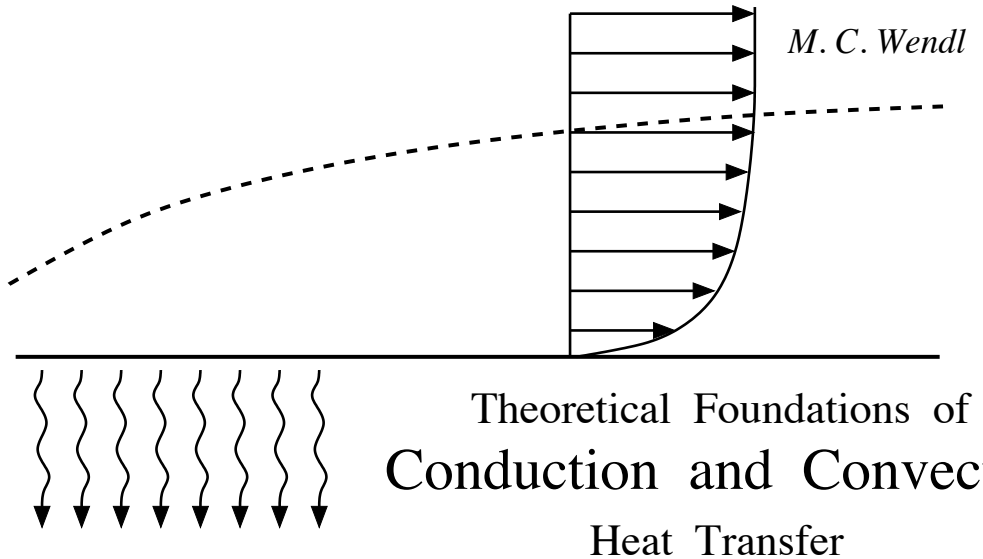


M. C. Wendl

Theoretical Foundations of Conduction and Convection Heat Transfer

Theoretical Foundations of

Conduction and Convection

Heat Transfer

Michael C. Wendl

The Wendl Foundation
Saint Louis, U.S.A.
2012

Theoretical Foundations of Conduction and Convection Heat Transfer

The Wendl Foundation is a 501(c)(3) public charity and promotes educational opportunities, in part, by producing and disseminating free, commercial–grade academic texts. The Foundation considers expert contributions for publication in all educational curricula at the primary, secondary, and college levels. Please contact:

The Wendl Foundation
attn: Textbook Committee
The Lindbergh Building, Box 310
7321 S. Lindbergh Blvd.
Saint Louis, MO U.S.A. 63125
`www.wendlfoundation.org`

Contents

Preface

The material in this monograph delves into the specific topics of conduction and convection heat transfer and should be a useful continuation for those who have completed introductory studies in thermodynamics, incompressible flow, and heat transfer and who also have a good understanding of basic calculus and differential equations. It is a "modern" treatment in the sense that the presentation is given primarily from the analytical point of view. Many older texts (e.g. McAdams, 1942) tend more toward empirics, but the main problem with this perspective is that it limits the potential for basic conceptual learning and understanding. In general, a science is advanced to the degree that its underlying phenomena can be quantified mathematically, rather than being restricted to lesser qualitative or empirical descriptions. Of course, there are still many problems where the level of mathematical difficulty exceeds what could reasonably be presented in a venue like this. In those few cases that are examined here, we will fall back to traditional treatments based on approximation (including numerical) and dimensionless analysis.

Several factors have motivated this text. First, advanced monographs tend to be written essentially as reference works and many are far too hefty for classroom use in terms of the amount of material presented. Here, the content is roughly what can be discussed in proper detail within a one–semester advanced undergraduate or a graduate survey course in conduction and convection within engineering, physics, or applied mathematics curricula. Focusing the scope in this way means omitting certain aspects. For example, we will not place heavy emphasis on coordinate system esoterica, anisotropic media, problems outside the continuum realm, numerical methods, etc. Each of these topics commands at least one dedicated course of its own. We will concentrate instead on presenting a broad conceptual survey of *fundamental* problems and the associated mathematical ideas and methods for examining them *in toto*. While all have some relevance to applications, we instead emphasize the concepts of formulation and mathematical techniques of solution.

These mathematical ideas and methods frame the second motivator. Certain aspects of the math will undoubtedly be new to the reader, but the combination of prerequisites mentioned above with the detailed presentation style used here should ensure a successful understanding of the developments. We purposely do not often fall back on extremely terse, general results commonly found in research papers and monographs; they are usually too intimidating and inaccessible for the student, and consequently tend to impede the learning process. We opt, rather, to expand these concepts in long–hand, often in appendices, so that the mathematical operations are presented in a manner that can be precisely followed and digested step–by–step and which are consequently rendered much more clear to the reader. Because this is a free book, there is no editorial pressure to limit page count and you will thus find the level of mathematical detail to be vastly more comprehensive than in a typical commercial publication. Again, this is a text meant primarily for *learning* rather than reference, though it may have some value in the latter context, too. I believe this style to be consistent with the philosophy that modern science and engineering work calls for

its practitioners to develop a comfortable relationship with advanced mathematics, a point that is certainly stated more authoritatively by the great theoretical physicist Richard Feynman in the epigraph below. "Real world" problems are invariably more difficult than those encountered in any academic course and the situation will be utterly hopeless unless one has laid a solid foundation in both concepts and the underlying mathematical methods to build upon. Some texts will end on this note, limiting their scope to the formalities of obtaining solutions, but we feel there is yet more to which the reader should be exposed. For example, there are often practical and interesting algorithmic issues in evaluating a closed–form solution, once obtained, for specific sets of parameters, as well as more esoteric aspects of the mathematical theory, e.g. solution uniqueness. We cover such topics in some detail. Our intent is that the reader obtain a truly well–rounded presentation of the fundamentals of the subject.

The last motivator for this book is cost. Retail prices of college texts have risen at a rate of about 7% per year for over two decades, far faster than inflation. These increases tend to push the affordability of proper learning materials out of reach for many students. Much has been written about this problem, much blame has been placed, and many programs have already been established to create and disseminate free or low–cost materials, for example the MIT Open Course Ware project. Hopefully, this text will contribute in some humble way to this effort.

The material presented here is but a tiny slice of the broad "universe" of conduction and convection heat transfer. The research literature and advanced reference texts (e.g. Carslaw and Jaeger, 1959) represent the complete compendium of what is currently known about these topics and it is hoped that this book helps form a preliminary link for the reader to those resources, one that might be additionally strengthened by even further study. Sections having a "star" designation are included for the sake of analytical completeness, but can be omitted without loss of continuity. The material here is largely self–contained, but we often give references to sources where additional relevant information can be found. Finally, any mistakes in this text are mine alone and I would greatly appreciate notice, so as to make the necessary corrections in future editions.

Michael C. Wendl
Saint Louis, USA
May 2012

The burden of the lecture is just to emphasize the fact that it is impossible to explain honestly the beauty of the laws of nature in a way that people can feel, without their having some deep understanding of mathematics. I am sorry, but this seems to be the case.

Richard Feynman

Table of Notation

The table below gives the general notation convention used throughout, as well as designations for dimensionless parameters, relevant acronyms, etc. Any exceptions will be noted. Symbolic modifiers are sometimes used in more than one way, for example an "overbar" can represent an average, as well as an integral–transformed quantity. In these instances, context will be clear.

Dependent and Independent Variables		Material Properties	
T, θ	temperature	k	thermal conductivity
x, y, z, r	coordinates	c	specific heat
t	time	α	thermal diffusivity
q''	heat flux	ρ	density
q	heat transfer rate	μ	dynamic viscosity
$\dot{q}$	heat generation rate	ν	kinematic viscosity (μ/ρ)
Q	heat energy		
Mathematical Functions		**Dimensionless Numbers**	
sin, cos, tan	trigonometric functions	Bi	Biot number
sinh, cosh, tanh	hyperbolic trig. functions	Kn	Knudsen number
I_j, J_j, K_j	Bessel functions	Re	Reynolds number
ζ_j	eigen–values	Pr	Prandtl number
Ψ, Γ, Ω	eigen–functions	Br	Brinkman number
erf	Gaussian error function	Nu	Nusselt number
		Fr	Froude number
Miscellaneous		**Acronyms**	
d, ∂, D	derivative signs	PDE	partial differential equation
h	convection coefficient	ODE	ordinary differential equation
φ, η, ξ, χ	non–specific variables	SOV	separation of variables
		IBP	integration by parts

CHAPTER 1

Introduction

Heat Transfer can be described as the process of energy transmission due to a gradient in temperature T, which is a measurable quantity.[1.1] Thus, determining the temperature distribution for a given problem is the desired solution since other quantities of interest, such as heat flux can be derived from it. The subject of heat transfer, which deals with *non–equilibrium* processes, is essentially an extension of an introductory course in thermodynamics, which is usually limited to *equilibrium* states. For example, a typical problem in thermodynamics might be to determine the final equilibrium T when an annealed steel machine part is quenched in a vat of water. The extension of this problem in heat transfer might be to find the rate of cooling of the steel (T as a function of time and location), which is required to predict the resulting hardness in various regions of the part.

1.1. Conduction and Convection

We will be concerned here with two main modes of heat transfer: conduction and convection.

- *Conduction* occurs through microscopic mechanisms, such as lattice vibrations and electron movement. There is no *bulk* motion of the medium — it is strictly a diffusion process. Example: the heat felt when holding one end of a long copper bar in a open fire.

- *Convection* is thermal transport via bulk motion of the medium. Example: cooling effect realized by standing in front of a fan after sprinting the 400m.

The typical heat transfer mode through solid matter is clearly conduction because of its stationary nature. Conversely, fluids (meaning liquids and gases) are more complicated. If the fluid is at rest, heat is transferred exclusively via conduction, as well. But, if motion is present, transfer is generally by both conduction and convection, though we refer to this more complex phenomenon simply as "convection".

There are other important modes of heat transfer that we will not have occasion to discuss, such as radiation (heat transfer by way of electromagnetic phenomena) and condensation and boiling (heat transfer involving a phase change).

1.2. The Continuum Assumption

One of the foundational concepts of all modern mechanics[1.2] is that of the *continuum assumption*. On a practical level, it means that physical properties are taken to be continuously distributed

[1.1] Temperature is not an entirely trivial concept to define in terms of first principles. Here, we will take the simplistic, but typical approach of understanding temperature merely as a quantity that indicates thermal energy.

[1.2] This includes, for example, fluid mechanics, solid mechanics (e.g. beam theory, elasticity, etc.), electrodynamics, thermodynamics, etc.

through space and time, such that these properties can be represented in the familiar terms of mathematical functions. In turn, these functions can be manipulated in the usual ways without further regard to physical limits. For example, say property ϕ varies in the x direction. Taking the derivative $d\phi/dx$ is meaningful *mathematically*, but the underlying principle involves a limit $\Delta x \to 0$, which *physically* takes us far below even the length scale of the molecules of our embodied system. How is this seeming contradiction resolved?

If we were to look at a mass of fluid or solid at the microscopic level, what we would see are individual molecules interacting with each other. We are not actually interested in the behavior of these individual molecules, but rather want to understand the overall (or macroscopic) behavior of the system as a whole. That is, it is the macroscopic properties such as density, temperature, or pressure drop that are of physical interest. What we are doing from the mathematical perspective is taking averages over small elemental *physical* volumes. These volumes must be large enough such that they contain enough molecules at any instant in time to yield a statistically significant average of the property of interest, yet they must also be small enough so that the statistical average does not vary over the volume. It should be a constant. If these conditions are met, the properties will have meaningful point values. In other words, they will be, to a very good approximation, continuous functions of space and time. This is the so–called *continuum assumption*.

To illustrate this concept, consider the density ρ of a fluid, defined as mass per unit volume. Without loss of generality take the volume as a cube of side length S and assume the medium in question is a fluid. Now, envision what a plot of the density would look like as a function of S (Fig. 1.2). At very small S, say comparable to molecule size, the actual number of molecules and

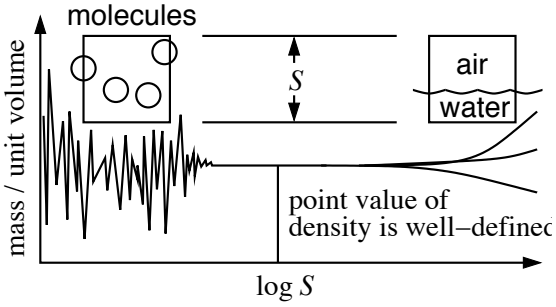


FIGURE 1.1. *Defining density as a point function according to the continuum assumption.*

thus the mass within the volume varies erratically — "density" in the sense we would like to define this concept is clearly ill–posed. There is clearly a lower limit for S which restricts the continuum assumption. Conversely, very large values of S are also quite obviously meaningless. For example, we could imagine a scale that includes heterogeneous matter, e.g. an air–water interface. Increasing S so that the cube goes deeper into the water would further increase density, whereas growing it further into the air would have the opposite effect.

The continuum assumption is clearly an issue of whether we can identify S large enough to have a statistically significant number of molecules, while still being vastly smaller than the finite scales of problems of interest. At first glance, this would seem to be the case. For example, a cubic meter of air at standard temperature and pressure ($15\,C$, $101.3\,kPa$) contains on the order of 10^{25} molecules. Therefore, the number of molecules in a volume about the size of a grain of sand, about

10^{-12} cubic meters, would still be order 10^{13}. The more rigorous assessment of the continuum assumption is made via the *Knudsen number*

$$Kn \;=\; \frac{\lambda}{L} \;=\; \frac{\text{mean free path of molecule}}{\text{length scale of problem}}\,, \tag{1.1}$$

where $Kn \ll 1$ indicates a valid continuum assumption (Schetz and Fuhs, 1999). Here, λ is the average distance traveled by a molecule before interacting with another molecule of the medium, whereby a low Kn indicates that the molecular "closeness" relative to the scale of the problem is very high. The kinetic theory of gases suggests that $\lambda \sim 10^{-7} m$, meaning that most problems in the traditional areas of heat transfer are well–suited to continuum analysis (Burmeister, 1983). We will not discuss either of the two general scenarios, which according to Eq. (1.1), violate the continuum model: rarefied media, where particles are spaced very far apart, and so–called nano–scale problems, where the length scale of the configuration itself becomes comparable to $10^{-7}\, m$.

With these ideas in place, we can define several primitive entities in terms of the continuum assumption. Here, we consider an arbitrary shaped differential volume of δV. As with the familiar use of the limit in calculus, we will define these entities in *mathematical* terms of $\delta V \to 0$, even though we realize *physically* that δV can only approach a very small, finite size.

Formally, we define *density* as mass per unit volume

$$\rho \;=\; \lim_{\delta V \to 0} \frac{\delta m}{\delta V}\,. \tag{1.2}$$

The dimensions of this quantity are mass per length to the third power.

