MEMS 5403
March 22, 2017

Homework #5 Solutions

1. (10 pts) A fluid of kinematic viscosity ν is in a semi–infinite domain $y \geq 0$ bounded by a flat plate at $y = 0$. This plate oscillates in its own plane at a frequency of ω, leading to a velocity distribution similar to what is qualitatively shown in Fig. 1, i.e. where the flow is uni–directional. If the velocity $u = u(y, t)$ is defined formally by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \qquad u(y \to \infty, t) = 0$$

$$u(y, t = 0) = 0 \qquad u(y = 0, t) = U_0 \sin(\omega t)$$

and we define dimensionless variables as

$$u^* = \frac{u}{U_0} \qquad t^* = \omega t \qquad y^* = \frac{y}{\sqrt{\nu/\omega}},$$

show that the problem can be stated non–dimensionally in the parameter–free form

$$\frac{\partial u^*}{\partial t^*} = \frac{\partial^2 u^*}{\partial y^{*2}} \qquad u^*(y^* \to \infty, t^*) = 0$$

$$u(y^*, t^* = 0) = 0 \qquad u(y^* = 0, t^*) = \sin(t^*).$$

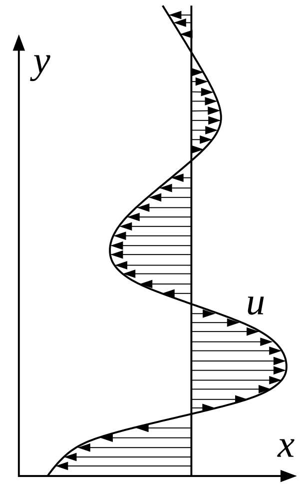

Figure 1: *Oscillating flow.*

Solution: By chain–rule, operators have the form

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{dt^*}{dt} = \omega \frac{\partial}{\partial t^*} \qquad \frac{\partial^2}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y^*}\frac{dy^*}{dy}\right) = \cdots = \frac{\omega}{\nu}\frac{\partial}{\partial y^{*2}},$$

so that the equation becomes

$$\omega U_0 \frac{\partial u^*}{\partial t^*} = \nu \cdot \frac{\omega}{\nu} \cdot U_0 \frac{\partial u^*}{\partial y^{*2}} \qquad \text{or} \qquad \frac{\partial u^*}{\partial t^*} = \frac{\partial u^*}{\partial y^{*2}}.$$

Each dimensionless variable is implied in a boundary condition, e.g. $t = 0$ implies $t^* = 0$ and the argument of the sin( ) function is precisely $t^* = \omega t$, whereby the dimensionless statement of the problem is obtained, as shown.

2. (10 pts) The flow configuration in Question 1 can be represented as the superposition of a transient "start–up" component and a steady, actually *quasi–steady* persistent component. The latter property is due to the time–dependent oscillating boundary condition at $y = 0$. Show that the quasi–steady component of the solution is

$$u^* = e^{-\sqrt{2}\, y^*/2} \sin\left(t^* - \frac{\sqrt{2}\, y^*}{2}\right).$$

Hint: Transform the problem to the complex domain, assuming the quasi–steady solution has the form $U^* = f(y^*) \cdot e^{i t^*}$. For instance, the oscillating boundary condition is

$$u^*(y^* = 0, t^*) = \sin(t^*) = \Im(e^{i t^*}),$$

according to Euler's formula, $e^{i\,t^*} = \cos(t^*) + i\,\sin(t^*)$, where $\Im(\,)$ indicates the imaginary component. After obtaining U^*, the physical solution can be recovered as $u^* = \Im(U^*)$.

