MEMS 5403
April 8, 2017

Homework #6 Solutions

1. (10 pts) The Karman–Pohlhausen integral method is a clever way to obtain good approximate solutions to the boundary layer development flow problem for velocity distribution $u(x, y)$ in the x–direction on a flat plate. Here, the integral takes the form

$$\frac{d}{dx}\int_0^{\delta} u\big(u_\infty - u\big)\,dy = \nu \left.\frac{\partial u}{\partial y}\right|_{y=0} ,$$

where u_∞ is the freestream velocity, ν is viscosity, $\delta = \delta(x)$ is the boundary layer thickness, and y is the direction perpendicular to the plate. Following the approach of solving for $u(x, y)$ as a function of the combined variable y/δ, a trial velocity profile in the form of a polynomial is often used. Instead, using a trial profile in the form of

$$\frac{u}{u_\infty} = a_0 + a_1 \sin\left(\frac{\pi\, y}{2\,\delta}\right) ,$$

where a_0 and a_1 are undetermined constants, solve for the boundary layer growth in the form of

$$\delta = \frac{C\, x}{\sqrt{Re_x}} ,$$

specifically determining the constant C. Compare this constant to the value of 5.0 and 4.64 from the exact and polynomial–approximate solutions, respectively.

Solution: Given 2 point boundary conditions

$$u\big|_{y=0} = 0 \qquad \text{and} \qquad u\big|_{y=\delta} = u_\infty ,$$

we find

$$0 = a_0 + a_1 \sin(0) \quad \therefore a_0 = 0 \qquad\qquad 1 = a_1 \sin\left(\frac{\pi}{2}\right) \quad \therefore a_1 = 1 ,$$

so that the basic form of the trial profile is

$$u = u_\infty \sin\left(\frac{\pi\, y}{2\,\delta}\right) .$$

Substituting this form into the Karman–Pohlhausen integral equation, we find

$$\begin{aligned}
\frac{d}{dx}\int_0^{\delta} u_\infty \sin\left(\frac{\pi\, y}{2\,\delta}\right)\left[u_\infty - u_\infty \sin\left(\frac{\pi\, y}{2\,\delta}\right)\right] dy &= \nu \frac{\partial}{\partial y}\left.\left[u_\infty \sin\left(\frac{\pi\, y}{2\,\delta}\right)\right]\right|_{y=0} \\
u_\infty^2 \frac{d}{dx}\int_0^{\delta} \sin\left(\frac{\pi\, y}{2\,\delta}\right)\left[1 - \sin\left(\frac{\pi\, y}{2\,\delta}\right)\right] dy &= \frac{\nu\, u_\infty\, \pi}{2\,\delta}\cos(0) \\
\frac{d}{dx}\int_0^{\delta}\left[\sin\left(\frac{\pi\, y}{2\,\delta}\right) - \sin^2\left(\frac{\pi\, y}{2\,\delta}\right)\right] dy &= \frac{\nu\, \pi}{2\, u_\infty\, \delta} .
\end{aligned}$$

The integral on the left is straightforward, especially in that the integral for $\sin^2(\cdot)$ is available in tables. We find

$$\frac{d}{dx}\left[-\frac{2\,\delta}{\pi}\,\cos\left(\frac{\pi\,y}{2\,\delta}\right)\,-\,\frac{y}{2}\,+\,\frac{2\,\delta}{4\,\pi}\,\sin\left(\frac{2\,\pi\,y}{2\,\delta}\right)\right]\Bigg|_0^{\delta} \;=\; \frac{\nu\,\pi}{2\,u_\infty\,\delta}$$

$$\frac{d}{dx}\left[-\,\frac{2\,\delta}{\pi}\,\cancelto{0}{\cos\left(\frac{\pi}{2}\right)}\,-\,\frac{\delta}{2}\,+\,\cancelto{0}{\frac{\delta\,\sin(\pi)}{2\,\pi}}\,+\,\frac{2\,\delta}{\pi}\,\cos(0)\,+\,\cancelto{0}{\frac{0}{2}}\,-\,\cancelto{0}{\frac{\delta\,\sin(0)}{2\,\pi}}\right] \;=\; \frac{\nu\,\pi}{2\,u_\infty\,\delta}$$

$$\frac{d}{dx}\left[\left(\frac{2}{\pi}\,-\,\frac{1}{2}\right)\,\delta\right] \;=\; \frac{\nu\,\pi}{2\,u_\infty\,\delta}$$

$$\frac{4\,-\,\pi}{2\,\pi}\,\cdot\,\frac{d\,\delta}{dx} \;=\; \frac{\nu\,\pi}{2\,u_\infty\,\delta}$$

This equation is a separable differential equation that can be solved as

$$\delta\,d\delta \;=\; \frac{2\,\nu\,\pi^2}{2\,u_\infty\,(4\,-\,\pi)}\;dx$$

$$\frac{\delta^2}{2} \;=\; \frac{2\,\nu\,\pi^2}{2\,u_\infty\,(4\,-\,\pi)}\;x\,+\,C_1\,,$$

but the integration constant here must be $C_1 = 0$, because the boundary layer thickness is $\delta = 0$ at the leading edge $x = 0$. Consequently,

$$\delta \;=\; \sqrt{\frac{4\,\nu\,\pi^2\,x}{2\,u_\infty\,(4\,-\,\pi)}} \;=\; \sqrt{\frac{2\,\nu\,\pi^2\,x}{u_\infty\,(4\,-\,\pi)}} \;=\; \sqrt{\frac{2\,\pi^2}{(4\,-\,\pi)}\;\frac{\nu}{u_\infty\,x}}\;\;x\,,$$

which, when recognizing that the second term under the $\sqrt{\cdot}$ is the inverse of the Reynolds number, can be simplified to

$$\delta \;=\; \sqrt{\frac{2\,\pi^2}{(4\,-\,\pi)}}\;\cdot\;\frac{x}{\sqrt{Re_x}} \;\approx\; \frac{4.80\,x}{\sqrt{Re_x}}\,.$$

Clearly, this trial profile gives a slightly closer answer to the rate constant of 5.0 from the exact solution, i.e. about a 4% under–prediction, as opposed to the polynomial–approximate value of 4.64, which under–predicts by about 7%.