The entity of velocity should be familiar from particle dynamics. It is a vector quantity that gives the speed of a particle in three orthogonal coordinate directions: $\mathbf{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$, where (u, v, w) represent the component magnitudes along the unit vectors $(\hat{i}, \hat{j}, \hat{k})$ in the coordinate directions (x, y, z). This is also the form we will use to denote the velocity distribution in fluids. However, we must realize that, like our other quantities, velocity is actually defined according to the continuum assumption. That is, *velocity* is the collective momentum of all the particles in δV divided by the total mass of these particles (Panton, 1984). Mathematically, we write this as

$$\mathbf{V} \;=\; \lim_{\delta V \to 0} \frac{\Sigma\, m_i\, \mathbf{V}_i}{\Sigma\, m_i}\,, \tag{1.3}$$

i.e. velocity is nothing more than momentum per unit mass. This definition permits us to use velocity as a continuum function defined at all points in the fluid.

1.3. Anatomy of Heat Transfer Problems

Heat transfer is a consequence of variations in the scalar field of temperature. Therefore, we consider a heat transfer problem solved if we know the temperature distribution, $T = T(\mathbf{r}, t)$, where, in the general case, T is a function of 3 space dimensions, $\mathbf{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, and time, t. Often, a problem will depend on only a subset of these independent coordinates, but there can be many other contributors, e.g. fluid motion, $\mathbf{V}$, and energy generation, $\dot{q}$. That is to say, the temperature problem is often more intimidating in terms of its dependencies:

$$T \;=\; T(\mathbf{r}, t, \mathbf{V}, \dot{q}, \dots)\,.$$

Presuming the distribution of T can actually be derived by suitable methods, the heat transfer follows as a direct consequence of its gradient, ∇T, whether it be via conduction (according to

Fourier's Law or other appropriate phenomenological relationships) or convection, where the so–called "no–slip" boundary condition of velocity (Panton, 1984) will be presumed to apply for our purposes, so that the temperature gradient at the boundary implies a convection coefficient.

Although heat transfer comprises a large fraction of the material presented here, we will give greater *schrift* to T, both because it is the more basic entity and because it generally poses the primary technical problem. The latter ranges from the moderately difficult (i.e. those problems of a mostly "academic" nature that may be physically improbable, or even wholly contrived) to the almost impossible (most "real–world" problems). The processes of deriving T are what we will pay special devotion to, including assessments of when a problem can be solved exactly and when suitable approximation methods must be employed.

1.4. Concept of Conservation of Energy

The cornerstone of heat transfer is the law of conservation of energy, which is described in terms of a specific volumetric space called the *control volume* (CV) and a bounding surface called the *control surface* (CS) that encloses the CV (Fig. 1.2). In later sections, we will also introduce the

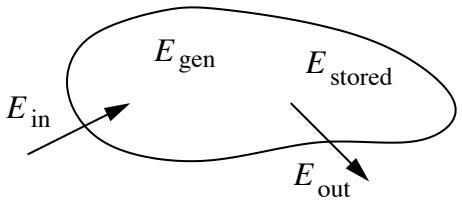


FIGURE 1.2. *Control volume and control surface schematic.*

conservation laws for mass and momentum, given the dependence of convection on fluid motion. Conservation of energy can be stated symbolically as

$$\frac{dE_{stored}}{dt} = \dot{E}_{stored} = \dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out} \,, \tag{1.4}$$

where E represents energy and the dot notation connotes a rate process.[1.3] In other words, the rate of energy increase in the control volume is equal to the rate at which it is generated internally[1.4] plus the rate at which it comes into the control volume minus the rate at which it leaves. This energy can be of various types — usually we mean thermal energy, but it can also be mechanical work. The terms $\dot{E}_{in}$ and $\dot{E}_{out}$ describe phenomena occurring at (across) the CS, while $\dot{E}_{gen}$ is a volumetric term associated with the CV. Note that the equation integrated over any time period Δt must also hold true:

$$\Delta E_{stored} = E_{in} + E_{gen} - E_{out} \,. \tag{1.5}$$

[1.3] We have not said anything about describing these $\dot{E}$ quantities yet.

[1.4] Energy can be generated by a variety of means, for example nuclear decay. One of the most common situations is electrical dissipation via current flowing through a wire having non–zero electrical resistance.

1.5. Brief Historical Perspective and the Methodological Triumvirate

Heat transfer, like many other branches of mechanics, has a complicated history of development with various fits and starts. It is highly intertwined with the development of fluid mechanics, a not–surprising fact, given the intimate relationship between convection and fluid motion. Like fluid mechanics, the "theoretical" and the "practical" were historically rather separate entities. For instance, Joseph Fourier had already made significant progress on analytical models of heat by the early 1800s, which were valuable in theoretical terms, but not widely used for applications. In fact, such methods were not really even part of the educational lexicon for engineers and physicists of the day, a point underscored in a quote made originally by Thornton Fry of Bell Telephone Laboratories in a 1941 report to Congress on the use of mathematics in industry: "The theory of linear differential equations, for example, is a subject by which the average *well–trained* engineer of 1890 would have been completely baffled" (Fry, 1948).[1.5] Indeed, much of the science of heat transfer was empirical up to the Second World War. The experimental approach played a large role in researching basic physical mechanisms of heat transfer and was used almost exclusively for engineering design and construction, a feat enabled primarily by the ability to systematically extrapolate test data using the techniques of dimensional analysis (Kays and Crawford, 1980). This state of affairs is reflected in textbooks of that time (e.g. McAdams, 1942), which while discussing some analytical elements of heat conduction, were largely empirical in their presentations of convection.

Analytical approaches had continued to mature, and, since that time, have been rapidly integrated into the mainstream of the science of heat transfer. Even more recently, say in the past 30 to 40 years, computational methods have also become increasingly important, a practicality enabled by spectacular advances in both hardware and programming languages. In essence, computation has become sufficiently cheap for routine usage by scientists and engineers.[1.6] These two additional developments have led to what might be called the "methodological triumvirate" available to today's *well–trained* practitioner, both for researching the basic physics of heat transfer and for applying its principles to engineering problems. The 3 components are:

- the experimental approach
- the analytical approach
- the computational approach

and, of course, the material we present here focuses on the middle component.

In a very real sense, the analytical perspective is the best point of departure for deeper study of the subject because it enables the clearest and most direct quantification of its principles. At some level, this is related to the fact that it is the only approach which is not, to some degree, *ad hoc*. What we mean by this statement is the following. Once a problem has been stated analytically, i.e. framed in proper and well–posed mathematical terms with all assumptions, boundary conditions, etc. specified, and, presuming this proposition can actually be solved, the result is *general*. There is now a known rule that precisely dictates how the various inputs of a problem, including velocity, type of medium, size of domain, etc., affect the output, that being usually temperature and/or heat transfer, for our purposes. In effect, it unlocks the intimate knowledge of precisely how the system behaves by simply being able to shunt new sets of parameters through the "solution rule". The

[1.5] Emphasis added.

[1.6] Tannehill et al. (1997) discuss these trends at length. They cite figures showing the relative computational cost dropping roughly 5 orders of magnitude from 1955 to 1995. That trend continues.

usefulness for both basic understanding, as well as application to practical problems is obvious. Such "economies of science" are considered to be one of the most important practical benefits of mathematics for problems in the physical world (Mach, 1956).

Conversely, computational solutions, and to a much larger degree, experimental measurements are *ad hoc* in the sense that they furnish a necessary answer for only one particular set of parameters. Discerning trends and assessing general operating characteristics, as one would routinely do via parametric studies, require repeatedly performing the associated analyses. This may be a formality for the working engineer or physicist, depending on available resources, but can be a significantly higher barrier for the student. Of course, there are many advantages to computation and experiment (Table 1.1) and we might say that a true expert on the subject of heat transfer will be highly knowledgeable in all 3 areas.

TABLE 1.1. Characteristics of the Triumvirate of Scientific Methods

Method	Advantages and Disadvantages
Experimental[†]	Potential for most realistic modeling of a problem of interest, but at typically enormously larger cost than other 2 methods; significant technical hurdles and equipment requirements; *ad hoc* results, although these are generalizable to certain other combinations of parameters via dimensionless analysis
Analytical	Entirely general, but issues of mathematical tractability limit realism in terms of physics and geometries; well–established methods exist for linear problems, but no general approach for non–linear problems yet known
Computational	Can treat non–linear physics and complex geometries; solutions are *ad hoc* but computational runs can often be reconfigured without much difficulty; costs continue to fall; expectation of independent validation due to various approximations and known limitations in modeling certain complex phenomena such as turbulence

[†]Here we include the "observational", which can be thought of as dealing with spontaneous or naturally–occurring inquiries, e.g. sampling ocean temperatures to infer convection patterns or other geothermal aspects of the oceans (e.g. von Arx, 1962)

A great deal more could be written here on the *interdependence* of these methods upon one another. Here, we will say only that the usefulness of analytical methods extends significantly past what meets the eye. For example, they are often used in the validation role for numerical methods mentioned in Table 1.1. They are also crucial in the nascent "finite analytic" approach that casts analytical solutions within finite sub–domains of a problem and might be combined with numerical aspects to develop very efficient and very accurate analytical–numerical methods. We hope this is sufficient justification to study this worthwhile subject.

Much has been written on the history and development of heat transfer as a scientific, engineering, and mathematical endeavor (e.g. Eckert, 1981; Narasimhan, 1999). Goldstein (1969) and Tani (1977) summarize the more basic history of fluid mechanics, which is the foundation of convection, and a particularly comprehensive examination is given in the book by Eckert (2006).

1.6. Exercises

1.1 A large class of problems in heat conduction involves solving ordinary differential equations having constant coefficients. Show by any suitable method that the general solution of

$$\frac{d^2T}{dx^2} \, + \, 5 \, \frac{dT}{dx} \, + \, 4 \, T \, = \, 0$$

is $T(x) = C_1 \, e^{-4x} + C_2 \, e^{-x}$, where C_1 and C_2 are constants. Furnish step–by–step details of your solution process.

1.2 A particular class of mathematical methods for more sophisticated heat conduction problems involves techniques to transform a partial differential equation for temperature, T, into an auxiliary ordinary differential equation for a transformed variable, $\overline{T}$. Given the ordinary equation

$$\frac{d\overline{T}}{dt} \, + \, \alpha \, \zeta_n^2 \, \overline{T} \, = \, G(\zeta_n, t) \, ,$$

where α is the thermal diffusivity, ζ_n is a constant, t is time, and G is an arbitrary function, show that

$$\overline{T} \, = \, e^{- \, \alpha \, \zeta_n^2 \, t} \left[\overline{F}(\zeta_n) \, + \, \int_0^t e^{\alpha \, \zeta_n^2 \, t'} \, G(\zeta_n, t') \, dt' \right]$$

is a solution to this problem, where $\overline{F}(\zeta_n)$ is associated with an initial condition.

CHAPTER 2

Fundamentals of Heat Conduction

HEAT TRANSFER VIA CONDUCTION is a diffusion phenomenon. Here we will introduce the basic concepts of diffusive conduction, including governing equations and boundary conditions, that describe this important mode of heat transport.

2.1. Diffusion and Fourier's Law

Diffusion is the process by which an entity, in this case heat energy, is transported by way of random molecular motion. Diffusive effects can be illustrated in many ways, a traditional experiment being to have a container of water and an iodine solution separated by a partition. Assume the partition can be removed without causing any actual motion of the fluids. At time $t = 0$, the two fluids are in their initial, unmixed state and the boundary between the two is sharp and distinct. For $t > 0$, we start to visually observe color progressively more on the water side and less on the iodine side. After a long time, $t \gg 0$, the color of the overall solution is uniform. Iodine molecules have clearly been transported throughout the container, though not by any motion of the fluid.

Diffusion can be discussed on a molecular basis in terms of the so–called "random walk" phenomenon, but we are concerned once again with the macroscopic viewpoint. Empirically, we find that, like many other diffusion processes, the rate of heat transfer is proportional to a gradient in the direction of the transport. In the case of conduction, the basic governing law is Fourier's Law of Heat Conduction, which in one dimension takes the form

$$q'' = -k(T)\,\frac{dT}{dx}\,, \tag{2.1}$$

where q'' is heat flux, k is thermal conductivity,[2.1] and dT/dx is temperature gradient along an independent coordinate x. Eq. (2.1) is a *phenomenological law* derived from numerous experimental observations rather than first principles. It holds for all the physical configurations we shall study here, but additional terms are required for more complicated materials, for example some animal

[2.1] Eq. (2.1) shows conductivity being a function of the temperature, $k(T)$, which it is, in general. However, this contingency renders conduction problems non–linear. In many situations k may only be a suitably weak function of T, or temperature differences might not be severe enough that variable conductivity makes an appreciable contribution. Here, conductivity can be taken as a constant, in which case we write this equation simply as

$$q'' = -k\,\frac{dT}{dx}\,. \tag{2.2}$$

If A_c is the cross–sectional area, then the heat transfer is

$$q = -k\,A_c\,\frac{dT}{dx}\,. \tag{2.3}$$

tissues.[2.2] In many cases of interest, the thermal conductivity can be reasonably approximated as constant. When its dependence on temperature must be considered, i.e. $k = k(T)$, the resulting problem becomes non–linear and commensurately more difficult mathematically.

Heat flux is clearly a directional quantity. For homogeneous, isotropic materials[2.3] the general three–dimensional form written in vector notation is

$$\mathbf{q}'' = -k\,\nabla T = -k\left(\hat{i}\,\frac{\partial T}{\partial x} + \hat{j}\,\frac{\partial T}{\partial y} + \hat{k}\,\frac{\partial T}{\partial z}\right). \tag{2.4}$$

Although the negative sign in Eqs. (2.1) and (2.4) looks rather strange, it is required for consistency of the physics. Heat energy is conducted along a temperature gradient. That is, energy flows from a high temperature region to one of lower temperature. For example, in Fig. 2.1 energy flows in the positive x direction, therefore q'' must be a positive quantity. The temperature gradient in this

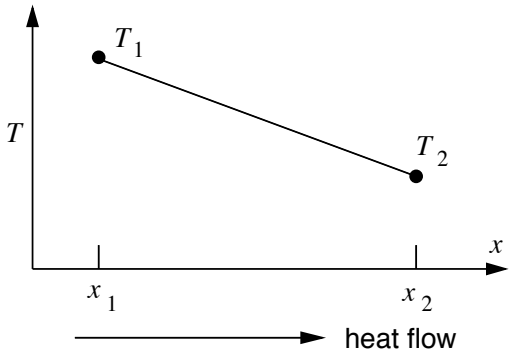


FIGURE 2.1. *Heat is conducted along a negative gradient from high temperature toward low temperature, shown in the positive direction here.*

case is linear, i.e.

$$\frac{dT}{dx} = \frac{T_2 - T_1}{x_2 - x_1},$$

which is clearly a negative quantity, since $x_2 - x_1$ is positive, but $T_2 - T_1$ is negative. In fact, $T_2 - T_1$ *must* be negative, otherwise the resulting gradient would not allow energy to flow as shown. Since k is defined as positive, the leading negative sign is clearly required for consistency. In this sense, the temperature gradient is very similar to the pressure gradient in pipe flow: movement proceeds along a negative gradient.

2.2. Thermal Properties of Matter

We mentioned the thermal conductivity k as a material parameter in Fourier's Law. Two other quantities are important to the problems we wish to study: the volumetric heat capacity, ρc_p, and the thermal diffusivity, α. Units for these quantities are listed in Table 2.1. Other properties will arise as well, including many of those associated with fluid mechanics problems and surface properties for radiation. These will be introduced as needed.

