall (i.e. at $y \to \infty$).

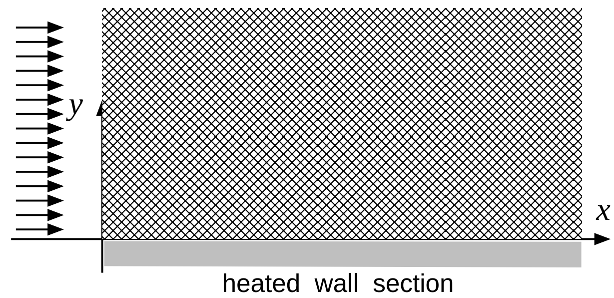

Fig. 1: *Porous media convection.*

(a) (20 pts) Here, velocity components are $(u, v) = (u_\infty, 0)$ and heat transfer is governed by a boundary–layer–type partial differential equation (PDE) for conservation of energy

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \, .$$

Defining dimensionless temperature and a similarity variable (transform), respectively, as

$$\theta = \frac{T - T_S}{T_\infty - T_S} \qquad \text{and} \qquad \varphi = y \sqrt{\frac{u_\infty}{\nu \, x}} \, ,$$

show that the PDE transforms to an ordinary differential equation (ODE) of the form

$$\frac{d^2\theta}{d\varphi^2} + \frac{\varphi \, Pr}{2} \cdot \frac{d\theta}{d\varphi} = 0 \, ,$$

where $Pr = \nu/\alpha$ is the Prandtl number.

Solution: The similarity variable φ is the same as that used for simple boundary layer flow, from which we already found

$$\frac{\partial}{\partial x} = -\frac{\varphi}{2\,x} \frac{d}{d\varphi} \qquad \frac{\partial}{\partial y} = \sqrt{\frac{u_\infty}{\nu \, x}} \frac{d}{d\varphi} \qquad \frac{\partial^2}{\partial y^2} = \frac{u_\infty}{\nu \, x} \frac{d^2}{d\varphi^2} \, .$$

After transforming the differential terms, substituting the flow velocity components, $(u, v) = (u_\infty, 0)$, and using $\Delta T = T_\infty - T_S$ for shorthand, we find

$$u_\infty \left(-\frac{\varphi \, \Delta T}{2\,x} \frac{d\theta}{d\varphi} \right) + 0 \cdot \Delta T \sqrt{\frac{u_\infty}{\nu \, x}} \frac{d\theta}{d\varphi} = \alpha \frac{u_\infty \, \Delta T}{\nu \, x} \frac{d^2\theta}{d\varphi^2}$$

$$\frac{\alpha}{\nu} \frac{d^2\theta}{d\varphi^2} + \frac{\varphi}{2} \frac{d\theta}{d\varphi} = 0 \, ,$$

from which the proposition follows by multiplying through by Pr.

(b) (10 pts) In using similarity approaches for boundary layer problems (as opposed to integral methods like the Karman–Pohlhausen procedure), we do not explicitly invoke any length scales, for example the boundary layer thickness, δ. This implies we must specify a boundary condition for $y \to \infty$ rather than at $y = \delta$. State the 2 boundary conditions for θ, one at $y \to \infty$ and one at $y = 0$, respectively.

Solution: Given the definition of θ in part (a) and the nature of the boundaries stated above, $T(y \to \infty) = T_\infty$ and $T(y = 0) = T_S$, respectively, we note the corresponding values of the similarity variable are

$$\varphi \;=\; 0 \cdot \sqrt{\frac{u_\infty}{\nu\, x}} \;=\; 0 \qquad \text{at} \qquad y \;=\; 0$$

$$\varphi \;\to\; \infty \cdot \sqrt{\frac{u_\infty}{\nu\, x}} \;\to\; \infty \qquad \text{at} \qquad y \;\to\; \infty\,,$$

so that we immediately find

$$\theta\,\Big|_{\varphi=0} = \frac{T_S \,-\, T_S}{T_\infty \,-\, T_S} \;=\; 0 \qquad\qquad \theta\,\Big|_{\varphi\to\infty} = \frac{T_\infty \,-\, T_S}{T_\infty \,-\, T_S} \;=\; 1\,,$$

i.e. $\theta(\infty) = 1$ and $\theta(0) = 0$.

