**MEMS 5403**
**Midterm Exam: March 21, 2017** **Name:** ____________

**Solutions**

1. The so–called Dirichlet problem for $T(x,t)$ can be stated in the familiar form:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}\,, \qquad T(0,t) = T(L,t) = 0, \qquad T(x,0) = F(x),$$

where $\alpha$ is the diffusion coefficient and $F(x)$ is some prescribed initial condition. Consider the case of a step function defined in the domain $0 \le x \le L$ (Fig. 1):

$$F(x) = \begin{cases} 0 & : \quad 0 \le x < L/2 \\ T_0 & : \quad L/2 \le x \le L \end{cases}$$

where $T_0$ is a constant.

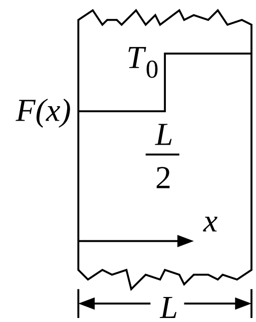


Fig. 1: *Step* $F(x)$

(a) (20 pts) Show that the solution can be expressed in the basic form

$$T(x,t) \;=\; \frac{2\,T_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\left(\frac{n\,\pi}{2}\right) \;-\; \cos\left(n\,\pi\right)\right] \sin\left(\zeta_n x\right)\; e^{-\alpha \zeta_n^2 t} \qquad \zeta_n \;=\; \frac{n\,\pi}{L}\,,$$

by first invoking the standard "general solution" and then evaluating the mode coefficients for the given $F(x)$.

*Solution:* The standard "general solution" is

$$T(x,t) \;=\; \sum_{n=1}^{\infty} C_n\, \sin\left(\zeta_n x\right)\; e^{-\alpha \zeta_n^2 t} \qquad\qquad C_n \;=\; \frac{2}{L}\int_0^L F(x)\, \sin(\zeta_n x)\, dx\,,$$

where the eigenvalues are $\zeta_n = n\,\pi/L$. The mode coefficients are readily evaluated by splitting the domain integral into the 2 equal parts suggested by $F(x)$, specifically

$$\begin{aligned}
C_n &= \frac{2}{L}\left[\int_0^{L/2} 0 \cdot \sin(\zeta_n x)\, dx \;+\; \int_{L/2}^{L} T_0 \cdot \sin(\zeta_n x)\, dx\right] \\
&= \frac{2\,T_0}{L}\int_{L/2}^{L} \sin(\zeta_n x)\, dx \;=\; \frac{2\,T_0}{L}\left[-\,\frac{1}{\zeta_n}\, \cos(\zeta_n x)\right]\Bigg|_{L/2}^{L} \\
&= -\,\frac{2\,T_0}{L} \cdot \frac{L}{n\,\pi}\left[\cos\left(\frac{n\,\pi}{L}\cdot L\right) \;-\; \cos\left(\frac{n\,\pi}{L}\cdot\frac{L}{2}\right)\right] \\
&= -\,\frac{2\,T_0}{n\,\pi}\left[\cos\left(n\,\pi\right) \;-\; \cos\left(\frac{n\,\pi}{2}\right)\right] \\
&= \frac{2\,T_0}{n\,\pi}\left[\cos\left(\frac{n\,\pi}{2}\right) \;-\; \cos\left(n\,\pi\right)\right]\,,
\end{aligned}$$

implying that the solution can be written as stated in the problem, i.e. by factoring the constant $2\,T_0/\pi$.

