MEMS 5403
February 12, 2017

Homework #3 Solutions

1. (10 pts) A 1–D domain is irradiated, causing a volumetric heating of

$$\dot{q}(x) = \dot{q}_0 \left(1 - \frac{x}{L}\right) \frac{W}{m^3},$$

where q_0 is a constant (Fig. 1). The domain is perfectly insulated at $x = 0$, while at $x = L$ the temperature is held at $T = 0$. At steady conditions, this scenario is govern by the 1–D equation

$$\frac{d^2T}{dx^2} + \frac{\dot{q}(x)}{k} = 0,$$

where k is the conductor's thermal conductivity. Show that the steady temperature distribution is

$$T(x) = \frac{\dot{q}_0}{k} \left(\frac{x^3}{6L} - \frac{x^2}{2} + \frac{L^2}{3} \right).$$

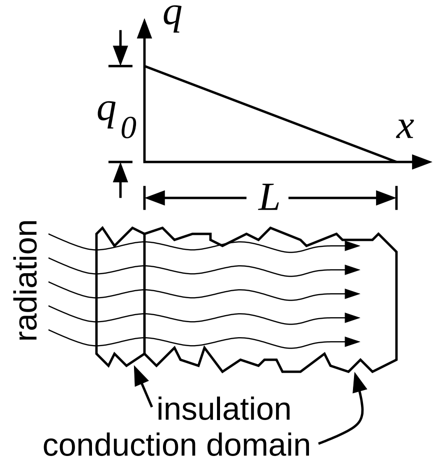

Figure 1: *Heat generation.*

Solution: This problem can be solved by direct integration, as shown, followed by application of the boundary conditions. Specifically,

$$\frac{d^2T}{dx^2} = -\frac{\dot{q}_0}{k}\left(1 - \frac{x}{L}\right) = \frac{\dot{q}_0}{kL}\,x - \frac{\dot{q}_0}{k}$$

$$\frac{dT}{dx} = \frac{\dot{q}_0}{2kL}\,x^2 - \frac{\dot{q}_0}{k}\,x + C_1$$

$$T(x) = \frac{\dot{q}_0}{6kL}\,x^3 - \frac{\dot{q}_0}{2k}\,x^2 + C_1 x + C_2,$$

whereby applying the two boundary conditions, we find

$$\left.\frac{dT}{dx}\right|_{x=0} = \frac{\dot{q}_0}{2kL}\,0^2 - \frac{\dot{q}_0}{k}\,0 + C_1 = 0 \qquad C_1 = 0$$

$$T(L) = \frac{\dot{q}_0}{6kL}\,L^3 - \frac{\dot{q}_0}{2k}\,L^2 + C_2 = 0 \qquad C_2 = \frac{\dot{q}_0 L^2}{3k},$$

from which it is clear that

$$T(x) = \frac{\dot{q}_0}{k} \left(\frac{x^3}{6L} - \frac{x^2}{2} + \frac{L^2}{3} \right).$$

2. (10 pts) Suppose that, for the configuration in Question 1, the irradiation is halted and at the same time the insulation is removed from the left boundary and the temperature at this boundary is set to $T = 0$. Also, the temperature at the right boundary continues to be maintained at $T = 0$. If this entire process takes place instantaneously at $t = 0$, demonstrate

that the unsteady temperature distribution, $T(x,t)$, in the domain for $t \geq 0$ (taking the steady $T(x)$ from Question 1 as the initial distribution) is

$$T(x,t) \;=\; \frac{2\,\dot{q}_0\,L^2}{k} \sum_{n=1}^{\infty} \left(\frac{1}{n^3\,\pi^3} \;+\; \frac{1}{3\,n\,\pi} \right) \sin\left(\zeta_n x\right)\; e^{-\alpha \zeta_n^2 t} \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L}\;.$$

Solution: This scenario is the so–called Dirichlet problem in unsteady conduction, for which the general solution is

$$T(x,t) \;=\; \sum_{n=1}^{\infty} C_n\, \sin\left(\zeta_n x\right)\; e^{-\alpha \zeta_n^2 t} \qquad C_n \;=\; \frac{2}{L} \int_0^L F(x)\, \sin(\zeta_n x)\, dx \qquad \zeta_n \;=\; \frac{n\,\pi}{L}\;.$$