(c) (10 pts) The ODE and its boundary conditions for this configuration comprise a linear equation that is *separable*. Demonstrate that the solution for $\theta(\varphi)$ is

$$\theta(\varphi) \;=\; \text{erf}\left(\frac{\varphi\,\sqrt{Pr}}{2}\right)\,,$$

where erf is the so–called "error function". *Hint:* It may help to define an auxiliary variable $\beta = d\theta/d\varphi$ and the following relations may be useful

$$\text{erf}(\zeta) \;=\; \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-\,\zeta^2}\, d\zeta \qquad\qquad \text{erf}(0) \;=\; 0 \qquad\qquad \text{erf}(\infty) \;=\; 1\,.$$

Solution: Using β, we find

$$\begin{aligned}
\frac{d\beta}{d\varphi} \;+\; \frac{\varphi\,Pr}{2}\cdot\beta \;&=\; 0 \\
\frac{d\beta}{\beta} \;&=\; -\,\frac{\varphi\,Pr}{2}\,d\varphi \\
\int_{\beta(0)}^{\beta(\varphi)} \frac{d\beta}{\beta} \;&=\; -\int_0^\varphi \frac{\varphi\,Pr}{2}\,d\varphi \\
\ln\Big[\beta(\varphi)\Big] \;-\; \ln\Big[\beta(0)\Big] \;=\; \ln\left[\frac{\beta(\varphi)}{\beta(0)}\right] \;&=\; -\,\frac{\varphi^2\,Pr}{4} \\
\beta(\varphi) \;=\; \frac{d\theta}{d\varphi} \;&=\; \underbrace{C_0}_{\beta(0)}\; e^{-\,\varphi^2\,Pr/4}
\end{aligned}$$

$$\int_{\theta(0)}^{\theta(\varphi)} d\theta \;=\; C_0 \int_0^{\varphi} e^{-\,(\varphi\,\sqrt{Pr}\,/2)^2}\, d\varphi$$

Here, we make a change of variables on the right–hand–side, letting

$$\zeta \;=\; \frac{\varphi\,\sqrt{Pr}}{2} \qquad \frac{d\zeta}{d\varphi} \;=\; \frac{\sqrt{Pr}}{2} \qquad \text{and change limits:} \quad (0,\varphi) \;\to\; (0,\zeta)$$

so that we have

$$\theta(\varphi) \;-\; \theta(0) \;\;=\;\; C_0 \cdot \frac{2}{\sqrt{Pr}} \cdot \frac{\sqrt{\pi}}{2} \cdot \underbrace{\frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-\,\zeta^2}\, d\zeta}_{\text{erf}(\zeta)} \;=\; C_1\,\text{erf}(\zeta)$$

$$\theta(\varphi) \;-\; 0 \;\;=\;\; C_1\,\text{erf}\left(\frac{\varphi\,\sqrt{Pr}}{2}\right) \qquad\qquad \text{invoked: } \theta(0) = 0$$

$$\theta(\infty) \;=\; 1 \;\;=\;\; C_1\,\text{erf}(\infty) \qquad\qquad \text{invoked: } \theta(\infty) = 1 \text{ implying } C_1 = 1$$

and finally

$$\theta(\varphi) \;=\; \text{erf}\left(\frac{\varphi\,\sqrt{Pr}}{2}\right)\,.$$

(d) (20 pts) Show that the local Nusselt number at the heated wall, Nu_x, can be expressed as

$$Nu_x \;=\; \frac{h\,x}{k} \;=\; \sqrt{\frac{Pe_x}{\pi}}\;,$$

where $Pe_x = u_\infty\,x/\alpha$ is the Peclet number. *Hint:* Formulate in terms of the conventional conduction–convection equivalence "at the wall"

$$-\,k\;\left.\frac{\partial T}{\partial y}\right|_{y=0} = \;h\,(T_s \;-\; T_\infty)$$

and use the solution for the dimensionless temperature $\theta(\varphi)$ from above. The following relation (derivative) may be useful

$$\frac{\partial}{\partial y}\Big(\text{erf}(\zeta)\Big) \;=\; \frac{2}{\sqrt{\pi}}\,e^{-\,\zeta^2} \cdot \frac{\partial \zeta}{\partial y}$$

Solution: It is useful to write out the entire solution for the temperature distribution from above in long–hand, i.e.

$$T \;=\; (T_\infty \;-\; T_S)\,\text{erf}\left(\frac{y}{2}\sqrt{\frac{u_\infty}{\alpha\,x}}\;\right) \;+\; T_S\,,$$

where we have already absorbed Pr into the expression under the radical. Invoking the conduction–convection equivalence, taking the derivative, and simplifying, we find

$$-\,k\,(T_\infty \;-\; T_S)\,\frac{2}{\sqrt{\pi}}\;\left[\left.e^{-\,y^2\,u_\infty/(4\,\alpha\,x)}\right|_{y=0}\right]\frac{1}{2}\sqrt{\frac{u_\infty}{\alpha\,x}} \;\;=\;\; h\,(T_s \;-\; T_\infty)$$

$$\frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty}{\alpha\, x}} \;=\; h$$

$$\frac{k}{\sqrt{\pi}}\sqrt{\frac{u_\infty\, x^2}{\alpha\, x}} \;=\; h\, x$$

$$\frac{1}{\sqrt{\pi}}\sqrt{\underbrace{\frac{u_\infty\, x}{\alpha}}_{Pe_x}} \;=\; \underbrace{\frac{h\, x}{k}}_{Nu_x}$$

$$\sqrt{\frac{Pe_x}{\pi}} \;=\; Nu_x$$

2. For low Prandtl number fluids, the thermal boundary layer will develop far more rapidly than the momentum boundary layer, i.e. $\delta_t \gg \delta$. In such cases, it is reasonable to model a constant velocity profile, u_∞, across the thermal boundary layer (Fig. 2). The freestream temperature is T_∞ and the wall at $y = 0$ is a constant temperature, T_S.