(b) (20 pts) The convergence rate of the series is strongly affected by $t$ because it governs how fast the exponential term decays. In fact, for sufficiently large values, the whole series can be approximated by just the first term. If we take the 1–term approximation to be sufficiently accurate when the so–called *Fourier number*, $Fo$, exceeds 0.2, i.e.

$$Fo \;=\; \frac{\alpha\, t}{L^2} \;>\; 0.2\,,$$

show that the centerline temperature decays roughly as

$$T\,\Big|_{x=L/2} \;\approx\; \frac{2\,T_0}{\pi}\, e^{-Fo\cdot\pi^2}\,.$$

*Solution:* Taking just the first term of the solution, we find

$$\begin{aligned} T(x,t) \;&\approx\; \frac{2\,T_0}{\pi}\left[\cos\left(\frac{\pi}{2}\right) \;-\; \cos(\pi)\right]\sin(\zeta_1 x)\; e^{-\alpha\zeta_1^2 t} \qquad\qquad \zeta_1 \;=\; \frac{\pi}{L}\,, \\ &\approx\; \frac{2\,T_0}{\pi}\,[0\;-\;-1]\sin(\zeta_1 x)\; e^{-\alpha\zeta_1^2 t} \;=\; \frac{2\,T_0}{\pi}\,\sin(\zeta_1 x)\; e^{-\alpha\zeta_1^2 t} \end{aligned}$$

Substituting the eigen–value and evaluating at $L/2$, a little algebra confirms the proposition

$$\begin{aligned} T\,\Big|_{x=L/2} \;&\approx\; \frac{2\,T_0}{\pi}\,\sin\left(\frac{\pi}{L}\cdot\frac{L}{2}\right)\, e^{-\alpha(\pi/L)^2 t} \\ &\approx\; \frac{2\,T_0}{\pi}\,\sin\left(\frac{\pi}{2}\right)\, e^{-\pi^2\cdot\alpha\, t/L^2} \\ &\approx\; \frac{2\,T_0}{\pi}\, e^{-Fo\cdot\pi^2}\,. \end{aligned}$$

(c) (20 pts) The form of the solution given in part (a) is actually not best–suited for numerical evaluation because many of the "modes" vanish. Suppose you wish to implement a computer program to evaluate the solution in the most *efficient* way, where these vanishing modes are already removed. (In other words, you do not want to waste CPU cycles on computing modes that add no information to the sum.) Show that the solution can be re–written in the more numerically desirable form

$$T(x,t) \;=\; \frac{2\,T_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\zeta_{2n-1}x)\; e^{-\alpha\zeta_{2n-1}^2 t}}{2\,n\;-\;1} \;-\; \frac{4\,T_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\zeta_{4n-2}x)\; e^{-\alpha\zeta_{4n-2}^2 t}}{4\,n\;-\;2}\,.$$

*Hint:* It may be useful to make a table to discern the pattern of vanishing modes and the coefficients of the modes that do not vanish.

*Solution:* Following the hint, we make such a table for the first several modes

| $n$ | $n\,\pi/2$ | $n\,\pi$ | $\cos(n\,\pi/2)$ | $\cos(n\,\pi)$ | $\cos(n\,\pi/2) - \cos(n\,\pi)$ |
|---|---|---|---|---|---|
| 1 | $\pi/2$ | $\pi$ | 0 | $-1$ | 1 |
| 2 | $\pi$ | $2\,\pi$ | $-1$ | 1 | $-2$ |
| 3 | $3\,\pi/2$ | $3\,\pi$ | 0 | $-1$ | 1 |
| 4 | $2\,\pi$ | $4\,\pi$ | 1 | 1 | 0 |
| 5 | $5\,\pi/2$ | $5\,\pi$ | 0 | $-1$ | 1 |
| 6 | $3\,\pi$ | $6\,\pi$ | $-1$ | 1 | $-2$ |
| 7 | $7\,\pi/2$ | $7\,\pi$ | 0 | $-1$ | 1 |
| 8 | $4\,\pi$ | $8\,\pi$ | 1 | 1 | 0 |
| 9 | $9\,\pi/2$ | $9\,\pi$ | 0 | $-1$ | 1 |
| 10 | $5\,\pi$ | $10\,\pi$ | $-1$ | 1 | $-2$ |
| 11 | $11\,\pi/2$ | $11\,\pi$ | 0 | $-1$ | 1 |
| 12 | $6\,\pi$ | $12\,\pi$ | 1 | 1 | 0 |