[2.2] This situation is quite analogous to Newtonian versus non–Newtonian fluids. Recall that for a Newtonian fluid, shear stress and rate of strain are linearly related, where the constant of proportionality is the viscosity. For non–Newtonian fluids, additional terms result in much more complicated relationships.

[2.3] These are materials for which the thermal conductivity is independent of direction.

TABLE 2.1. Thermal Properties

Property	Notation	Units
thermal conductivity	k	$W/(m\,K)$
volumetric heat capacity	$\rho\, c_p$	$(kg/m^3) \times J/(kg\,K) = J/(m^3\,K)$
thermal diffusivity	$\alpha = k/(\rho\, c_p)$	m^2/s

Recalling that the physical basis for conductivity is at the molecular and atomic levels, we would suspect that solids would generally have the highest conductivities, followed by liquids, and finally gases. This is in fact the case and is largely due to differences in the molecular spacings (Fig. 2.2). Actual values are available in many references (e.g. Weast and Astle, 1982).

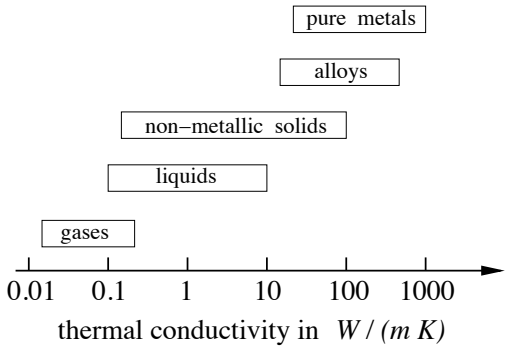


FIGURE 2.2. *Approximate ranges of thermal conductivity for various classifications of matter.*

2.3. Derivation of the Conduction Equation

It was mentioned in the opening sentences of Chapter 1 that the goal of heat transfer is to determine the temperature distribution. In the general case, this will depend on all three spatial dimensions and on time, so that $T = T(\mathbf{r}, t)$. We introduced Fourier's Law in Eq. (2.1), but this alone does not provide a foundation for calculating T. Instead, we must base our theoretical framework on the conservation law for energy, which was introduced only at a conceptual level in Chapter 1.

We extend the conceptual treatment using the classic differential approach; The resulting equation will be valid for every differential point in a problem domain. This is characteristic of differential formulations: the conservation law must be satisfied simultaneously for all $(\mathbf{r}, t)$. In this sense, the differential formulation is *exact*, however, it typically presents a more challenging mathematical situation than the integral approach.[2.4] We define a differential element with properties of interest defined in the center. Any properties depending upon a flux can be extrapolated to the boundaries using truncated 1–term Taylor series.[2.5]

[2.4]The integral approach is not typically as important in heat transfer as in fluid mechanics, but will be introduced later for certain configurations.

[2.5]Terms of second–order and higher can all be neglected since they involve products of the differential quantities, e.g. $(\delta x)^2$, which are exceedingly small.

Define the differential element according to the volume $\delta x \times \delta y \times \delta z$ (Fig. 2.3). According to

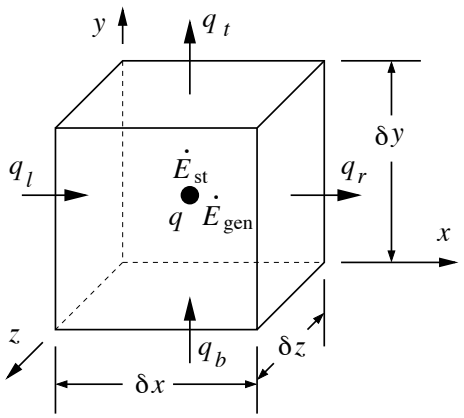


FIGURE 2.3. *Differential volume showing energy generation $\dot{E}_{gen}$ and energy storage $\dot{E}_{st}$ terms in the center and heat fluxes at the boundaries. Diagram omits fluxes pointing in the positive z–direction, q_n at the near face and q_f at the far face (at $z = 0$).*

the concept of conduction, the material is either a solid, or a non–moving fluid. We formulate all quantities at a given instant of time: the energy generation and storage terms, $\dot{E}_{gen}$ and $\dot{E}_{st}$, represented as volumetric entities in the element, and all heat conduction terms at the boundaries. Expressing these terms as 1–term Taylor series expansions with respect to their associated values defined in the center, we find

$$q_l \;=\; q_x \;-\; \frac{\partial q_x}{\partial x}\,\frac{\delta x}{2} \qquad \text{and} \qquad q_r \;=\; q_x \;+\; \frac{\partial q_x}{\partial x}\,\frac{\delta x}{2}\,, \tag{2.5}$$

$$q_b \;=\; q_y \;-\; \frac{\partial q_y}{\partial y}\,\frac{\delta y}{2} \qquad \text{and} \qquad q_t \;=\; q_y \;+\; \frac{\partial q_y}{\partial y}\,\frac{\delta y}{2}\,, \tag{2.6}$$

$$q_f \;=\; q_z \;-\; \frac{\partial q_z}{\partial z}\,\frac{\delta z}{2} \qquad \text{and} \qquad q_n \;=\; q_z \;+\; \frac{\partial q_z}{\partial z}\,\frac{\delta z}{2}\,, \tag{2.7}$$

where higher–order terms have been omitted. Energy generation is simply

$$\dot{E}_{gen} \;=\; \dot{q}\;\delta x\;\delta y\;\delta z\,, \tag{2.8}$$

where $\dot{q}$ is the per unit volume rate of generation.[2.6] Finally, the rate of change of thermal energy stored in the element can be expressed as

$$\dot{E}_{st} \;=\; \rho\,c_p\,\frac{\partial T}{\partial t}\;\delta x\;\delta y\;\delta z \tag{2.9}$$

based on the dimensional argument using the volumetric heat capacity $\rho\,c_p$ in units of $J/(m^3\,K)$ given in Table 2.1.

[2.6]There does not seem to be a standard notation for heat generation. Here, we follow the choice of many introductory texts, e.g. Incropera and Dewitt (2002), in using $\dot{q}$. This should not be confused with heat flux q''. Other texts, like Özişik (1980) use g, which are also easily confused with other quantities, like gravitational acceleration.

These terms can be used directly in the conceptual conservation law in Eq. (1.4) on pp. 4, where the rate of energy entering the element is $\dot{E}_{in} = q_b + q_f + q_l$ and the rate of energy leaving is $\dot{E}_{out} = q_n + q_r + q_t$. This operation yields

$$\rho\, c_p\, \frac{\partial T}{\partial t}\, \delta x\, \delta y\, \delta z \;=\; \dot{q}\, \delta x\, \delta y\, \delta z \;+\; q_b \;+\; q_f \;+\; q_l \;-\; (q_n \;+\; q_r \;+\; q_t)\;, \tag{2.10}$$

which becomes

$$\begin{aligned}\rho c_p \frac{\partial T}{\partial t}\delta x\delta y\delta z = \dot{q}\delta x\delta y\delta z + \left(q_y \;-\; \frac{\partial q_y}{\partial y}\;\frac{\delta y}{2}\right) + \left(q_z \;-\; \frac{\partial q_z}{\partial z}\;\frac{\delta z}{2}\right) + \left(q_x \;-\; \frac{\partial q_x}{\partial x}\;\frac{\delta x}{2}\right) \\ - \left(q_z \;+\; \frac{\partial q_z}{\partial z}\;\frac{\delta z}{2}\right) - \left(q_x \;+\; \frac{\partial q_x}{\partial x}\;\frac{\delta x}{2}\right) - \left(q_y \;+\; \frac{\partial q_y}{\partial y}\;\frac{\delta y}{2}\right)\;,\end{aligned} \tag{2.11}$$

after substituting Eqs. (2.5) through (2.7) for the heat conduction terms. We then cancel and combine terms appropriately to get

$$\rho\, c_p\, \frac{\partial T}{\partial t}\, \delta x\, \delta y\, \delta z \;=\; \dot{q}\, \delta x\, \delta y\, \delta z \;-\; \left(\frac{\partial q_x}{\partial x}\, \delta x \;+\; \frac{\partial q_y}{\partial y}\, \delta y \;+\; \frac{\partial q_z}{\partial z}\, \delta z\right)\;. \tag{2.12}$$

Eq. (2.12) represents the limit in terms of what can be derived strictly from theory. It cannot be solved, because T and $\mathbf{q}$ are both unknowns.[2.7] However, we have additional relationships between T and $\mathbf{q}$ in the form of Fourier's Law. In light of Eq. (2.4), we can express the heat terms as

$$q_x \;=\; -k\, \frac{\partial T}{\partial x}\, \delta y\, \delta z \qquad \text{and} \qquad q_y \;=\; -k\, \frac{\partial T}{\partial y}\, \delta x\, \delta z \qquad \text{and} \qquad q_z \;=\; -k\, \frac{\partial T}{\partial z}\, \delta x\, \delta y\;, \tag{2.13}$$

which can be substituted into Eq. (2.12) to obtain

$$\begin{aligned}\rho\, c_p\, \frac{\partial T}{\partial t}\, \delta x\, \delta y\, \delta z \;=\; \dot{q}\, \delta x\, \delta y\, \delta z \;-\; \Bigg[\frac{\partial}{\partial x}\left(-k\, \frac{\partial T}{\partial x}\, \delta y\, \delta z\right)\, \delta x \\ + \frac{\partial}{\partial y}\left(-k\, \frac{\partial T}{\partial y}\, \delta x\, \delta z\right)\, \delta y \;+\; \frac{\partial}{\partial z}\left(-k\, \frac{\partial T}{\partial z}\, \delta x\, \delta y\right)\, \delta z\Bigg]\;.\end{aligned} \tag{2.14}$$

All volume terms $\delta x \times \delta y \times \delta z$ cancel and all double–negatives cancel, leaving

$$\rho\, c_p\, \frac{\partial T}{\partial t} \;=\; \dot{q} \;+\; \frac{\partial}{\partial x}\left(k\, \frac{\partial T}{\partial x}\right) \;+\; \frac{\partial}{\partial y}\left(k\, \frac{\partial T}{\partial y}\right) \;+\; \frac{\partial}{\partial z}\left(k\, \frac{\partial T}{\partial z}\right) \tag{2.15}$$

as the general three–dimensional time–dependent form of the heat conduction equation, or in vector notation with functional dependencies annotated

$$\rho\, c_p\, \frac{\partial T(\mathbf{r}, t)}{\partial t} \;=\; \dot{q}(\mathbf{r}, t) \;+\; \nabla \cdot \Big[k(T)\, \nabla T(\mathbf{r}, t)\Big]\;. \tag{2.16}$$

Eq. (2.15) states the conduction equation explicitly in Cartesian (rectangular) coordinates, which will be sufficient for the majority of the problems we will study, while Eq. (2.16) is a more general statement in terms of the ∇ operator. The latter implicitly specifies all coordinate systems, given the associated forms of ∇, and obtaining this equation explicitly in another system is relatively straightforward using coordinate transformation.[2.8]

[2.7] We assume that heat generation $\dot{q}$ would be prescribed, or would be measurable for a problem.

[2.8] Further discussion of coordinate transformation, particularly the conduction equation in various orthogonal coordinate systems, is found in many sources e.g. Özişik (1980). Other sources have extensive tabulation of operators in various coordinate systems e.g. Panton (1984).

2.4. Discussion of the Conduction Equation and its Special Cases

Eq. (2.16) expresses conservation of energy in terms of the single dependent variable of temperature T. We can solve this in principle because there is one equation and exactly one unknown and we will largely examine it in the context of a *linear partial differential equation*. Linearity is satisfied if T does not appear in the form of any products of itself, or its derivatives, involve any special functions of itself or its derivatives, etc. For example, if k is a constant and $\dot{q}$ is, at most, a linear function of T, Eq. (2.16) is linear. The mathematical advantage to such problems is that the theory of linear equations is extremely well developed and there are many general methods for solution. Conversely, there is no corresponding comprehensive theory of non–linear equations. Fortunately, there are methods for special cases of interest, especially problems for which $k = k(T)$. We will examine this particular class of problems in detail. Additionally, if the source term $\dot{q}$ vanishes, the equation is *homogeneous* and this simplification entails certain additional advantages for solution that we shall study in detail.

For the class of linear conduction problems, we will take k as a constant, whereby it can be moved outside of the derivatives yielding

$$\frac{1}{\alpha}\,\frac{\partial T}{\partial t} \;=\; \frac{\dot{q}}{k} \;+\; \frac{\partial^2 T}{\partial x^2} \;+\; \frac{\partial^2 T}{\partial y^2} \;+\; \frac{\partial^2 T}{\partial z^2}\,, \tag{2.17}$$

where $\dot{q}$ is again the energy generation rate per unit volume (W/m^3) and α is the thermal diffusivity (Table 2.1). Further simplifications are possible, for example if conduction is also steady, Eq. (2.17) reduces to

$$\frac{\dot{q}}{k} \;+\; \frac{\partial^2 T}{\partial x^2} \;+\; \frac{\partial^2 T}{\partial y^2} \;+\; \frac{\partial^2 T}{\partial z^2} \;=\; 0\,. \tag{2.18}$$

Moreover, if energy generation is absent, we obtain

$$\frac{\partial^2 T}{\partial x^2} \;+\; \frac{\partial^2 T}{\partial y^2} \;+\; \frac{\partial^2 T}{\partial z^2} \;=\; 0\,, \tag{2.19}$$

which is *Laplace's equation* and which also can be written in general vector notation as $\nabla^2 T = 0$, where ∇^2 is the Laplacian operator. This equation may simplify further if T is only a function of fewer than the three independent coordinates.

2.5. Boundary and Initial Conditions

The various forms of the conduction equation we have just discussed govern the physics of how heat is conducted in the interior of some pre–defined domain. However, this still does not completely specify the problem. We must also have some prescription of temperature (or its derivatives) on the boundaries of the domain — these are called *boundary conditions*. Moreover, if the problem is also unsteady, we must also know the *initial condition* for the problem. That is, we must have the value of T (or its derivative) at some specific time.

2.5.1. Initial Conditions. Conduction problems involve second derivatives of their spatial coordinates, meaning there are two required boundary conditions for each coordinate. Conversely, there is only one initial condition needed for any unsteady problem because the temporal term

appears as a first derivative. Almost always, the latter takes the form of prescribing T itself at some instant of time, usually by convention at $t = 0$, which can be written in a few different ways

$$T(\mathbf{r}, t)\,|_{t=0} \;=\; T(\mathbf{r}, 0) \;=\; T_0(\mathbf{r})\,, \tag{2.20}$$

where again $\mathbf{r} = (x, y, z)$ or other coordinates, as appropriate.

2.5.2. Boundary Conditions of the First Kind: Dirichlet. Boundary conditions, on the other hand, can take a number of forms, depending upon the physics of the problem. The most obvious case is similar to the initial condition, where a specific value of T is specified at a coordinate location. For example we could write

$$T(x, y, z, t)\,|_{x=0} \;=\; T_s \qquad \text{or} \qquad T(x, y, z, t)\,|_{x=L} \;=\; T_s\,, \tag{2.21}$$

where T_s is the temperature at the surface coinciding with the coordinate location $x = 0$ or $x = L$, respectively. Eq. (2.21) is commonly known as a boundary condition of the first kind, or a *Dirichlet* boundary condition, and is common to heat transfer problems. For instance, if the boundary is in contact with a medium undergoing a phase change, the constant temperature boundary condition is a good model.

