MEMS 5403
January 30, 2017

Homework #2 Solutions

1. (10 pts) A fin having triangular cross–section with protrusion length L, width w, and thickness t at the root is made of material having thermal conductivity k. It transfers heat to the surroundings at temperature T_∞ with a convection coefficient of h (Fig. 1). Assuming steady state and a Biot number sufficiently low such that temperature varies in x only, i.e. $T = T(x)$ and noting the slightly unusual, but very convenient coordinate system choice, show that the usual transformation of $\theta(x) = T(x) - T_\infty$, taken along with the conditions $t \ll w$ and $t \ll L$, leads to the governing ordinary differential equation

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \frac{m^2}{x}\theta = 0 \qquad m^2 = \frac{2\,h\,L}{k\,t}$$

for the temperature distribution.

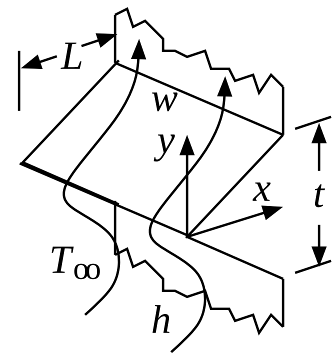

Figure 1: *Triangle fin.*

Solution: Given the change–of–variables transformation, the general equation for 1–D conduction in a variable–area fin is

$$\frac{d}{dx}\left(A_c\,\frac{d\theta}{dx}\right) - \frac{h}{k}\cdot\frac{dA_s}{dx}\,\theta = 0\,,$$

where A_c and A_s are the local cross–sectional area and fin surface area, respectively. The unusual choice of coordinate system, i.e. with the origin at the fin tip pointing towards the base, results in straightforward quantification of these variables. Observing the fin's geometry is characterized by the line

$$y = \frac{t}{2\,L}\cdot x\,,$$

The cross–sectional area as a function of x is

$$A_c = 2\,y\,w = 2\left(\frac{t}{2\,L}\cdot x\right)w = \frac{t\,w}{L}\cdot x\,.$$

Because t is small compared to L and w, the heat transfer from the triangular–shaped edges is small, so the fin area is taken to be comprised only of the top and bottom surfaces. Strictly speaking, the surface area of the fin as a function of x under this approximation would be

$$A_s = 2\,w\,\sqrt{x^2 + y^2} = 2\,w\,\sqrt{x^2 + \frac{t^2\,x^2}{4\,L^2}} = 2\,w\,\sqrt{1 + \frac{t^2}{4\,L^2}}\cdot x\,,$$

however, because $t \ll L$, the second term under the radical is small compared to unity, whereby

$$A_s = 2\,w\,x$$

to a good approximation. The rate of change of fin area required in the conduction equation is then $dA_s/dx = 2\,w$. Substituting these entities, we find

$$\frac{d}{dx}\left(\frac{t\,w}{L}\cdot x\cdot\frac{d\theta}{dx}\right) \;-\; \frac{h}{k}\cdot 2\,w\,\theta \;=\; 0$$

$$\frac{t\,w}{L}\cdot\frac{d}{dx}\left(x\,\frac{d\theta}{dx}\right) \;-\; \frac{2\,w\,h}{k}\cdot\theta \;=\; 0$$

$$\frac{d}{dx}\left(x\,\frac{d\theta}{dx}\right) \;-\; \frac{2\,h\,L}{k\,t}\cdot\theta \;=\; 0$$

$$x\,\frac{d^2\theta}{dx^2} \;+\; \frac{d\theta}{dx} \;-\; m^2\,\theta \;=\; 0\,,$$

from which it is clear that we obtain the equation in the given, canonical form by dividing through by x.

2. (10 pts) In Question 1, $x = 0$ is a regular singular point, implying that the solution to the governing ODE can be cast in the form of an extended power series

$$\theta \;=\; x^{\varphi}\cdot\sum_{i=0}^{\infty} a_i\,x^i \;=\; \sum_{i=0}^{\infty} a_i\,x^{\varphi+i}\,,$$

where φ is an undetermined constant. Given the requirement that $a_0 \neq 0$, show that $\varphi = 0$ and furthermore that the general solution will involve the modified Bessel function $I_0(2\,m\sqrt{x}\,)$, where the form of I_0 is

$$I_0(x) \;=\; \sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^{2i}\frac{1}{(i!)^2}\,.$$

