MEMS 5403
Midterm Exam: March 23, 2016 **Name:** ____________________

Solutions

1. The basic unsteady, homogeneous Dirichlet heat conduction problem in a domain $0 \le x \le L$ for $t \ge 0$ is stated as

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad T(0,t) = T(L,t) = 0 \qquad T(x,0) = F(x)\,,$$

where $T = T(x,t)$ is the temperature distribution and α is the thermal diffusivity. The corresponding general solution, obtainable by the method of separation of variables, can be expressed as

$$T(x,t) = \frac{2}{L}\sum_{n=1}^{\infty} C_n \sin(\zeta_n x)\, e^{-\alpha \zeta_n^2 t} \qquad C_n = \int_0^L F(x) \sin(\zeta_n x)\, dx \qquad \zeta_n = \frac{n\pi}{L}\,.$$

Here, we consider the configuration in which the initial temperature distribution is

$$F(x) = T_0 \sin\left(\frac{\pi x}{L}\right)\,,$$

where T_0 is a (constant) reference temperature.

(a) (20 pts) Perform the integral for C_n to show that the exact solution for this configuration is the single–term expression[1]

$$T(x,t) = T_0 \sin\left(\frac{\pi x}{L}\right) e^{-\alpha(\pi/L)^2 t}$$

Solution: The integral can be expressed as

$$C_n = T_0 \int_0^L \sin\left(\frac{1\,\pi x}{L}\right) \sin\left(\frac{n\,\pi x}{L}\right) dx\,,$$

from which we can recognize immediately by the orthogonality property, or by checking the reference integral below, that $C_1 \neq 0$, but $C_2 = C_3 = C_4 = \cdots = 0$. The first mode, $n = 1$, is the only one that does not vanish, whereby

$$\begin{aligned} C_1 &= T_0 \int_0^L \sin\left(\frac{1\,\pi x}{L}\right) \sin\left(\frac{1\,\pi x}{L}\right) dx = T_0 \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \\ &= \left[\frac{x}{2} - \frac{L}{4\pi}\sin\left(\frac{2\pi x}{L}\right)\right]\Bigg|_0^L = \frac{T_0 L}{2}\,. \end{aligned}$$

[1] Integrals that may be helpful here are:

$$\begin{aligned} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx &= \left[\frac{\sin[(n-m)\pi x/L]}{2(n-m)\pi/L} - \frac{\sin[(n+m)\pi x/L]}{2(n+m)\pi/L}\right]\Bigg|_0^L \qquad m \neq n \\ &= \left[\frac{x}{2} - \frac{L}{4n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]\Bigg|_0^L \qquad m = n \end{aligned}$$

Given $\zeta_n = n\,\pi/L$, the solution can then be expressed in long–hand as

$$\begin{aligned} T(x,t) &= \frac{2}{L}\left[C_1 \sin\left(\frac{1\,\pi\,x}{L}\right) e^{-\alpha(1\,\pi/L)^2\,t} + C_2 \sin\left(\frac{2\,\pi\,x}{L}\right) e^{-\alpha(2\,\pi/L)^2\,t} + \cdots\right] \\ &= \frac{2}{L}\left[\frac{T_0\,L}{2} \sin\left(\frac{1\,\pi\,x}{L}\right) e^{-\alpha(1\,\pi/L)^2\,t} + 0 + \cdots\right] \\ &= T_0 \sin\left(\frac{\pi\,x}{L}\right) e^{-\alpha(\pi/L)^2\,t} \end{aligned}$$

(b) (10 pts) The simplicity of the solution, i.e. that it can be expressed as a single term, may seem a little surprising. *Briefly* remark on this observation.

Solution: The separation of variables method gives the solution of $T(x,t)$ in the form of a Fourier series. In this particular case, the initial condition is congruent with one and only one of the Fourier mode shapes, so only that single mode appears in the solution.

(c) (20 pts) Because the boundary conditions and the intial condition, $F(x)$, are symmetric in this particular case, $T(x,t)$ itself will be symmetric about the centerline for $t \geq 0$. Show mathematically that another implication of symmetry is that there is no heat flux across the centerline, i.e. $q''|_{x=L/2} = 0$.