2.5.3. Boundary Conditions of the Second Kind: Neumann. A boundary condition of the second kind, or *Neumann* boundary condition, involves the derivative of T. For instance, Fourier's Law in Eq. (2.1) on pp. 8 is a ready–made Neumann boundary condition when applied at a physical boundary, e.g.

$$q''_s \;=\; -k\,\frac{\partial T}{\partial x}\,\bigg|_{x=L}\,. \tag{2.22}$$

In other words, the heat flux q''_s at the $x = L$ boundary is related to the temperature gradient $\partial T/\partial x$ at $x = L$ as prescribed by this equation. The special case of the *adiabatic* (perfectly insulated) boundary is given by

$$\frac{\partial T}{\partial x}\,\bigg|_{x=L} \;=\; 0\,. \tag{2.23}$$

2.5.4. Boundary Conditions of the Third Kind: Robbins. Boundary conditions of the third type, also called *Robbins* boundary conditions, specify a balance between energy conduction at the boundary and the rate at which this energy is convected away. The latter is specified by *Newton's Law of Cooling*:

$$q''_s \;=\; h\,(T|_{x=L} - T_\infty)\,, \tag{2.24}$$

where T_∞ is the freestream temperature of the flow and h is the convection heat transfer coefficient. Eq. (2.24) implies the definition of the entity h, such that its product with the temperature difference gives the heat flux.[2.9] In fact, h is a complicated quantity that depends on the flow and energy equations — it is the main focus of convection heat transfer to be discussed later. The form of a Robbins boundary condition is something of a combination of the first two types of boundary

[2.9] Clearly, h must have units of $W/(m^2\,K)$ so that its product with a temperature difference yields heat flux. If h can be expressed as an average value for an area A, Newton's Cooling Law can also be written in the useful form $q_s = h\,A\,(T|_{x=L} - T_\infty)$ for the actual heat transfer rate.

condition in that it contains both temperature itself and its derivative. Equating the expressions in Eqs. (2.22) and (2.24), we find[2.10]

$$-k \left.\frac{\partial T}{\partial x}\right|_{x=L} = h \left(T|_{x=L} - T_\infty\right) . \tag{2.25}$$

2.6. Exercises

2.1 Derive the statement of conservation of mass in the cylindrical coordindate system, c.f. Eq. (I.1) on pp. 182, by analyzing a differential "sector" element.

[2.10] There is an equation of identical form, but somewhat different physical meaning, that arises when considering heat transfer strictly on the fluid side. This is discussed in detail in §7.4 on pp. 92.

CHAPTER 3

One–Dimensional Steady Conduction: Fins

In the simplest case, conduction heat transfer occurs only along one primary dimension, say the x direction, and the problem is steady. Almost all such problems are elementary from a mathematical perspective. For example, if $\dot{q} = 0$, then we see from Eq. (2.19) that the governing equation is $d^2T/dx^2 = 0$, which is readily integrated to find T. Here, we will examine a class of more interesting problems, primarily in the context of "extended surfaces", i.e. fins, and the conditions under which the one–dimensional model could be reasonably expected to apply.

3.1. The One–Dimensional Model and the Biot Number

While there are several different physical scenarios under which the one–dimensional model is reasonable, their commonality is that the gradient in one of the coordinate directions is much larger than those in the other directions. Consider a simple two–dimensional domain shown in Fig. 3.1 and assume that the temperature on the boundaries is given by the equations

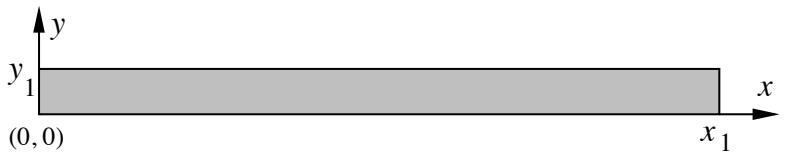


FIGURE 3.1. *Two–dimensional rectangular domain with a side length in one dimension much longer than the other, i.e.* $x_1 \gg y_1$.

$$T\,|_{x=0} = T\,|_{y=0} = 0 \quad \text{and} \quad T\,|_{x=x_1} = T\,|_{y=y_1} = T_1\,.$$

According to discussions in Chapter 2, particularly Eq. (2.19) on pp. 13, we could deduce that this problem is governed formally by the conduction equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0\,. \tag{3.1}$$

However, if we look at the rough sizes of temperature gradients using a simple finite–difference approximation,

$$\frac{\partial T}{\partial x} \approx \frac{T_1 - 0}{x_1 - 0} \quad \text{and} \quad \frac{\partial T}{\partial y} \approx \frac{T_1 - 0}{y_1 - 0}\,,$$

it is clear that gradients in the x direction will be small compared to those in the y direction. Thus, such a problem could be satisfactorily approximated as one–dimensional. In fact, we often recognize such cases on a dimensional basis, that is, whenever two of the length dimensions are much larger than the third.

The subtlety in Fig. 3.1 is that, while the temperature itself varies, i.e. is not constant in the x–direction, the length scales are such that the rate of change of T with respect to x is much smaller than the rate in the y direction. Another category of problems in which the one–dimensional model is valid is the more obvious case where the temperature in two of the coordinate directions is essentially constant. There is a formal, dimensionless method for identifying such instances. Consider the slab geometry in Fig. 3.2, where the left–boundary, right–boundary, and freestream convection temperatures are, T_0, T_L, and T_∞, respectively, and $T_0 > T_L > T_\infty$. The assumption

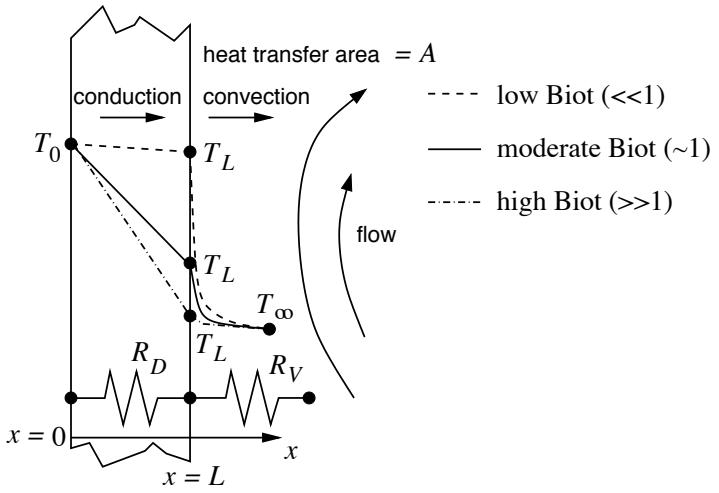


FIGURE 3.2. *Dimensionless assessment of constant temperature in a slab of thickness L based on the difference between a given left–boundary temperature, T_0, and 3 possible values of a right–boundary temperature, T_L, along with a freestream temperature, T_∞, of a convecting flow. Corresponding resistors from the circuit analogy are also shown: R_D and R_V for conduction and convection, respectively.*

of "almost" constant temperature within the conductor clearly depends on how close T_L is to T_0. By a straightforward conservation of energy argument, the conduction and convection fluxes at the $x = L$ boundary must be equivalent. Writing these respective fluxes using an approximate finite–difference form of Eq. (2.1) on pp. 8 and Newton's Law of Cooling in Eq. (2.24) on pp. 14, respectively, we find

$$-\,k\,\frac{T_L - T_0}{L} \;\approx\; h\,(T_L - T_\infty) \tag{3.2}$$

which is readily re–arranged to reveal the *Biot Number*, Bi:

$$\frac{T_0 - T_L}{T_L - T_\infty} \;\approx\; \frac{h\,L}{k} \;=\; \frac{L/(kA)}{1/(hA)} \;=\; \frac{R_D}{R_V} \;=\; Bi\,. \tag{3.3}$$

The temperature ratio in this expression clearly shows that a state of constant temperature is approached progressively better by smaller values of Bi. Although formally we would require $Bi \ll 1$, most texts advise $Bi < 0.1$ as an acceptable threshold (Holman, 2010; Bergman et al., 2011). The second ratio, hL/k, is the Biot number definition,[3.1] from which actual calculations can

[3.1] Note that k in Eq. (3.3) is the thermal conductivity of the conductor, not the flowing fluid.

be made, presuming h is known. The third and fourth ratios furnish a physical interpretation of the Biot number, where $R_D = L/(kA)$ and $R_V = 1/(hA)$ are the resistor model values from the conduction circuit analogy.[3.2] In other words, the Biot number is

$$Bi \;=\; \frac{h\,L}{k} \;=\; \frac{\text{resistance of conduction}}{\text{resistance of convection}}\;, \tag{3.4}$$

and, if this figure is small, convection is the rate determining factor. Equivalently, the conductivity is very high so that temperature gradients in the conductor are low enough such that their product, the heat transfer, correctly matches the heat transfer by convection at the boundary.

3.2. Derivation of the Fin Equation

Extended surfaces, or "fins", are often used to increase heat transfer, for example as applied to air conditioner coils, I/C chips, heat exchangers, car radiators, lawn mower engines, pipes, etc. Why is this the case? If we recall the concept of Newton's Law of Cooling introduced in Eq. (2.24) on pp. 14, we can write a generic form of this convection equation as

$$q \;=\; h\,A_s\,\left(T\,\Big|_{surface} - \,T_\infty\right)\;, \tag{3.5}$$

where h is once again the convective heat transfer coefficient and A_s is the surface area available for heat transfer. How can the various components of this equation be modified to increase q?

- T_∞: This is the ambient temperature of the surrounding fluid environment and, as such, cannot usually be changed much, if at all. For example, if air is the working fluid then the atmospheric air temperature is essentially a constant.

- $T\,|_{surface}$: Similar to T_{fluid}, the device temperature and thus the resulting surface temperature is usually constrained by the operating range of the device.

- h: This can be increased to a degree, however, it still may be highly constrained, for example adding a blower to increase convection may be possible, but it may be impractical due to weight, size restrictions, electrical considerations, aerodynamics, etc.

- A_s: It's easy to increase this by huge margins using fins and Eq. (3.5) indicates there is a proportional increase in the heat transfer.

Of course, to properly analyze such systems, we must first have the appropriate equation describing the conservation of energy. Here, we develop the so–called fin equation, which is a special case of the general conduction equation that explicitly incorporates a convection boundary condition. The conceptual justification for this special treatment is as follows. Fins are normally fashioned from high–conductivity material and they are typically employed within gas rather than liquid media, as the former have lower convection coefficients. Moreover, maximizing area logically implies

[3.2] Recall the simple one–dimensional model $d^2T/dx^2 = 0$ implies, whose general solution is $T(x) = C_1\,x + C_2$, and whose gradient is constant. With respect to the notation in Fig. 3.2, this gradient is $(T_L - T_0)/L$, so that Fourier's Law becomes

$$q \;=\; -\,k\,A\,\frac{dT}{dx} \;=\; -\,k\,A\,\frac{T_L - T_0}{L} \;=\; k\,A\,\frac{T_0 - T_L}{L} \;=\; \Delta T\,\frac{k\,A}{L}\;,$$

which is analogous to the linear voltage drop in DC circuits, $I = \Delta V/R$, for a flowing current. Newton's Law of Cooling is a direct analog, i.e. $q = hA(T_L - T_\infty) = hA\Delta T$. The circuit analog model is discussed extensively in introductory texts (e.g. Holman, 2010; Bergman et al., 2011).

maximizing surface area to volume ratios, meaning that the cross–sectional length scales tend to be small (Fig. 3.3). These factors all imply a low Bi per Eq. (3.4). That is to say, to a

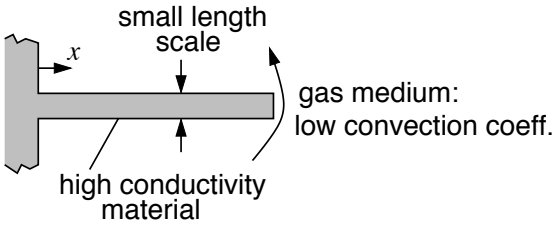


FIGURE 3.3. *Elements of good fin design imply low cross–sectional Biot number according to Eq.* (3.4)*: small length scale, L, small convection coefficient, h, of gas media, and large thermal conductivity, k, of the fin material.*

very good approximation, the temperature in any cross–sectional slice through the fin is constant. Consequently, we take temperature to vary only along the fin's length, e.g. in the x direction in Fig. 3.3. Heat conduction in the fin satisfies the one–dimensional idealization.

In the usual fashion, we analyze a differential length dx of a fin as shown in Fig. 3.4. Here,

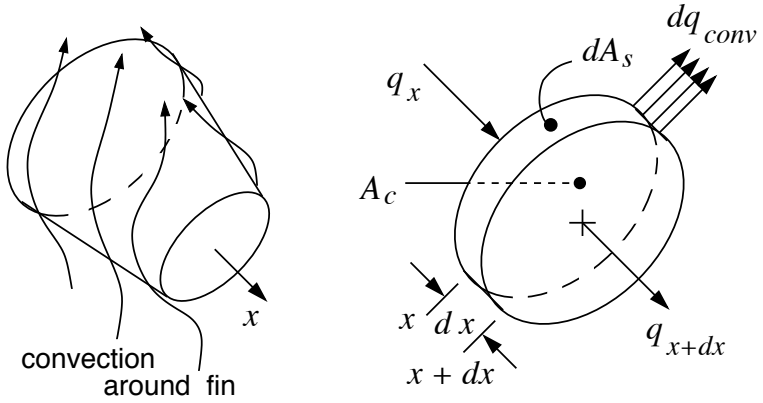


FIGURE 3.4. *Schematic of a general, variable–area fin showing convection (left) and differential element of thickness dx (right). Cross–sectional shape is arbitrary, although it is shown here as roughly circular. Temperature in this cross–section is taken as constant.*

energy is conducted along the fin in the x direction and dissipated by convection along the entire outer boundary in what we will take as a steady–state process. The local cross–sectional area at x is A_c, however, we assume this can vary along the axis, so we properly write this as a function of x, as in $A_c(x)$. The local differential boundary area exposed to convection is dA_s.

Heat is conducted into the element at x at a rate of q_x and is conducted out of the element at location $x + dx$ at a rate q_{x+dx}. Heat is convected away at the radial boundary at a rate dq_{conv}. For energy to be conserved, we write the simple conservation equation[3.3]

$$q_x \;=\; q_{x+dx} \;+\; dq_{conv}\,. \tag{3.6}$$

[3.3]This is clearly a form of Eq. (1.4) on pp. 4: $\dot{E}_{stored} = \dot{E}_{in} + \dot{E}_{gen} - \dot{E}_{out}$, where $\dot{E}_{stored} = \dot{E}_{gen} = 0$.