Solution: Letting $U^* = f(y^*) \cdot e^{i\,t^*}$ and substituting this into the governing equation, we find

$$\begin{aligned} i \cdot f(y^*)\, e^{i\,t^*} &= f''(y^*)\, e^{i\,t^*} \\ i \cdot f(y^*) &= f''(y^*), \end{aligned}$$

which is effectviely a second–order ODE with constant coefficients in the form $f'' - i \cdot f = 0$. The corresponding auxiliary equation is $\varphi^2 + 0\,\varphi^1 - i\,\varphi^0 = 0$, or, in simplified form

$$\varphi^2 - i = (\varphi + \sqrt{i}\,) \cdot (\varphi - \sqrt{i}\,) = 0\,,$$

whose roots are clearly $\varphi = \pm\sqrt{i}$. Working further with $\sqrt{i}$ will not be very practical, so let us convert that to a more conventional representation in the form $a + i\,b$. If $a + i\,b = \sqrt{i}$ then $(a + i\,b)^2 = i$, implying

$$\begin{aligned} a^2 + 2\,a\,b\,i + b^2\,i^2 &= i \\ (a^2 - b^2) + 2\,a\,b\,i &= 0 + 1\,i\,. \end{aligned}$$

Solving the two equations $a^2 - b^2 = 0$ and $2ab = 1$ for the two unknowns, we find $a = b = \sqrt{2}/2$, whereby

$$\sqrt{i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$$

Consequently, considering the roots $\pm\sqrt{i}$, the general solution for f is

$$f = C_1\, e^{(\sqrt{2}+\sqrt{2}\,i)y^*/2} + C_2\, e^{-\,(\sqrt{2}+\sqrt{2}\,i)y^*/2}\,,$$

where constants C_1 and C_2 remain to be evaluated from the two boundary conditions. It is clear from the presumed form of the solution, $U^* = f(y^*) \cdot e^{i\,t^*}$, that the "far field" boundary condition, $u^*(y^* \to \infty, t^*) = 0$, implies that

$$f(y^* \to \infty) = 0 \qquad \therefore \quad C_1 = 0\,,$$

since $e^{i\,t^*}$ is not zero. Re–assembling what is left, we have

$$\begin{aligned} U^* &= C_2\, e^{-\,(\sqrt{2}+\sqrt{2}\,i)y^*/2} \cdot e^{i\,t^*} \\ \therefore\ u^* &= \Im\Big(C_2\, e^{-\,(\sqrt{2}+\sqrt{2}\,i)y^*/2} \cdot e^{i\,t^*}\Big) \end{aligned}$$

and applying the oscillating boundary condition at $y^* = 0$ means that

$$\begin{aligned} u^*(y^* = 0, t^*) = \sin(t^*) &= \Im\Big(C_2\, e^{-\,(\sqrt{2}+\sqrt{2}\,i)\cdot 0/2} \cdot e^{i\,t^*}\Big) \\ &= \Im\Big(C_2\, e^{i\,t^*}\Big) = \Im\Big(C_2\,\cos(t^*) + i\,C_2\,\sin(t^*)\Big) \\ &= C_2\,\sin(t^*)\,, \end{aligned}$$

from which we conclude $C_2 = 1$. Substituting this back into the expression for u^*, we find

$$
\begin{aligned}
u^* &= \Im\left(e^{-(\sqrt{2}+\sqrt{2}\,i)y^*/2} \cdot e^{i\,t^*}\right) \\
&= \Im\left(e^{-\sqrt{2}\,y^*/2 - i\cdot\sqrt{2}\,y^*/2 + i\cdot t^*}\right) = \Im\left(e^{-\sqrt{2}\,y^*/2} \cdot e^{i\,(t^* - \sqrt{2}\,y^*/2)}\right) \\
&= \Im\left[e^{-\sqrt{2}\,y^*/2} \cos\left(t^* - \frac{\sqrt{2}\,y^*}{2}\right) + i \cdot e^{-\sqrt{2}\,y^*/2} \sin\left(t^* - \frac{\sqrt{2}\,y^*}{2}\right)\right] \\
\therefore \quad u^* &= e^{-\sqrt{2}\,y^*/2} \sin\left(t^* - \frac{\sqrt{2}\,y^*}{2}\right) .
\end{aligned}
$$