Solution: This proposition depends upon Fourier's Law of conduction and, ultimately, in showing that the temperature *gradient* vanishes at $x = L/2$ for all time. This is readily demonstrated as

$$\begin{aligned} q''_x\Big|_{x=L/2} &= -\,k\,\frac{\partial T}{\partial x}\Big|_{x=L/2} \\ &= -\,k\left[\frac{T_0\,\pi}{L} \cos\left(\frac{\pi\,x}{L}\right) e^{-\alpha(\pi/L)^2\,t}\right]\Big|_{x=L/2} \\ &= -\,\frac{k\,T_0\,\pi}{L}\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} e^{-\alpha(\pi/L)^2\,t} \\ &= 0\,. \end{aligned}$$

(d) (20 pts) The symmetry observation indicates that this configuration is equivalent to a half–domain of $L/2 \leq x \leq L$ which has an adiabatic boundary condition on the left side, $q''|_{x=L/2} = 0$, as was just demonstrated. Here, all the heat leaves the domain at the right boundary only. Calculate the rate of heat transfer, q, over this boundary (i.e. in units of W, *not* the heat flux q''_x in W/m^2) if its area is $A = L \times L$.

Solution: Evaluating q''_x from the previous problem at $x = L$, we find

$$\begin{aligned} q''_x\Big|_{x=L} &= -\,\frac{k\,T_0\,\pi}{L} \cos\left(\frac{\pi\,x}{L}\right) e^{-\alpha(\pi/L)^2\,t}\Big|_{x=L} \\ &= -\,\frac{k\,T_0\,\pi}{L}\underbrace{\cos(\pi)}_{=-1} e^{-\alpha(\pi/L)^2\,t} = \frac{k\,T_0\,\pi}{L} e^{-\alpha(\pi/L)^2\,t}\,, \end{aligned}$$

which means that the rate of heat transfer is $q = q'' \cdot A$, or

$$q \;=\; \frac{k\,T_0\,\pi}{L}\,e^{-\,\alpha(\pi/L)^2\,t}\cdot L^2 \;=\; k\,T_0\,L\,\pi\,e^{-\,\alpha(\pi/L)^2\,t}\,.$$

(e) (20 pts) The form of this particular solution indicates a simple first–order "time–constant" type of behavior, i.e. where a response is proportional to $e^{-\,t/\tau}$. In fact, it is not difficult to see that

$$\tau \;=\; \frac{L^2}{\alpha\,\pi^2}$$

in this case. This property means it is relatively straightforward to calculate the amount of heat energy, Q (in units of J) transferred through the right–side boundary as a function of time. Formulating this differentially as $dQ \;=\; q\cdot dt$, show that a general expression for $Q(t)$ is

$$Q(t) \;=\; \frac{k\,T_0\,L^3}{\alpha\,\pi}\,\left(1 \;-\; e^{-\,t/\tau}\right)\,.$$

Note that $Q(0) = 0$ might be needed in your derivation, i.e. no heat has yet been transferred at $t = 0$.

Solution: The differential statement implies

$$\begin{aligned}
\int_{Q(0)}^{Q(t)} dQ \;&=\; \int_0^t k\,T_0\,L\,\pi\,e^{-\,\alpha(\pi/L)^2\,t}\,dt \;=\; k\,T_0\,L\,\pi\int_0^t e^{-\,t/\tau}\,dt \\
Q\Big|_{Q(0)}^{Q(t)} \;&=\; -\,k\,T_0\,L\,\pi\,\tau\,e^{-\,t/\tau}\Big|_0^t \\
Q(t) \;-\; \underbrace{Q(0)}_{=\,0} \;&=\; -\,k\,T_0\,L\,\pi\,\tau\,\left(e^{-\,t/\tau} \;-\; 1\right) \\
Q(t) \;&=\; k\,T_0\,L\,\pi\,\tau\,\left(1 \;-\; e^{-\,t/\tau}\right) \;=\; k\,T_0\,L\,\pi\,\frac{L^2}{\alpha\,\pi^2}\,\left(1 \;-\; e^{-\,t/\tau}\right) \\
\therefore\;\; Q(t) \;&=\; \frac{k\,T_0\,L^3}{\alpha\,\pi}\,\left(1 \;-\; e^{-\,t/\tau}\right)
\end{aligned}$$

(f) (10 pts) For a particular physical instantiation of this problem, the conducting domain is made out of copper, having $\alpha \approx 0.0001\,m^2/s$ and $k \approx 380\,W/(m\,K)$, and has a dimension $L = 0.03\,m$. If $T_0 = 100$ above the boundary temperature, find the heat energy that has been transferred after $\tau = 1$ time constant.

Solution:

$$Q(t=\tau) \;=\; \frac{380\cdot 100\cdot 0.03^3}{0.0001\cdot\pi}\cdot\left(1 \;-\; e^{-\,1}\right) \;\approx\; 2064\,J\,.$$