The mode coefficients can be determined from the steady temperature distribution from Question 1, that being

$$F(x) \;=\; \frac{\dot{q}_0}{k} \left(\frac{x^3}{6\,L} \;-\; \frac{x^2}{2} \;+\; \frac{L^2}{3} \right)\;,$$

whereby we find[1]

$$\begin{aligned}
C_n \;&=\; \frac{2}{L} \int_0^L \frac{\dot{q}_0}{k} \left(\frac{x^3}{6\,L} \;-\; \frac{x^2}{2} \;+\; \frac{L^2}{3} \right) \sin(\zeta_n x)\, dx \\
&=\; \frac{2\,\dot{q}_0}{L\,k} \int_0^L \left(\frac{x^3\, \sin(\zeta_n x)}{6\,L} \;-\; \frac{x^2\, \sin(\zeta_n x)}{2} \;+\; \frac{L^2\, \sin(\zeta_n x)}{3} \right)\, dx \\
&=\; \frac{2\,\dot{q}_0}{L\,k} \left[\frac{1}{6\,L} \left(\frac{3\,\zeta_n^2\, x^2 \;-\; 6}{\zeta_n^4}\, \sin(\zeta_n x) \;-\; \frac{\zeta_n^2\, x^3 \;-\; 6\,x}{\zeta_n^3}\, \cos(\zeta_n x) \right) \right. \\
&\qquad\qquad -\, \frac{1}{2} \left(\frac{2\,x}{\zeta_n^2}\, \sin(\zeta_n x) \;-\; \frac{\zeta_n^2\, x^2 \;-\; 2}{\zeta_n^3}\, \cos(\zeta_n x) \right) \\
&\qquad\qquad\qquad \left. +\; \frac{L^2}{3} \left(-\, \frac{1}{\zeta_n}\, \cos(\zeta_n x) \right) \right] \Bigg|_0^L\;.
\end{aligned}$$

In order that we do not waste too much space in writing needlessly, let us observe that all terms having $\sin(\zeta_n x)$ will vanish for both the top and bottom limits, the former because $\sin(n\pi)$ is always zero by virtue of its argument being a multiple of π and the latter being $\sin(0)$, which is identically zero. Using this observation with some algebraic simplification, we continue

[1] With the help of integral tables, e.g. "CRC Standard Mathematical Tables"

$$
\begin{aligned}
C_n \;&=\; \frac{2\,\dot{q}_0}{L\,k}\left[\left(-\,\frac{\zeta_n^2\,x^3 \,-\, 6\,x}{6\,L\,\zeta_n^3} \,+\, \frac{\zeta_n^2\,x^2 \,-\, 2}{2\,\zeta_n^3} \,-\, \frac{L^2}{3\,\zeta_n}\right)\cos(\zeta_n x)\right]\Bigg|_0^L \\
&=\; \frac{2\,\dot{q}_0}{L\,k}\left[\underbrace{\left(-\,\frac{\zeta_n^2\,L^3 \,-\, 6\,L}{6\,L\,\zeta_n^3} \,+\, \frac{\zeta_n^2\,L^2 \,-\, 2}{2\,\zeta_n^3} \,-\, \frac{L^2}{3\,\zeta_n}\right)}_{\text{sums to zero}}\cos(n\pi)\right. \\
&\qquad\qquad \left. -\,\left(-\,\frac{\zeta_n^2\cdot 0^3 \,-\, 6\cdot 0}{6\,L\,\zeta_n^3} \,+\, \frac{\zeta_n^2\cdot 0^2 \,-\, 2}{2\,\zeta_n^3} \,-\, \frac{L^2}{3\,\zeta_n}\right)\cos(0)\right]\,,
\end{aligned}
$$

where we again pause to note some simplifications, specifically $\cos(0) = 1$ and the set of terms in the first set of round brackets vanishes, as shown. Incorporating $\zeta_n = n\pi/L$ and doing some final simplification, we find

$$
\begin{aligned}
C_n \;&=\; -\,\frac{2\,\dot{q}_0}{L\,k}\left(-\,\frac{2}{2\,\zeta_n^3} \,-\, \frac{L^2}{3\,\zeta_n}\right) \;=\; \frac{2\,\dot{q}_0}{L\,k}\left(\frac{1}{\zeta_n^3} \,+\, \frac{L^2}{3\,\zeta_n}\right) \\
&=\; \frac{2\,\dot{q}_0}{L\,k}\left(\frac{L^3}{n^3\,\pi^3} \,+\, \frac{L^3}{3\,n\,\pi}\right) \;=\; \frac{2\,\dot{q}_0\,L^2}{k}\left(\frac{1}{n^3\,\pi^3} \,+\, \frac{1}{3\,n\,\pi}\right)\,,
\end{aligned}
$$

whereby the entire solution can be written as

$$
T(x,t) \;=\; \frac{2\,\dot{q}_0\,L^2}{k}\sum_{n=1}^{\infty}\left(\frac{1}{n^3\,\pi^3} \,+\, \frac{1}{3\,n\,\pi}\right)\sin\left(\zeta_n x\right)\, e^{-\alpha\zeta_n^2 t}\;.
$$