We immediately recognize q_x from Fourier's Law in Eq. (2.2) on pp. 8, which can be written for this problem as

$$q_x \;=\; -k\, A_c(x)\, \frac{dT}{dx}\,, \tag{3.7}$$

where $A_c(x)$ is again the cross–sectional area of the fin that varies with x. Note that we are explicitly assuming a constant thermal conductivity. We can expand q_{x+dx} according to a truncated 1–term Taylor series[3.4] to obtain

$$q_{x+dx} \;=\; q_x \;+\; \frac{dq_x}{dx}\, dx \;=\; -k\, A_c(x)\, \frac{dT}{dx} \;-\; k\, \frac{d}{dx}\left(A_c(x)\, \frac{dT}{dx} \right)\, dx\,, \tag{3.8}$$

where we have removed k from under the derivative in the second term, since it is assumed constant. Finally, we write the convection term once again according to Newton's Law of Cooling as

$$dq_{conv} \;=\; h\, dA_s\; (T \;-\; T_\infty)\;, \tag{3.9}$$

where T_∞ is the temperature of the fluid absorbing the heat. It is important to note again that the assumption of a constant temperature cross–section means that the internal cross–section temperature and the boundary temperature are one in the same, T. We can now substitute these components back into our original conservation law in Eq. (3.6) to obtain

$$h\, dA_s\; (T \;-\; T_\infty) \;-\; k\, \frac{d}{dx}\left(A_c(x)\, \frac{dT}{dx} \right)\, dx \;=\; 0\,. \tag{3.10}$$

If we divide by $k\, dx$ and change sign, we get the canonical form

$$\frac{d}{dx}\left(A_c(x)\, \frac{dT}{dx} \right) \;-\; \frac{h}{k}\, \frac{dA_s}{dx}\; (T \;-\; T_\infty) \;=\; 0\,. \tag{3.11}$$

This equation is the *generalized fin equation* for one–dimensional conduction. It is a non–homogeneous, second–order equation which requires 2 boundary conditions for solution of $T(x)$. Differential equations are more manageable in homogeneous form[3.5] and, in this case, we can convert Eq. (3.11) to homogeneous form using a simple change of variables: $\theta(x) = T(x) - T_\infty$, which results in

$$\frac{d}{dx}\left(A_c(x)\, \frac{d\theta}{dx} \right) \;-\; \frac{h}{k}\, \frac{dA_s}{dx}\, \theta \;=\; 0\,. \tag{3.12}$$

This expression is also known as the generalized fin equation.

[3.4] Recall we did this with the derivation of the conduction equation in Chapter 2 for flux terms as well.

[3.5] The term involving T_∞ renders Eq. (3.11) non–homogeneous. We would like to remove that by a change of variables $\theta(x) = T(x) - T_\infty$. Since T_∞ is a constant, derivatives of T can be recast directly in terms of θ, for example

$$\frac{dT}{dx} = \frac{d(\theta + T_\infty)}{dx} = \frac{d\theta}{dx} + \frac{dT_\infty}{dx} = \frac{d\theta}{dx}\,.$$

Further derivatives clearly show the same behavior.

3.3. Special Case: Constant Cross–Section Fins

Notice that a non–trivial aspect of Eq. (3.12) is that we must know the function $A_c(x)$ before we can solve the problem. That is, Eq. (3.12) depends on the geometric attribute of how the cross–section varies along the fin. Before we study some of these more general cases, let us briefly review the most straightforward configuration: a uniform cross–section. Under these conditions, the total surface area A_s is simply the perimeter of the cross–section P multiplied by the length x, i.e. $A_s = P\,x$ (as can be inferred from e.g. Fig. 3.5), so that $dA_s/dx = P$ by simple differentiation. Moreover, for constant cross–section $A_c(x) \to A_c$ is a constant and can be moved outside of the differential term in Eq. (3.12). With these two simplifications, Eq. (3.12) reduces to

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0 \qquad \text{where} \qquad m^2 = \frac{h\,P}{k\,A_c} \tag{3.13}$$

is a composite, constant parameter made up of the flow, geometric, and material attributes of the fin. Because its coefficients are constant, Eq. (3.13) is readily solvable using the method of the auxiliary equation (Ross, 1965), which in this case is $\varphi^2 - m^2 = (\varphi + m)(\varphi - m) = 0$. The roots are obviously $\varphi = \pm m$, implying the general solution[3.6]

$$\theta(x) = C_1\,e^{mx} + C_2\,e^{-mx}\,. \tag{3.15}$$

The constants C_1 and C_2 must be evaluated using specific boundary conditions.

3.4. Boundary Conditions

Both Eq. (3.12) and its special case of Eq. (3.13) and are second order and require 2 boundary conditions, one at the base of the fin, often $x = 0$, and one at the end of the fin at $x = L$ (Fig. 3.5). The first boundary condition is straightforward: it is almost always taken as a known

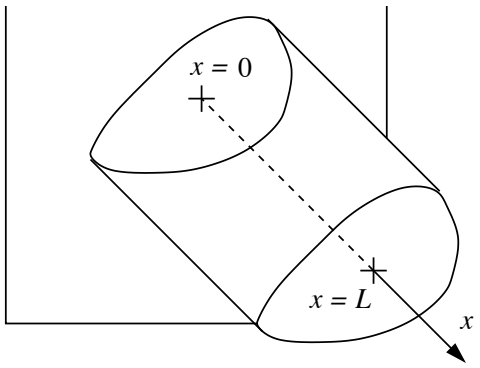


FIGURE 3.5. *Boundary locations for fins.*

temperature at the fin base, i.e. $T(0) = T_b$ assuming a boundary at $x = 0$, because we assume

[3.6] The general solution can likewise be written in terms of the hyperbolic trigonometric functions according to the identities

$$\cosh\varphi = \frac{e^{\varphi} + e^{-\varphi}}{2} \qquad \text{and} \qquad \sinh\varphi = \frac{e^{\varphi} - e^{-\varphi}}{2} \tag{3.14}$$

and these could furnish a more compact representation, depending upon the boundary conditions. See for example Eq. (3.18) on pp. 23.

the operating temperature of the device to which the fin is attached is known. This translates to $\theta(0) = T(0) - T_\infty = T_b - T_\infty = \theta_b$. If we substitute this into Eq. (3.15), we find that one of the boundary equations for the special case of constant cross–section is

$$\theta(0) \;=\; \theta_b \;=\; C_1 \;+\; C_2\,. \tag{3.16}$$

At the end of the fin, $x = L$, there are 4 cases that are of interest here. Listed in general order of mathematical difficulty, they are:

Very long fin: Fins convect heat to the surrounding fluid along their length. If we assume that the fin tip is very long, i.e. $L \to \infty$, eventually *all* of the heat will be convected away, so that the temperature at the very end *must* be equal to T_∞. That is, there is no longer any temperature gradient because there is no more heat to be transferred. According to our original substitution $\theta(x) = T(x) - T_\infty$, meaning $\theta(L) \to 0$. If we substitute this into Eq. (3.15), it is clear that the first term would be unbounded, so we must conclude that $C_1 = 0$. We can then use the boundary condition at the base to find $C_2 = \theta_b$, so that the solution is

$$\theta(x) \;=\; \theta_b\, e^{-mx}\,. \tag{3.17}$$

Negligible fin tip heat transfer: In this case, we assume the end of the fin is insulated, so that $q = 0$ at $x = L$. Writing this in terms of Fourier's Law, we obtain a boundary condition of

$$\left.\frac{d\theta}{dx}\right|_{x=L} \;=\; 0\,.$$

Again, application to the special case of constant cross–sections is straightforward. Take the derivative of Eq. (3.15) and substitute this into the boundary condition to obtain the equation

$$\frac{d\theta}{dx} \;=\; C_1\, m\, e^{mx} \;-\; C_2\, m\, e^{-mx} \qquad \text{gives} \qquad C_1\, e^{mL} \;-\; C_2\, e^{-mL} \;=\; 0\,.$$

This equation, along with the boundary condition from $x = 0$ in Eq. (3.16) allows us to solve for the constants. For example, we have $C_1 = \theta_b - C_2$ from Eq. (3.16), which can be substituted into the above equation

$$(\theta_b - C_2)\; e^{mL} \;-\; C_2\, e^{-mL} \;=\; \theta_b\, e^{mL} \;-\; C_2\left(e^{mL} + e^{-mL}\right) \;=\; 0\,,$$

so that C_2 can be solved as

$$C_2 \;=\; \theta_b\, \frac{e^{mL}}{e^{mL} + e^{-mL}}\,.$$

Substitute this back into Eq. (3.16) to solve for C_1 as

$$C_1 \;=\; \theta_b \;-\; \theta_b\, \frac{e^{mL}}{e^{mL} + e^{-mL}} \;=\; \theta_b \left(1 \;-\; \frac{e^{mL}}{e^{mL} + e^{-mL}}\right) \;=\; \theta_b\, \frac{e^{-mL}}{e^{mL} + e^{-mL}}\,.$$

Finally, we can substitute C_1 and C_2 into Eq. (3.15) to obtain the exact solution

$$\begin{aligned}
\theta(x) \;&=\; \theta_b\, \frac{e^{-mL}}{e^{mL} + e^{-mL}}\, e^{mx} \;+\; \theta_b\, \frac{e^{mL}}{e^{mL} + e^{-mL}}\, e^{-mx} \\
&=\; \theta_b\, \frac{e^{-mL}e^{mx} \;+\; e^{mL}e^{-mx}}{e^{mL} + e^{-mL}} \\
&=\; \theta_b\, \frac{e^{m(L-x)} \;+\; e^{-m(L-x)}}{e^{mL} + e^{-mL}}\,.
\end{aligned}$$

According to the identity for cosh in Eq. (3.14), as footnoted on pp. 21, we can write this result in the more simple form

$$\theta(x) \;=\; \theta_b \,\frac{\cosh\left[m(L-x)\right]}{\cosh\left(mL\right)}\,, \tag{3.18}$$

which is the exact solution.

Fin tip temperature is known: That is, we have a standard Dirichlet boundary condition at $x = L$, similar to that at the base $x = 0$ in the form $\theta(L) = \theta_t$. The procedure to evaluate C_1 and C_2 for this case is identical to what we have just seen, although the algebra is somewhat more involved. The final solution is

$$\theta(x) \;=\; \frac{\theta_t\,\sinh\left(mx\right) \;+\; \theta_b\,\sinh\left[m(L-x)\right]}{\sinh\left(mL\right)}\,, \tag{3.19}$$

where the sinh identity in Eq. (3.14), has been used for maximum simplification.

Convection from fin tip: Rate of energy transferred to the fluid by convection at the tip of the fin equals the rate at which energy arrives at the tip via conduction. We write the basic equation, which can be simplified using our standard change of variables, as

$$h\,A_c\left(T(L)-T_\infty\right) \;=\; -\,k\,A_c\,\frac{dT}{dx}\,\bigg|_{x=L} \qquad\rightarrow\qquad h\,\theta(L) \;=\; -\,k\,\frac{d\theta}{dx}\,\bigg|_{x=L}\,.$$

This case is the most tedious to solve, but the final solution is

$$\theta(x) \;=\; \theta_b\,\frac{\cosh\left[m(L-x)\right] \;+\; \lambda\,\sinh\left[m(L-x)\right]}{\cosh\left(mL\right) \;+\; \lambda\,\sinh\left(mL\right)}\,, \tag{3.20}$$

where $\lambda = h/(mk)$ is a constant that depends upon the characteristics of the boundary condition, specifically h and k.

3.5. Analysis of the Common Annular Fin Using Frobenius' Method

The previous few sections give a fairly complete picture of the temperature distribution in the special case of constant cross–section fins. However, in a broad class of applications, the cross–section actually varies. Perhaps the most familiar example is the annular fin, as seen on IC engine cylinders, numerous types of piping and tubing systems, electric motor housings, firearm barrels and mortar tubes, etc. (Fig. 3.6). This configuration is extremely common because of the natural

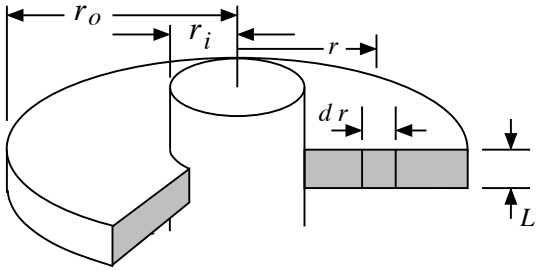


FIGURE 3.6. *Cut–away view of an annular fin of constant thickness L.*

conformity to the usually heat–generating devices for which fins are often needed. It is clear in Fig. 3.6 that the cross–sectional area increases with the radius, r.

Analysis of this configuration starts with quantifying the geometric parameters A_c and A_s in Eq. (3.12) on pp. 20. Note for this section we will switch notation from "x" to "r" for the independent variable to represent the cylindrical nature of this configuration in a more familiar way. The fin itself is defined as the annular material in the domain $r_i \le r \le r_o$. Consider a portion of the fin up to radius r. The cross–sectional area at r is the outer face of the element, i.e. $A_c = 2\,\pi\,r\,L$. The surface area consists of the top and bottom faces of the fin, from r_i up to coordinate location r, so that $A_s = 2(\pi\,r^2 - \pi\,r_i^2)$ and therefore

$$\frac{dA_s}{dr} \;=\; 4\,\pi\,r\,.$$

Substituting these quantities into Eq. (3.12) on pp. 20, we find

$$\frac{d}{dr}\left(2\,\pi\,r\,L\,\frac{d\theta}{dr}\right) \;-\; \frac{4\,\pi\,h}{k}\,r\,\theta \;=\; 0\,. \tag{3.21}$$

We can develop this into the standard form

$$\frac{d^2\theta}{dr^2} \;+\; \frac{1}{r}\,\frac{d\theta}{dr} \;-\; m^2\,\theta \;=\; 0 \qquad \text{where} \qquad m^2 \;=\; \frac{2\,h}{k\,L} \tag{3.22}$$

is again a constant comprised of the flow, geometric, and material parameters. Note the resemblance to Eq. (3.13) on pp. 21 for constant cross–section fins, with the exception of an additional first derivative term, which has a variable coefficient, r^{-1}. Eq. (3.22) is a type of *Bessel differential equation*, for which the extended power series method of solution (see appendix A) yields[3.7]

$$\theta(r) \;=\; \frac{I_0(m\,r)\,K_1(m\,r_o) \;+\; K_0(m\,r)\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\theta_b \tag{3.23}$$

when the prescribed boundary conditions are $\theta(r_i) = \theta_b$ and an insulated (adiabatic) fin tip. Here, I_0 and K_0 are the modified Bessel functions of the first and second kind of order zero, respectively.