Solution: Taking two derivatives, we find

$$\frac{d\theta}{dx} \;=\; \sum_{i=0}^{\infty}(\varphi\,+\,i)\,a_i\,x^{\varphi\,+\,i\,-\,1} \qquad \text{and} \qquad \frac{d^2\theta}{dx^2} \;=\; \sum_{i=0}^{\infty}(\varphi\,+\,i)\,(\varphi\,+\,i\,-\,1)\,a_i\,x^{\varphi\,+\,i\,-\,2}\,,$$

whereby substitution into the governing equation yields

$$\Big[(\varphi\,+\,0)\,(\varphi\,+\,0\,-\,1)\,a_0\,x^0 \;+\; (\varphi\,+\,1)\,(\varphi\,+\,1\,-\,1)\,a_1\,x^1\,+$$
$$(\varphi\,+\,2)\,(\varphi\,+\,2\,-\,1)\,a_2\,x^2 \;+\; (\varphi\,+\,3)\,(\varphi\,+\,3\,-\,1)\,a_3\,x^3 \;+\;\cdots\Big]\,x^{\varphi\,-\,2}\,+$$
$$\Big[(\varphi\,+\,0)\,a_0\,x^0 \;+\; (\varphi\,+\,1)\,a_1\,x^1 \;+\; (\varphi\,+\,2)\,a_2\,x^2 \;+\; (\varphi\,+\,3)\,a_3\,x^3 \;+\;\cdots\Big]\,x^{\varphi\,-\,1\,-\,1}$$
$$-\,m^2\Big[a_0\,x^0 \;+\; a_1\,x^1 \;+\; a_2\,x^2 \;+\; a_3\,x^3 \;+\;\cdots\Big]\,x^{\varphi\,-\,1} \;=\; 0\,.$$

Gathering terms in like powers of x and recogizing that $x \neq 0$, we find a series of algebraic expressions that all must equate to zero. For $i = 0$, i.e. the so–called indicial equation, we have

$$\varphi\,(\varphi\,-\,1)\,a_0 \;+\; \varphi\,a_0 \;=\; \varphi^2\,a_0 \;=\; 0 \qquad \therefore \quad \varphi \;=\; 0$$

in the form of a double root. Substituting $\varphi = 0$ in all subsequent components, we find

$$i = 1: \quad 1 \cdot 0\, a_1 \,+\, 1\, a_1 \,-\, m^2\, a_0 \,=\, 0 \qquad 1\, a_1 \,=\, m^2\, a_0 \qquad \therefore \quad a_1 \,=\, \frac{m^2}{1 \cdot 1}\, a_0$$

$$i = 2: \quad 2 \cdot 1\, a_2 \,+\, 2\, a_2 \,-\, m^2\, a_1 \,=\, 0 \qquad 4\, a_2 \,=\, m^2\, a_1 \qquad \therefore \quad a_2 \,=\, \frac{m^2}{2 \cdot 2}\, a_1$$

$$i = 3: \quad 3 \cdot 2\, a_3 \,+\, 3\, a_3 \,-\, m^2\, a_2 \,=\, 0 \qquad 9\, a_3 \,=\, m^2\, a_2 \qquad \therefore \quad a_3 \,=\, \frac{m^2}{3 \cdot 3}\, a_2$$

$$i = 4: \quad 4 \cdot 3\, a_4 \,+\, 4\, a_4 \,-\, m^2\, a_3 \,=\, 0 \qquad 16\, a_4 \,=\, m^2\, a_3 \qquad \therefore \quad a_4 \,=\, \frac{m^2}{4 \cdot 4}\, a_3\,,$$

and so on and so forth. In fact, it is now relatively clear that the recursive pattern is

$$i\,(i \,-\, 1)\, a_i \,+\, i\, a_i \,-\, m^2\, a_{i-1} \,=\, 0 \qquad i^2\, a_i \,=\, m^2\, a_{i-1} \qquad \therefore \quad a_i \,=\, \frac{m^2}{i^2}\, a_{i-1}\,,$$

whereby doing a "cascading substitution" from equation to equation, we find

$$i = 1 \qquad a_1 \,=\, \frac{m^2}{1 \cdot 1}\, a_0$$

$$i = 2 \qquad a_2 \,=\, \frac{m^2}{2 \cdot 2} \cdot \frac{m^2}{1 \cdot 1}\, a_0$$

$$i = 3 \qquad a_3 \,=\, \frac{m^2}{3 \cdot 3} \cdot \frac{m^2}{2 \cdot 2} \cdot \frac{m^2}{1 \cdot 1}\, a_0$$

$$i = 4 \qquad a_4 \,=\, \frac{m^2}{4 \cdot 4} \cdot \frac{m^2}{3 \cdot 3} \cdot \frac{m^2}{2 \cdot 2} \cdot \frac{m^2}{1 \cdot 1}\, a_0\,,$$

and so forth. We can clearly write the general expression

$$a_i \,=\, \frac{(m^2)^i}{i! \cdot i!}\, a_0 \,=\, \frac{m^{2\,i}}{(i!)^2}\, a_0\,.$$

Finally, we substitute both $\varphi = 0$ and this result back into the original form for the extended power series solution. With some manipulation, we find

$$\theta \,=\, \sum_{i=0}^{\infty} a_i\, x^i \,=\, \sum_{i=0}^{\infty} \frac{m^{2\,i}}{(i!)^2}\, a_0\, x^i \,=\, a_0 \sum_{i=0}^{\infty} \frac{(m\,\sqrt{x}\,)^{2\,i}}{(i!)^2}$$

$$=\, a_0 \sum_{i=0}^{\infty} \left(\frac{2\,m\,\sqrt{x}}{2} \right)^{2\,i} \frac{1}{(i!)^2} \,=\, a_0\, I_0(2\,m\sqrt{x}\,)\,.$$