The pattern is now fairly obvious: odd modes have a coefficient of 1 and even modes alternate between $-2$ and 0. We want to drop the "zero modes", i.e. the $n \in \{4, 8, 12, \ldots\}$, suggesting we can write

$$T(x,t) \;=\; \frac{2\,T_0}{\pi} \sum_{n=1,3,5,7,\ldots}^{\infty} \frac{1}{n}\, \sin\left(\zeta_n x\right)\, e^{-\alpha \zeta_n^2 t} \;-\; \frac{4\,T_0}{\pi} \sum_{n=2,6,10,\ldots}^{\infty} \frac{1}{n}\, \sin\left(\zeta_n x\right)\, e^{-\alpha \zeta_n^2 t}\,,$$

where we have incorporated the coefficient of negative 2 in the second series. For the odd modes, we can remap indexes as

$$\begin{array}{ccccccccc} 1 & 3 & 5 & 7 & 9 & 11 & \cdots & & n \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots & & 2\,n - 1 \end{array}$$

and for the remaining modes, we have

$$\begin{array}{ccccccccc} 2 & 6 & 10 & 14 & 18 & 22 & \cdots & & n \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & & & \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots & & 4\,n - 2 \end{array}$$

which can be substituted into the above to obtain the answer.

2. Consider a solution for a certain unsteady 1–D conduction problem $0 \le x \le L$ for $t > 0$

$$T(x,t) \;=\; 2\,T_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,\pi}\, \sin\left(\zeta_n x\right)\, e^{-\alpha \zeta_n^2 t}\,,$$

where $T_0$ is a constant, $\alpha$ is the thermal diffusivity, and $\zeta_n = n\,\pi/L$ are the eigen–values.

(a) (20 pts) Show that the heat flux at the boundary $x = L$ is

$$q_x''(L,t) \;=\; \frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} e^{-\alpha \zeta_n^2 t}\,,$$

where $k$ is the thermal conductivity.

*Solution:* Applying Fourier's Law of conduction directly, we find

$$\begin{aligned}
q_x''(L,t) &= -\,k\ \left.\frac{\partial T}{\partial x}\right|_{x=L} \\
&= -\,k\ \left.\left[2\,T_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,\pi}\,\zeta_n\,\cos\left(\zeta_n x\right)\,e^{-\alpha\zeta_n^2 t}\right]\right|_{x=L} \\
&= -\,2\,k\,T_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\,\pi}\,\frac{n\,\pi}{L}\,\cos\left(\frac{n\,\pi\,L}{L}\right)\,e^{-\alpha\zeta_n^2 t} \\
&= -\,\frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} (-1)^{n+1} \cos(n\,\pi)\,e^{-\alpha\zeta_n^2 t} \\
&= -\,\frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} (-1)^{n+1}\,(-1)^n\,e^{-\alpha\zeta_n^2 t} \\
&= -\,\frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} (-1)^{2n+1}\,e^{-\alpha\zeta_n^2 t}\,.
\end{aligned}$$

Here, we notice that $2n+1$ is *always odd* for $n = 1, 2, 3, \ldots$, so that the coefficient of the exponential is *always* negative. The negative signs cancel, leaving

$$q_x''(L,t) = \frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} e^{-\alpha\zeta_n^2 t}\,.$$

(b) (20 pts) Evaluate the result in the previous question at time $t = L^2/(\alpha\,\pi^2)$, giving your answer in terms of $k$, $T_0$, $L$, and a numerical constant. The series converges very quickly in this case, so only take the first 3 terms.

*Solution:* Substituting the given time, we find

$$\begin{aligned}
q_x''(L, L^2/(\alpha\,\pi^2)) &= \frac{2\,k\,T_0}{L} \sum_{n=1}^{\infty} e^{-n^2} \\
&= \frac{2\,k\,T_0}{L}\left(e^{-1} + e^{-4} + e^{-9}\right) \\
&\approx \frac{0.773 \cdot k\,T_0}{L}\,.
\end{aligned}$$