3.6. Fin Heat Transfer and Performance Metrics

The quantity of practical interest for many real–world problems is the total rate of heat transferred by the fin to its surroundings, q. We have considered the steady–state case, i.e. Eq. (3.12), whereby the rate of energy entering the fin at its base via conduction must equal the rate of energy convected by the fin to its surroundings over its total surface area. In general, calculating q according to the latter viewpoint is more difficult because the temperature along the fin is not constant. Consequently, we would have to write a differential form of Newton's Law of Cooling, then integrate it about the entire surface area of the fin

$$q \;=\; \int_{A_s} h\left(T(x) - T_\infty\right)\,dA_s \;=\; \int_{A_s} h\,\theta(x)\,dA_s\,.$$

This would require proper consideration of heat transfer at the fin tip, as well, with the exception of the case of adiabatic tip conditions. It is rather more straightforward to use to the former point of view, i.e. calculating q using Fourier's Law applied at the fin base. For example, in the case of the constant cross–section long fin, we find

$$\frac{d\theta}{dx} \;=\; -\theta_b\,m\,e^{-mx}\,,$$

[3.7] This is commonly known as Frobenius' method. See appendix A pp. 129 for development and detailed solution method for Eq. (3.22).

which can be evaluated at $x = 0$ and substituted into Eq. (2.2) on pp. 8 (Fourier's Law) to yield

$$q \;=\; -\,k\,A_c\;(-\theta_b\,m) \;=\; \theta_b\,k\,A_c\,m\,.$$

Recalling the original definition of parameter m from Eq. (3.13), we can write the result in the more useful form

$$q \;=\; \theta_b\,\sqrt{h\,P\,k\,A_c}\;. \tag{3.24}$$

This equation indicates that the temperature difference at the base is the strongest parameter of the problem. In particular, doubling the temperature difference will double the heat transfer. Conversely, the other four variables are not as significant, since they appear as square roots. For example, to double the heat transfer, the convection coefficient would have to be increased by a factor of four and so forth. Likewise, for the constant cross–section with adiabatic tip, whose temperature distribution appears in Eq. (3.18) on pp. 23, the derivative is

$$\frac{d\theta}{dx} \;=\; -\,m\,\theta_b\,\frac{\sinh\,[m(L-x)]}{\cosh\,(mL)}\;,$$

which can again be evaluated at $x = 0$ and substituted into Fourier's Law to yield

$$q \;=\; -\,k\,A_c\left(-\,m\,\theta_b\,\frac{\sinh\,(mL)}{\cosh\,(mL)}\right) \;=\; \theta_b\,\sqrt{h\,P\,k\,A_c}\;\tanh\left(L\,\sqrt{\frac{h\,P}{k\,A_c}}\;\right)\;, \tag{3.25}$$

after again substituting m from Eq. (3.13). Heat transfer rates for the remaining boundary conditions of the constant cross–section fin are similarly derived.

The annular fin is not quite as straightforward, but it does likewise follow directly from applying Fourier's Law at the fin base, i.e. at $r = r_i$. The calculation is shown in appendix A, the answer being

$$q \;=\; 2\,\pi\,r_i\,\theta_b\,\sqrt{2\,L\,k\,h}\;\frac{K_1(m\,r_i)\,I_1(m\,r_o)\;-\;I_1(m\,r_i)\,K_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o)\;+\;I_1(m\,r_o)\,K_0(m\,r_i)}\;, \tag{3.26}$$

where, for this case, m is defined in Eq. (3.22) on pp. 24.

One of the main metrics that quantifies performance of fins is their *fin efficiency*, i.e. the ratio of the actual heat transfer, q, to the maximum possible heat transfer, $h\,A_s\,\theta_b$, which would take place if the entire fin was hypothetically maintained at the base temperature, so that convection was maximized all along the fin length. The form is

$$\varepsilon \;=\; \frac{q}{h\,A_s\,\theta_b}\;, \tag{3.27}$$

where again A_s is the total participating area of heat transfer of the fin. Once the fin heat transfer has been determined, it is a relatively simple matter of substitution to obtain ε for a given fin. For example, for the constant cross–section long fin, $A_s = P \cdot L$, where L is the fin length,[3.8] whereby substitution of Eq. (3.24) gives

$$\varepsilon \;=\; \frac{\theta_b\,\sqrt{h\,P\,k\,A_c}}{h\,L\,P\,\theta_b} \;=\; \frac{1}{L}\,\sqrt{\frac{k\,A_c}{h\,P}}\;. \tag{3.28}$$

[3.8] The fin tip area is not counted in A_s for the case of the "long" fin because there is identically no heat transfer at the tip by virtue of the fin tip being at ambient temperature, c.f. Eq. (3.17) on pp. 22. The same observation can be made for the adiabatic fin tip boundary condition.

An extremely common instance is the "thin" rectangular cross–section fin of width w and thickness b, where $w \gg b$. Here, $A_c = w \cdot b$ and $P = 2\,(w + b) \approx 2\,w$, so that

$$\varepsilon \;=\; \frac{1}{L}\,\sqrt{\frac{k\,b}{2\,h}}\;. \tag{3.29}$$

Similarly, for the annular fin with adiabatic fin tip (see footnote 3.8), we can substitute Eq. (3.26) and observe

$$\varepsilon \;=\; \frac{r_i}{r_o^2 - r_i^2}\,\sqrt{\frac{2\,L\,k}{h}}\,\frac{K_1(m\,r_i)\,I_1(m\,r_o) \;-\; I_1(m\,r_i)\,K_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\;. \tag{3.30}$$

3.7. Additional Remarks

Heat conduction in actual fins is generally a more complicated proposition than we have represented here in several respects. First, geometries are not always idealized shapes, such as the perfect annular fin discussed in §3.5. Although Fig. 3.6 is meant to be a cut–away view, it is a good example of how only a partial or a notched fin might be used to allow perhaps for clearance with another device. Such a case would entail angular dependence and two–additional boundary conditions. The temperature distribution could be quite different. A substantial Biot number could imply cross–sectional temperature dependencies and large temperature differences could mean conductivity changes should be considered, rendering a problem non–linear. Such contingencies will be explored in succeeding chapters.

More substantive difficulties occur when multiple fins are used in arrays, as is very frequently the case. Introductory texts generally present a superposition analysis where heat transfer is simply additive (Bergman et al., 2011), however this viewpoint totally neglects the effects that arrays have on the convective flow with which they interact. If operating temperatures are very high, there may be radiative effects, as well, with mutual exchanges of radiation among fins and with the outside environment. The broader aspects of fin design include many other factors like cost and manufacturability considerations, optimization for heat transfer per unit mass, etc.

3.8. Exercises

3.1 Prove Eq. (3.19) for the case of a constant cross–section fin having a prescribed fin tip temperature of θ_t.

3.2 A common ODE that arises in various problems in mechanics is Bessel's equation, a solution of which is the Bessel function of the first kind of order n

$$J_n(x) \;=\; \Big(\frac{x}{2}\Big)^n \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(i+n)!}\Big(\frac{x}{2}\Big)^{2i}\;,$$

where n is an integer. If a particular problem has a boundary condition of the second kind, the derivative of J_n must be determined. Show that $J_0' = -J_1$, for example by operating term–by–term on J_0 and reconstituting the result.

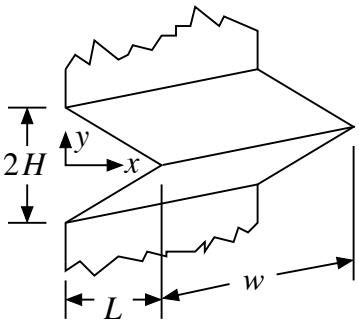
3.3

A fin has a triangular shape defined by the coordinates

$$\left|\frac{y}{H}\right| \;=\; 1 \;-\; \frac{x}{L}\,,$$

where the root has a thickness $2\,H$ and the extension length is L. Assume the fin is "long", i.e. w is much larger than the other 2 geometric parameters. Show that the efficiency of this fin is

$$\varepsilon \;=\; \frac{I_1(2\,m\,L)}{m\,L\cdot I_0(2\,m\,L)} \quad \text{where} \quad m \;=\; \sqrt{\frac{h}{k\,H}}\;\cdot$$



CHAPTER 4

One–Dimensional Unsteady Conduction

THERE ARE NUMEROUS SITUATIONS where unsteady effects are important, i.e. where they occur because of inherent operational transients in a device or system, or during the start–up or shut–down in such systems. Mathematically, these scenarios generally represent more complicated conduction problems than what we have examined thus far. Whereas steady 1–D problems can be reduced to ordinary differential equations, transient problems routinely lead to partial differential equations, which are typically more difficult to solve. In this chapter, we will survey various cases of transient conduction. First, let us briefly review the most elementary transient problem: zero–dimensional conduction.

4.1. Special Case: Zero–Dimensional Unsteady Conduction

Consider an arbitrarily–shaped domain having a uniform temperature, T, for example by virtue of a low Biot number (c.f. Eq. (3.4) on pp. 18) within a surrounding fluid at a temperature of T_∞ (Fig. 4.1). Heat transfer will occur when $T_\infty \neq T$, but let us assume $T > T_\infty$, without loss of

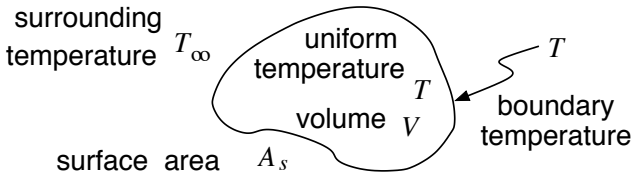


FIGURE 4.1. *Constant temperature domain: both the interior and the boundary are at a temperature T, while surroundings are at T_∞, where $T_\infty \neq T$.*

generality. According to the conservation of energy concept in Eq. (1.4) on pp. 4, we can observe that $\dot{E}_{stored} = -\dot{E}_{out}$, where $\dot{E}_{out} = hA_s(T - T_\infty)$ is the rate at which convection removes heat energy defined by Newton's law of Cooling. For the rate of change of stored energy, a dimensional argument (c.f. Eq. (2.9) on pp. 11) gives $E_{stored} = \rho V c\, dT/dt$ implying

$$\rho\, V\, c\, \frac{dT}{dt} \;=\; -\, h\, A_s\, (T - T_\infty)\,.$$

Or, making the standard change of variables as introduced in Chapter 3 (c.f. footnote 3.5 on pp. 20) to obtain a homogeneous equation, $\theta(t) = T(t) - T_\infty$, this becomes[4.1]

$$\frac{d\theta}{dt} \;=\; -\,\frac{h\, A_s}{\rho\, V\, c}\,\theta\,. \tag{4.1}$$

[4.1] Eq. (4.1) is routinely called the lumped capacitance model of transient conduction in introductory texts (e.g. Holman, 2010; Bergman et al., 2011).

Eq. (4.1) is a separable differential equation, which can be conveniently solved by gathering dependent terms, those involving θ, on the left, and independent ones, those of t, on the right, then integrating over an elapsed time, as

$$\begin{aligned} \frac{d\theta}{\theta} &= -\frac{h\,A_s}{\rho\,V\,c}\,dt \\ \int_{\theta_0}^{\theta}\frac{d\theta'}{\theta'} &= -\frac{h\,A_s}{\rho\,V\,c}\int_0^t dt' \\ \ln\theta'\Big|_{\theta_0}^{\theta} &= -\frac{h\,A_s}{\rho\,V\,c}\,t'\Big|_0^t \\ \Big(\ln\theta\;-\;\ln\theta_0\Big)\;=\;\ln\left(\frac{\theta}{\theta_0}\right) &= -\frac{h\,A_s}{\rho\,V\,c}\,t\,, \end{aligned}$$

the last step exploiting the logarithm identity $\ln a - \ln b = \ln(a/b)$. Exponentiating both sides yields the solution

$$\frac{\theta(t)}{\theta(0)} \;=\; \frac{T(t)-T_\infty}{T_i-T_\infty} \;=\; e^{-t/t_c}\,, \qquad \text{where} \qquad t_c \;=\; \frac{\rho\,V\,c}{h\,A_s} \tag{4.2}$$

is a time constant and $T(0) = T_i$ is the initial, obviously constant temperature of the domain. According to Eq. (4.2), temperature response is a simple exponential when spatial gradients are small enough to be neglected.

4.2. Casting the One–Dimensional Unsteady Configuration

As mentioned above, the zero–dimensional model of unsteady conduction is valid only under very restricted circumstances, conventionally taken to be satisfied roughly when $Bi < 0.1$ (Bergman et al., 2011). We might consider a domain such as that shown in Fig. 4.2, where one of the physical dimensions is much smaller than the other two, but where Bi nevertheless exceeds an

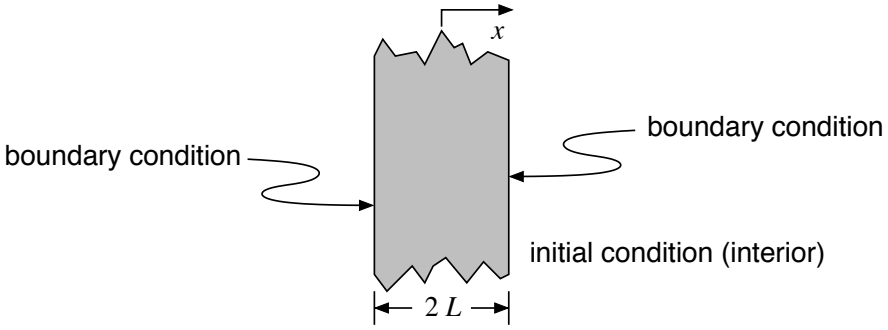


FIGURE 4.2. *One–dimensional transient problem, where $Bi \not< 0.1$, and the value of $2L$ is much smaller than the other two physical dimensions.*

acceptable small–value threshold.[4.2] If temperature differences across the 3 physical dimensions

[4.2]In the case of Fig. 4.2, we mean that $2L$ is significantly smaller than both the vertical height of the domain and its depth dimension, i.e. "into the paper". Here, the length scale for the problem can be deduced on dimensional grounds as the quotient of total volume of the domain and the total area available for heat transfer, the latter being

are comparable, or if they are larger in the x direction than in the other two, then temperature gradients will only matter in the x direction. Simplifying the general equation of heat conduction, Eq. (2.15) on pp. 12, commensurately, we get

$$\frac{\partial T}{\partial t} \;=\; \alpha\,\frac{\partial^2 T}{\partial x^2}\;, \tag{4.3}$$

where $T = T(x,t)$ and α is the thermal diffusivity (c.f. Table 2.1 on pp. 10). This is a partial differential equation rather than an ordinary differential equation, and is therefore usually more difficult to solve. It is linear in that there are no products of T or its derivatives, and it is also homogeneous, i.e. has no source term (c.f. §2.4). In general, we derive $T(x,t)$ according to the following procedure.

- Obtain the general solution of Eq. (4.3) via suitable advanced calculus techniques for linear differential equations.
- Specify an initial condition, which gives the temperature distribution at some known time, i.e. $T(x,0) = T_0(x)$. In special cases $T(x,0)$ may be a constant.
- Specify 2 boundary conditions, one each at $x = \pm L$, which may be of any of the 3 standard types discussed in §2.5. Formally, the total number of possible types of boundary problems then is 6, but this neglects the fact that symmetry considerations collapse matters further.[4.3] For example, the domain of $-L \le x \le L$ in Fig. 4.2 having Dirichlet boundary conditions at $x = \pm L$ is formally equal to the domain $0 \le x \le L$ having the same condition at $x = L$, but adiabatic conditions at $x = 0$. The former is symmetric about the plane $x = 0$, whereby there is no heat conduction across this plane, and this precisely matches conditions of the latter, which enforces a zero slope of T at $x = 0$. However, it is also possible to have additional forms of this problem if, for example, the problem is semi–infinite in one of the coordinate directions, e.g. $0 \le x < \infty$.

We will examine several representative problems.

