MEMS 5403
February 27, 2017

Homework #4 Solutions

1. (10 pts) Steady heat conduction occurs in a 2–D domain having width and height L and H, respectively, according to the Laplace equation

$$\nabla^2 T \;=\; 0 \qquad \text{where} \qquad T \;=\; T(x,y)\,,$$

and where the boundary conditions are given by

$$T(0,y) \;=\; T(x,0) \;=\; 0\,, \quad T(x,H) \;=\; T_0\,, \quad \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0\,,$$

where T_0 is a constant, as shown in Fig. 1. Determine the exact solution for $T(x,y)$ using the separation of variables method.

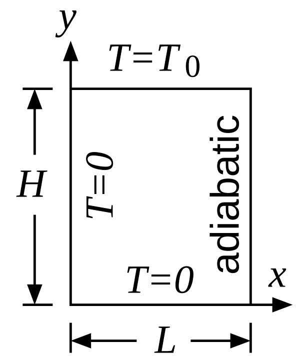

Figure 1: *2–D Conduction.*

Solution: Using the usual approach, i.e. assuming a separation of variables form of

$$T(x,y) \;=\; \Psi(x)\cdot\Gamma(y)\,,$$

substituting this into both the Laplace equation and the 3 boundary conditions that are homogeneous, separating into x–only and y–only parts, and invoking the negative constant in the form of ζ^2, we find the following two ordinary differential equations (ODEs)

$$x: \quad \Psi'' + \zeta^2\,\Psi \;=\; 0 \qquad \Psi(0) \;=\; 0 \qquad \Psi'(L) \;=\; 0$$
$$y: \quad \Gamma'' - \zeta^2\,\Gamma \;=\; 0 \qquad \Gamma(0) \;=\; 0\,,$$

where we leave the non–homogeneous 4th boundary for later evaluation. For x, we have the usual solution

$$\Psi(x) \;=\; C_1\,\sin(\zeta\,x) \;+\; C_2\,\cos(\zeta\,x)\,,$$

for which the boundary condition at $x=0$ indicates

$$\Psi(0) \;=\; 0 \;=\; C_1\,\sin 0 \;+\; C_2\,\cos 0 \qquad \rightarrow \qquad C_2 \;=\; 0\,.$$

The adiabatic boundary condition at $x=L$ depends on the first derivative and it indicates

$$\Psi'(L) \;=\; 0 \;=\; C_1\,\zeta\,\cos(\zeta\,L) \quad \rightarrow \quad C_1 \;\neq\; 0\,, \quad \zeta \;=\; 0 \text{ or } \zeta\,L \;=\; \frac{(2\,n-1)\,\pi}{2}\,,$$

which implies $\Psi_n(x) = C_n\,\sin(\zeta_n\,x)$ and that the set of eigenvalues is

$$\zeta_0 \;=\; 0 \quad \text{and} \quad \zeta_n \;=\; \frac{(2\,n-1)\,\pi}{2\,L} \quad \text{where} \quad n\in\{1,2,3,\ldots\}\,.$$

Because there is a possible "zero eigenvalue", ζ_0, we must go back and rework this mode in the x–direction ODE, where $\Psi_n'' + \zeta_n^2\,\Psi_n = 0$ now becomes simply $\Psi_0'' = 0$ once we substitute $\zeta_0 = 0$. Integrating twice, we find $\Psi_0(x) = A_1 x + A_2$, where the above boundary conditions still apply, i.e. $\Psi_0(0) = \Psi_0'(L) = 0$. Substituting, it is clear that $A_1 = A_2 = 0$, meaning that

$\Psi_0(x) = 0$, in other words the candidate "zero eigenvalue" mode does not actually exist for this set of boundary conditions.

For the ODE in the y direction, the boundary conditions are the same as the Neumann problem examined in class (i.e. Appendix F in the text), so we immediately know that

$$\Gamma_n(y) = B_n \sinh(\zeta_n y)$$

from the ODE and the application of $\Gamma(0) = 0$. Note that since the "zero eignmode" does not exist, $\Psi_0(x) = 0$, we simply skip any consideration of $\Gamma_0(y)$.

The physical solution is a linear combination of the individual modal solutions, meaning that we can write

$$T(x,y) \;=\; \sum_{n=1}^{\infty} C_n \sin(\zeta_n x) \sinh(\zeta_n y) \qquad \zeta_n \;=\; \frac{(2\,n-1)\,\pi}{2\,L}\,.$$

where we have subsumed all constants into the yet–to–be–determined set of mode coefficients, C_n. Here, we exploit the orthogonality property of the eigenfunctions to determine the C_n using the final remaining boundary condition, $T(x,H) = T_0$. We find

$$\begin{aligned}
T(x,H) \;=\; T_0 \;&=\; \sum_{m=1}^{\infty} C_m \sin(\zeta_m x) \sinh(\zeta_m H) \\
\int_0^L T_0 \sin(\zeta_n x)\, dx \;&=\; \int_0^L \sum_{m=1}^{\infty} C_m \sin(\zeta_m x) \sinh(\zeta_m H) \sin(\zeta_n x)\, dx \\
T_0 \int_0^L \sin(\zeta_n x)\, dx \;&=\; \sum_{m=1}^{\infty} C_m \sinh(\zeta_m H) \int_0^L \sin(\zeta_m x) \sin(\zeta_n x)\, dx \\
-\;\frac{T_0}{\zeta_n} \cos(\zeta_n x)\Big|_0^L \;&=\; C_n \sinh(\zeta_n H) \cdot \frac{L}{2} \\
C_n \;&=\; -\,\frac{2\,T_0}{\zeta_n L \sinh(\zeta_n H)} \Big[\cos(\zeta_n L) \,-\, \cos(0)\Big] \\
&=\; -\,\frac{2\,T_0}{\zeta_n L \sinh(\zeta_n H)} \left[\cos\left(\frac{(2\,n-1)\,\pi}{2}\right) \,-\, 1\right] \\
&=\; -\,\frac{2\,T_0}{\zeta_n L \sinh(\zeta_n H)} \Big[0 \,-\, 1\Big] \\
&=\; \frac{2\,T_0}{\zeta_n L \sinh(\zeta_n H)} \;=\; \frac{4\,T_0}{(2\,n \,-\, 1)\,\pi \sinh(\zeta_n H)}\,,
\end{aligned}$$

which, when taken with $T(x,y)$ and ζ_n above, completes the exact solution.