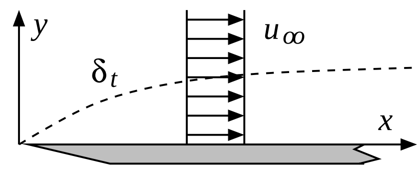

Fig. 2: *Low Pr flat plate.*

(a) (20 pts) If the dimensionless temperature distribution T^* is approximated according to the "trial profile"

$$T^* \;=\; \frac{T \;-\; T_S}{T_\infty \;-\; T_S} \;\approx\; \frac{3}{2}\left(\frac{y}{\delta_t}\right) \;-\; \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 ,$$

use the Kármán–Pohlhausen integral condition,

$$\frac{d}{dx}\int_0^{\delta_t} u\,(1 \;-\; T^*)\, dy \;=\; \alpha\, \frac{\partial T^*}{\partial y}\bigg|_{y=0} ,$$

to show that the growth law for δ_t is $\delta_t(x) = \sqrt{8\,\alpha\, x\,/u_\infty}$. *Hint:* Use the observation that, in this particular case, $\delta_t(0) = 0$ at the leading edge.

Solution: We substitute T^* into the Kármán–Pohlhausen relationship, finding

$$\frac{d}{dx}\int_0^{\delta_t} u_\infty\left[1 \;-\; \frac{3}{2}\left(\frac{y}{\delta_t}\right) \;+\; \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3\right] dy \;=\; \frac{3\,\alpha}{2\,\delta_t}$$

$$\frac{d}{dx}\left(u_\infty\left[y \;-\; \frac{3}{4}\left(\frac{y^2}{\delta_t}\right) \;+\; \frac{1}{8}\left(\frac{y^4}{\delta_t^3}\right)\right]\bigg|_0^{\delta_t}\right) \;=\; \frac{3\,\alpha}{2\,\delta_t}$$

$$u_\infty\,\frac{d}{dx}\left(\delta_t \;-\; \frac{3}{4}\,\delta_t \;+\; \frac{1}{8}\,\delta_t\right) \;=\; \frac{3\,\alpha}{2\,\delta_t}$$

$$\frac{3\,u_\infty}{8}\,\frac{d\delta_t}{dx} \;=\; \frac{3\,\alpha}{2\,\delta_t}$$

$$\delta_t\; d\delta_t \;=\; \frac{4\,\alpha}{u_\infty}\, dx\, .$$

This is a simple differential equation which can be integrated directly. If we take $\delta_t(0) = 0$ at the leading edge, then

$$\begin{aligned}\int_0^{\delta_t} \delta_t \, d\delta_t &= \int_0^x \frac{4\,\alpha}{u_\infty}\, dx \\ \frac{\delta_t^2}{2} &= \frac{4\,\alpha}{u_\infty}\, x \\ \delta_t &= \sqrt{\frac{8\,\alpha\, x}{u_\infty}} \;.\end{aligned}$$

(b) (20 pts) Determine the local Nusselt number, $Nu_x = hx/k$, in terms of the local Reynolds number, $Re_x = u_\infty x/\nu$, and the Prandtl number, $Pr = \nu/\alpha$, for this configuration under the above Kármán–Pohlhausen model. *Hint:* The no–slip boundary condition still applies at the plate surface.

Solution: Equating the heat flux expressions from Newton's Law of Cooling, $h(T_S - T_\infty)$, to that of Fourier's Law of Conduction, $-k\,\partial T/\partial y$ evaluated at $y = 0$, we find

$$\begin{aligned}h(T_S \,-\, T_\infty) &= -\,k\,\frac{3}{2\,\delta_t}\,(T_\infty \,-\, T_S) \\ \frac{h}{k} &= \frac{3}{2\,\delta_t} = \frac{3}{2\sqrt{8}}\sqrt{\frac{u_\infty}{\alpha\, x}} \\ \frac{h\,x}{k} &= \frac{3}{4\sqrt{2}}\sqrt{\frac{u_\infty\, x}{\nu}}\,\sqrt{\frac{\nu}{\alpha}} \\ Nu_x &\approx 0.53\,\sqrt{Re_x\, Pr}\end{aligned}$$