4.3. The Dirichlet Problem and the Technique of Separation of Variables

Let us first study the so–called Dirichlet problem in heat conduction, that is, the configuration where the boundaries at $x = \pm L$ in Fig. 4.2 are prescribed with conditions of the first kind. In this section, we focus specifically on introducing the method of separation of variables for solving a partial differential equation. This technique is one of the oldest and most important for finding explicit closed–form solutions of linear equations (Dettman, 1962; Özişik, 1980). Roughly speaking, the procedure involves separating the partial differential equation into a set of ordinary equations, solving these, and reconstructing the desired problem by the method of superposition. It is well–suited for homogeneous problems like Eq. (4.3).

---

the sum of the areas of the plane boundaries, each being of area A_c, at $x = \pm L$. The former is $V = 2\,L\,A_c$, while the latter is $A_s = 2\,A_c$, whereby the length scale is the half–thickness L. Consequently, the Biot number is $Bi = hL/k$, assuming a convection coefficient of h and domain thermal conductivity of k.

[4.3] The number of ways that 3 objects, the 3 possible types of boundary conditions, can be combined in 2 ways, i.e. at the two boundaries, allowing for repetition, e.g. both boundaries could have Dirichlet boundary conditions, is

$$\binom{3+2-1}{2} \;=\; \binom{4}{2} \;=\; \frac{4\cdot 3}{2} \;=\; 6\,.$$

For convenience, we will change the domain from $-L \le x \le L$ as shown in Fig. 4.2 to $0 \le x \le L$ for this section without loss of generality. We will also assume homogeneous Dirichlet boundary conditions $T(0,t) = T(L,t) = 0$, or equivalently, non–homogeneous conditions that can readily be converted to homogeneous form by a simple change of variables (c.f. footnote 3.5 on pp. 20). The problem is then well–suited to the separation of variables treatment, where we assume T can be written as the product of two univariate functions: $T(x,t) = \Psi(x)\,\Gamma(t)$. The detailed solution procedure is furnished in appendix B, which shows the general solution to be

$$T(x,t) \;=\; \sum_{n=1}^{\infty} C_n \,\sin\left(\zeta_n x\right)\, e^{-\alpha\zeta_n^2 t} \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L}\,, \tag{4.4}$$

where the mode coefficients are given by the integral

$$C_n \;=\; \frac{2}{L}\int_0^L F(x)\,\sin(\zeta_n x)\,dx\,. \tag{4.5}$$

The solution given by Eqs. (4.4) and (4.5) permits us to examine a wide range of initial conditions, $F(x)$, for the Dirichlet problem. For example, consider a constant temperature, which we can specify without loss of generality as $F(x) = 1$. The exact solution (see §B.4 in appendix B) is

$$T(x,t) \;=\; \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\,\pi}\,\sin\left(\zeta_n x\right)\, e^{-\alpha\zeta_n^2 t}\,.$$

Fig. 4.3 plots the temperature distributions for a billet of size $L = 10\,cm$ at elapsed times of 10

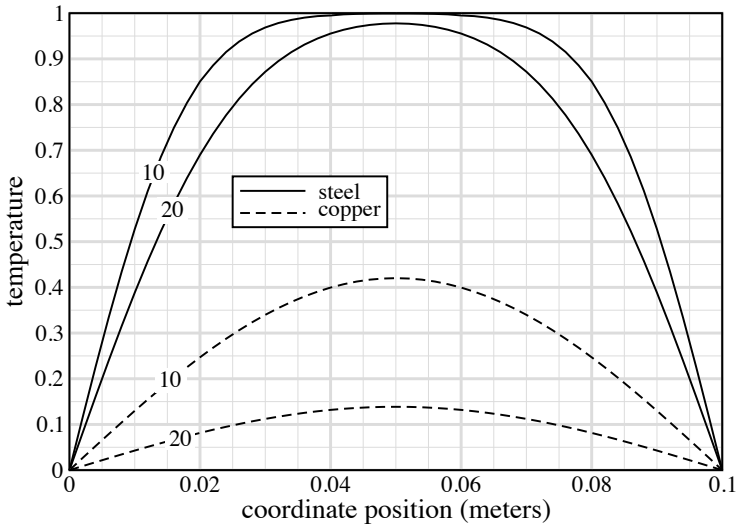


FIGURE 4.3. *Temperature distributions for steel and copper at 10 and 20 seconds elapsed time, given unit initial conditions, i.e.* $T(x,0) = 1$.

and 20 seconds for two materials: steel (1.5% carbon content) and unalloyed copper. The thermal diffusivities are $\alpha = 9.7 \times 10^{-6}\ m^2/s$ and $1.12 \times 10^{-4}\ m^2/s$, respectively (Özişik, 1985). Two notable properties are observed. First, the $T(x,t)$ curves are all symmetric about the centerline, here $x = 5\,cm$, as we would expect from the fact that the boundary conditions are symmetric *and* the initial conditions are symmetric. We would not expect this property if either one were

non–symmetric. Second, the copper billet, having a thermal diffusivity roughly 10–fold of the steel, shows much faster response to the imposed boundary conditions.

We might also consider another problem that is a direct extension to the elementary one–dimensional steady conduction problem, whose solution is $T(x) = C_1\, x\, +\, C_2$ (c.f. footnote 3.2 on pp. 18). Specifically, consider the same 10 cm billet where there is steady–state conduction and a temperature distribution of $T(x) = x/L$. This clearly implies boundary temperatures of $T(0) = 0$ and $T(L) = 1$. Now suppose the boundary temperature at L is changed to $T(L) = 0$, so that the problem is one of calculating the decay transient as the entire billet reaches thermal equilibrium. Framed in this way, we see the problem is identical to the previous one, except where $F(x) = x/L$. The exact solution (see §B.5 in appendix B) is

$$T(x,t)\; =\; \sum_{n=1}^{\infty}\; \frac{2\;(-1)^{n+1}}{n\,\pi}\; \sin\left(\zeta_n x\right)\; e^{-\alpha\zeta_n^2 t}\;.$$

Fig. 4.4 plots decay profiles for the billet assuming it is made of steel (α quoted above) for several

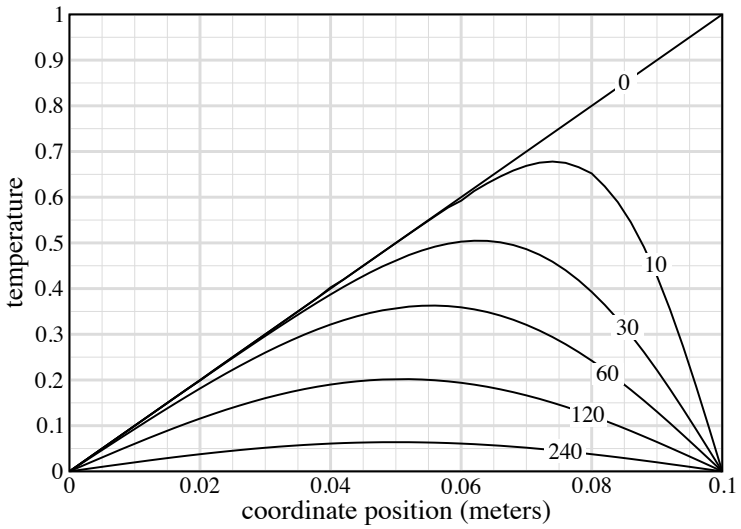


FIGURE 4.4. *Transient decay of the elementary one–dimensional conduction problem. Decay curves are shown for steel billet at 10, 30, 60, 120, and 240 seconds elapsed time.*

time points. Here, the decay is clearly not symmetric about the mid–plane, early in the process at least, and that aspect is due to the asymmetry of the initial condition. However, note that the profiles at later times tend increasingly toward symmetry. This behavior is inherent in the process because the end state, thermal equilibrium, is itself symmetric.

One additional subtlety of this problem is also worth mentioning. The temperature distribution for steady, one–dimensional conduction depends only on boundary conditions and is independent of the material properties of the conductor.[4.4] Conversely, once transients are present, the temperature profile is strongly dependent on material, specifically the thermal diffusivity. In particular, materials having higher values of α will respond faster to changes in temperature.

[4.4]Of course, the heat transfer, does depend on material properties according to Fourier's Law (Eq. (2.2) on pp. 8).

4.4. Dimensionless Reduction of the One–Dimensional Robbins Configuration

The previous section examined the so–called Dirichlet problem, in which the two boundary conditions are both of the "first kind". Recall that we broadly discussed three different kinds of boundary conditions in §2.5, and, although they are clearly distinct in a mathematical sense, there is a subtle hierarchy in terms of their generality in a physical sense for problems such as these. Consider that we have framed Dirichlet problems in the context of having some initial temperature distribution and then one or both of the boundary values is/are instantaneously changed to a different value. Here, the word "instantaneous" carries some obvious physical caveats. The actual procedural aspect would more typically be something like changing the surroundings within which the conductor is placed. For example, we might take the unit initial condition Dirichlet problem (discussed above, solution in §B.4 in appendix B) as a model of quenching an annealed part: The workpiece is heated for a long time, for which we assume it has some uniform temperature. Then, it is dropped into a cold liquid.

Here, we do not necessarily expect the surface temperature of the part to change immediately to match the liquid temperature. Instead, there is some fluid motion about the part with accompanying convection, so that a more appropriate boundary condition in such cases might be of the "third kind". In this sense, the Dirichlet boundary condition is a special case of the Robbins boundary condition for infinite convection coefficient. This interpretation is immediately clear from the concept of the circuit analogy, e.g. as referred to in Fig. 3.2 (pp. 17). That is, for the surface temperature of the conductor to instantly match the fluid temperature, the convection resistor must be "short–circuited", meaning its value, R_V, must be zero. Given $R_V \propto h^{-1}$, the implication is an infinite convection coefficient.

This discussion implies that there is a large class of problems where the Robbins "convection boundary condition" is applicable for some finite value of h. Aside from added mathematical complexity, as we shall see, the main practical hindrance is the dependence on a larger number of variables than in the Dirichlet problem. Consider from Eqs. (4.4) and (4.5) that the Dirichlet problem is stated as

$$T \;=\; T(x, t, \alpha, L, T_i)\,,$$

where we assume for this discussion that T_i is a uniform (constant) initial temperature. In other words, T depends on two independent coordinates, x and t, a material property, α, and geometric property, L, and an initial configuration, T_i — a total of 5 variables. There is no dependence on boundary temperature, because we assumed problems of the type where a simple change of variables could be applied to cast the problem as homogeneous.

Let us now consider the Robbins problem in this same way, i.e. where conditions of the "third kind" are specified at both boundaries for the domain shown in Fig. 4.2. If we limit our discussion to uniform initial conditions, $T(x, 0) = T_i$ and also assume the Robbins' conditions are identical at both boundaries, then the problem is symmetric about the mid–plane. Here, it is more convenient to consider the equivalent problem of the half–domain $0 \leq x \leq L$ with a symmetry boundary condition $\partial T/\partial x = 0$ at $x = 0$.[4.5] Consequently, the problem is described by the governing field

[4.5] Note that this is formally equivalent to an adiabatic boundary condition at the mid–plane, which is entirely consistent with the fact that no heat transfer occurs across $x = 0$ in a symmetric configuration.

equation in Eq. (4.3), the initial condition $T(x,0) = T_i$, and the boundary conditions

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \qquad \text{and} \qquad h\left[T(L,t) - T_\infty\right] = -k\left.\frac{\partial T}{\partial x}\right|_{x=L} .$$

Checking these expressions, one readily sees that, rather than the 5 variables above, the Robbins problem depends on 8 variables

$$T = T(x,t,\alpha,L,T_i,k,h,T_\infty) ,$$

where the three extra parameters are another conductor material property, k, a thermo–flow property, h, and a fluid state, T_∞.

It is often beneficial to non–dimensionalize a problem because the number of variables is thereby reduced.[4.6] For the Robbins problem, we define the following forms of dimensionless temperature, location, and time using the "star" notation (*):

$$\theta^* = \frac{T(x,t) - T_\infty}{T_i - T_\infty} = \frac{\theta(x,t)}{\theta_i} , \qquad x^* = \frac{x}{L} , \qquad t^* = \frac{\alpha\, t}{L^2} = Fo , \tag{4.6}$$

where the dimensionless time, t^*, is routinely called the *Fourier Number*, Fo.[4.7]

To apply the dimensionless treatment to the governing equation, Eq. (4.3), we first determine how derivatives behave in the dimensionless realm using the Chain Rule of Calculus, as in

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t^*}\frac{\partial t^*}{\partial t} = \frac{\alpha}{L^2}\frac{\partial}{\partial t^*} \qquad \text{and} \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x^*} . \tag{4.7}$$

The latter expression leads us to the required second derivative in x, as in

$$\frac{\partial^2}{\partial^2 x} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{L}\frac{\partial}{\partial x^*}\right) = \frac{1}{L}\frac{\partial}{\partial x^*}\frac{\partial x^*}{\partial x}\left(\frac{\partial}{\partial x^*}\right) = \frac{1}{L^2}\frac{\partial^2}{\partial^2 x^*} . \tag{4.8}$$

Using these expressions, Eq. (4.3) can be written directly in the dimensionless form

$$\frac{\partial \theta^*}{\partial t^*} = \frac{\partial^2 \theta^*}{\partial {x^*}^2} \tag{4.9}$$

Likewise, the initial condition becomes

$$\theta^*(x^*,0) = 1 \tag{4.10}$$

[4.6] This is not the only benefit of so–called dimensionless analysis. For example, it is also the basis of the principle of *dynamic similarity*, whereby two systems, a prototypical model and a full–scale device, are dynamically equivalent, permitting knowledge of one system to be extrapolated to the other, e.g. as in aerodynamic wind–tunnel testing. As a by–product, it also yields various dimensionless numbers that have physical significance, for example, the well–known Reynolds number for viscous fluid flow, c.f. Eq. (7.18) on pp. 90, which indicates the relative strengths of inertial and viscous forces. In some cases, it can even be pushed to reveal specific physical laws for some systems (Ong, 1991). See Barenblatt (1987) or Panton (1984) for extensive discussion.

[4.7] This is the slightly unfortunate nomenclature used in most texts. The Fourier number is not a dimensionless "number" in the same sense as, for example the Reynolds number for pipe flow, but rather a dimensionless coordinate.

and boundary conditions become[4.8]

$$\frac{\partial\theta^*}{\partial x^*}\bigg|_{x^*=0} = 0 \qquad \text{and} \qquad -Bi\cdot\theta^*(1,t^*) = \frac{\partial\theta^*}{\partial x^*}\bigg|_{x^*=1} . \tag{4.11}$$

The non–dimensionalizing process has now reduced the problem to

$$\theta^* = \theta^*(x^*, t^*, Bi) , \tag{4.12}$$

where the single parameter, the Biot number, arises in the boundary conditions.

4.5. The Non–Dimensionalized Robbins Problem

Eqs. (4.9) through (4.11) represent another linear partial differential equation system, which can again be solved using the separation of variables technique, much like the Dirichlet problem. However, there are some important differences from the Dirichlet problem and the solution procedure is shown in appendix C on pp. 141. The solution takes the form of an infinite series

$$\theta^*(x^*, t^*) = \sum_{n=1}^{\infty} \frac{4\,\sin\zeta_n}{2\,\zeta_n + \sin(2\,\zeta_n)}\,\cos\left(\zeta_n\,x^*\right)\,e^{-\zeta_n^2 t^*} , \tag{4.13}$$

where ζ_n are the eigen–values given by the roots of the transcendental equation

$$\zeta_n \tan\zeta_n = Bi . \tag{4.14}$$

Roots of Eq. (4.14) can be found numerically, e.g. as shown in §C.2 in appendix C.

One of the interesting aspects of the Robbins problem is that it is the direct generalization of the lumped capacitance model, which here we have called the zero–dimensional problem (c.f. §4.1 on pp. 28). That is, the Robbins solution considers the spatial gradient that the capacitance model neglects. As mentioned above, capacitance is conventionally taken as valid roughly when $Bi < 0.1$ (Bergman et al., 2011). These observations naturally prompt a comparison between the two theories.

Consider conduction in a large pane of soda lime glass of half–thickness $L = 0.004\,m$ (material properties in Table 4.1). If the convection coefficient at the boundary is $h = 70\,W/(m^2\,K)$, then the

TABLE 4.1. Material Properties of Soda Lime Glass at Room Temperature

Property	Value
k	$1.4\,W/(m\,K)$
ρ	$2500\,kg/m^3$
c_p	$750\,J/(kg\,K)$
α	$7.5\times10^{-7}\,m^2/s$

[4.8] For example, $T(x,t) = (T_i - T_\infty)\theta^* + T_\infty$ and T_i and T_∞ are both constants, so that

$$\frac{\partial T}{\partial x} = \frac{T_i - T_\infty}{L}\,\frac{\partial\theta^*}{\partial x^*} ,$$

from which we see the Robbins boundary condition can be cast as

$$h\left[T(L,t) - T_\infty\right] = -k\,\frac{T_i - T_\infty}{L}\,\frac{\partial\theta^*}{\partial x^*}\bigg|_{x^*=1} .$$

The final expression in Eq. (4.11) follows directly from arithmetic (c.f. the definition of the Biot number in Eq. (3.4) on pp. 18).

Biot number is $Bi = 70 \cdot 0.004/1.4 = 0.2$. The time constant in the capacitance model of Eq. (4.2) on pp. 29 can be calculated from the material properties and the geometric characteristics as

$$t_c \;=\; \frac{\rho\,(2\,L\cdot A_s)\,c}{h\cdot 2\,A_s} \;=\; \frac{\rho\,L\,c}{h} \;=\; \frac{2500\cdot 0.004\cdot 750}{70} \;\approx\; 107.14\,sec\,,$$

where the volume, V, of the domain is $2L$ multiplied by the surface area of the face, and the *total* surface area is twice the surface area of the face (note the left and right exposed faces, e.g. as shown in Fig. 4.2).

Fig. 4.5 shows $\theta^*(x^*, t^*)$ for both the Robbins solution and the lumped capacitance model, the

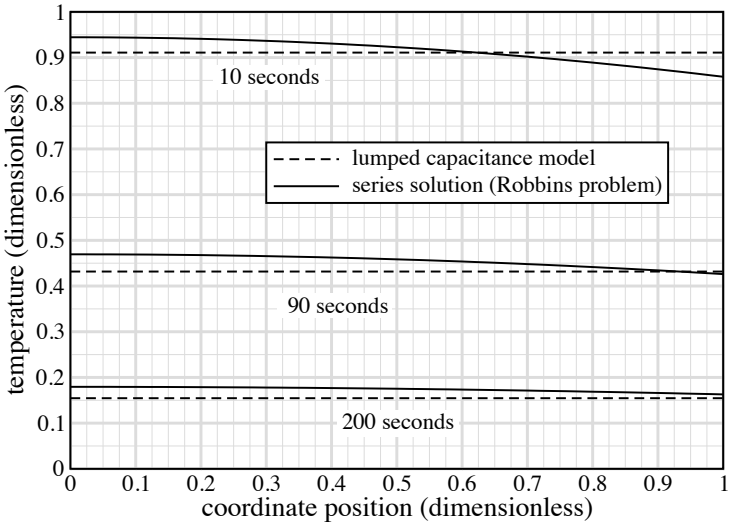


FIGURE 4.5. *Comparison of lumped capacitance (zero–dimensional) model, Eq.* (4.2)*, to the unsteady Robbins solution, Eq.* (4.13)*, for dimensionless temperature decay in a soda lime plate glass pane of half–thickness* $L = 0.004\,m$ *at a Biot number of* $Bi = 0.2$ *for several time points.*

latter obviously being independent of x^*, for several time points within the half–domain $0 \le x^* \le 1$. Note the the zero–slope at $x^* = 0$ for the Robbins solution, indicating a plane of symmetry (as the boundary condition requires). The Biot number in this calculation is comparable to the commonly recommended threshold maximum for the capacitance model of 0.1 (Bergman et al., 2011). Error is highest at the outset and lessens with elapsed time, finally vanishing in the limit of thermal equilibrium, at which point both models indicate $\theta^* = 0$. (In Fig. 4.5, the maximum error appears in the curve for 10 seconds elapsed time. Specifically, the capacitance model over–predicts the boundary temperature at $x^* = 1$ by roughly 6%.) Larger errors would accompany larger values of Bi as the gradients within the pane would increase, while those in the convection layer would decrease. In essence, we would see a quantification of what Fig. 3.2 on pp. 17 represents only qualitatively.

4.6. The Semi–Infinite Rayleigh Problem and the Similarity Technique

Another geometry which can be studied analytically is that of a semi–infinite solid, which serves as a useful idealization for many problems, e.g. any body which is very "thick" (Fig. 4.6). This

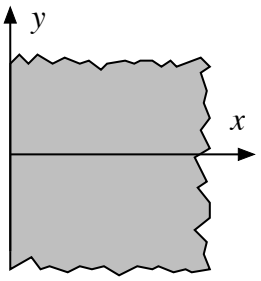


FIGURE 4.6. *Semi–infinite domain:* $0 \leq x < \infty$ *and* $-\infty < y < \infty$.

problem is described by the same governing law as the others, Eq. (4.3). Like the Robbins problem, we will assume a uniform initial temperature, but the boundary conditions are special. We require two conditions because of the second–order spatial derivative, but there is only one obvious location at which to specify a boundary condition, i.e. $x = 0$. The second boundary condition depends upon the following logical deduction: If the entire domain is initially at some constant temperature, say $T(x, 0) = T_i$, and truly $x \to \infty$, then it will always be possible to pick some x sufficiently large that the temperature at that location will still be T_i, i.e. a location where boundary effects prescribed at $x = 0$ have not yet been perceived. This idea is quantified as

$$T(x \to \infty, t) \;=\; T_i \; .$$

For the surface $x = 0$, we can specify any of the 3 standard types of boundary conditions we have studied, though a prescribed temperature, i.e. $T(0, t) = T_0$, is a classical case having analogs in other branches of mechanics.[4.9] That is, the boundary temperature is suddenly changed to some value T_0.

The mathematical significance of this problem is that there is a very clever transform to simplify the partial differential equation (PDE), Eq. (4.3), to an ordinary differential equation (ODE).[4.10] Called a *similarity transform*, it effectively converts the problem of two independent variables, x and t, into a problem of just one independent variable, the unified similarity transform variable φ (Barenblatt and Zel'Dovich, 1972; Hansen, 1967). Such a transform may be possible for problems that do not have naturally–identifiable scales, for example those having no clear length scale because of infinite dimensions. Determining what the similarity transform variable is for a specific problem is not trivial, nor may the parameter itself be unique. In this case, one parameter that works is $\varphi = x/\sqrt{\alpha\, t}$ (see appendix D) and the solution can be shown to be a function of this single,

[4.9] Eq. (4.3) on pp. 30 taken with the initial condition $T(x, 0) = 0$ and the boundary conditions

$$T(0, t) \;=\; T_0 \qquad \text{and} \qquad T(x \to \infty, t) \;=\; 0$$

has a well–known analog in fluid mechanics known variously as Stokes' First Problem (Schlichting, 1979) and the Rayleigh Problem (Panton, 1984). In the hydrodynamic version, a semi–infinite domain of fluid is initially at rest when the boundary at $x = 0$ is suddenly given a specified in–plane velocity. Its solution describes the time–dependent behavior of the flow as momentum diffuses in the x direction. This problem is discussed in numerous sources (Batchelor, 1967). We shall adopt the Rayleigh moniker here, despite the fact that the heat conduction version is not commonly referred to by either name.

[4.10] The technique we discuss here is only one such way of reducing a PDE to an ODE to simplify the solution process. We will examine another approach based on integral transforms in Chapter 6 on pp. 63.

combined variable, as

$$\frac{T(x,t) - T_0}{T_i - T_0} = \operatorname{erf}\left(\frac{x}{\sqrt{4\,\alpha\,t}}\right), \tag{4.15}$$

where "erf" is the so–called Gaussian error function (Andrews, 1985). Heat transfer is then readily derived by applying Fourier's Law (see Eq. (4.16) in appendix D), so that, at the surface $x = 0$, we find

$$q\Big|_{x=0} = \frac{k\,(T_0 - T_i)}{\sqrt{\pi\,\alpha\,t}}\,. \tag{4.16}$$

4.7. Additional Remarks

Unsteady conduction heat transfer occurs in numerous physical situations. If the Biot number is sufficiently small, the capacitance model can be used. It is mathematically straightforward and has the additional convenience of being independent of any particular spatial coordinate system. Otherwise, spatial gradients must be considered and the problem becomes more complicated. We have shown a number of representative examples here, solved using the separation of variables and similarity methods. There are many other scenarios for which similar unsteady, one–dimensional dynamics can be examined, including those in other orthogonal coordinate systems. In fact, separation of variables has been proved to work for homogeneous problems over a wide range of coordinate systems (Özişik, 1980). In cases where boundary conditions are steady and only the end behavior is of interest, i.e. the steady temperature distribution after a very long time, the unsteady term can simply be dropped, reverting the problem to simpler form.

We note finally that we have matched mathematical techniques and methods to specific problems based on some idea of simplicity and convenience. There are often several ways to solve a given problem, for example the Rayleigh conduction in §4.6 can be solved using separation of variables just as well (Özişik, 1980). All such problems can be solved with more general techniques, such as we will examine in upcoming chapters.

4.8. Exercises

4.1 Show that the total heat energy tranferred over an elapsed time period $0 \to t$ in a zero–dimensional unsteady conduction process is $Q = \rho\,V\,c\,\theta_0\big(1 - e^{-t/t_c}\big)$.

4.2 The Rayleigh problem can be solved with the so–called similarity method using the transformed similarity parameter $\varphi = x/\sqrt{t}$. It has been asserted that a successful similarity transform is not necessarily unique. Confirm this claim by solving the Rayleigh problem using the alternative dimensionless parameter $\varphi = x/\sqrt{\alpha\,t}$ and demonstrate that the solutions obtained using the two different parameters are in fact one and the same.

4.3 The Rayleigh problem for $T(x,t)$ in $0 \le x < \infty$ for $t \ge 0$ is defined as

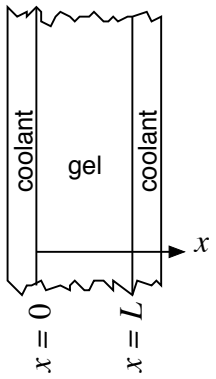
$$\frac{\partial T}{\partial t} \;=\; \alpha\,\frac{\partial^2 T}{\partial x^2} \qquad T(x,0) \;=\; T_i \qquad T(0,t) \;=\; T_0 \qquad T(x\to\infty,t) \;=\; T_i$$

and this problem can, in fact, be solved with the so–called similarity method using the transformed similarity parameter $\varphi = x/\sqrt{t}$. In particular, this transform reduces the PDE to an ODE and collapses the 2 boundary conditions for x and 1 initial condition for t into just 2 boundary values for φ:

$$\frac{d^2T}{d\varphi^2} \;+\; \frac{1}{2\,\alpha}\,\varphi\,\frac{dT}{d\varphi} \;=\; 0 \qquad T(\varphi=0) \;=\; T_0 \qquad T(\varphi\to\infty) \;=\; T_i\,.$$

Show that the similarity approach is *not* applicable to the associated Dirichlet problem, where the boundary condition for $x \to \infty$ is moved to a finite location $x = L$, i.e. $T(L,t) = T_i$.

4.4 DNA sequencing by the "di–deoxy" method depends on the concept of *molecular sieving* of charged DNA fragments through an electrically–conducting gel. One of the primary design problems of such devices is that the associated heat generation, $\dot{q}$, will break the molecular bonds (destroying the molecules) if temperature, T, becomes sufficiently high. The gel is defined in $0 \le x \le L$, where L is very small compared to the other 2 dimensions, so the model of 1–D conduction in the x direction is applicable, i.e.



$$\rho\,c\,\frac{\partial\,T}{\partial\,t} \;=\; \dot{q} \;+\; k\,\frac{\partial^2\,T}{\partial\,x^2}\,,$$

where c, ρ, and k are heat capacity, density, and conductivity, respectively. Here, we will consider the idea of modifying a device by adding coolant jackets that can precisely control the temperatures at the gel boundaries.

Taking room temperature as a reference, $T_r = 0$, consider the case where the cooling jackets are set so that the boundary temperatures on the gel are T_c *below* room temperature. Determine T for steady–state operating conditions if the maximum temperature in the gel is not to exceed T_c *above* room temperature. Give the answer as a function of the parameters T_c and L and the coordinate x.

4.5 Referring to exercise 4.4: At a reference time $t = 0$, the device power is cut, whereby $\dot{q} = 0$ and the coolant jackets are immediately removed, so that the boundaries at $x = 0$ and $x = L$ are simultaneously reset to room temperature, $T = 0$. Determine the transient temperature decay $T(x,t)$ within the gel for $t \ge 0$ and, in particular, simplify the mode coefficients, C_n, to the greatest degree possible.

CHAPTER 5

Multi–Dimensional Steady Conduction: Laplace's Equation

THE GENERAL FORM OF CONSERVATION OF ENERGY when heat transfer occurs exclusively by conduction was derived in Eq. (2.16) on pp. 12. We observed that if thermal conductivity is constant, conditions are steady, and there is no heat generation, then the equation simplifies to Laplace's equation, $\nabla^2 T = 0$, which was written in long–hand in rectangular coordinates as Eq. (2.19). This equation can be expressed in many other coordinate systems, via the associated forms of the ∇ operator (Panton, 1984). The general expression in vector notation implies that $T = T(x, y)$ if the dependence is two–dimensional and $T = T(x, y, z)$ if it is three–dimensional.

The Laplace equation describes dynamics that apply over a broad spectrum of physical problems, close thermofluid relatives being those of uni–directional flow and potential flow.[5.1] Within the discipline of conduction, there are an enormous number of practical problems governed by this equation, especially when considered in light of combinations of various possible boundary conditions. As with other special cases discussed earlier for conduction, we will examine several representative problems. Although multi–dimensional steady problems are physically of a different kind than those discussed in Chapter 4 (one–dimensional unsteady conduction), we can still apply the same mathematical tools to many of them here, chiefly separation of variables (SOV).

In §4.3, we had passingly commented that separation of variables is well–suited to homogeneous problems. Here, we shall see it is also readily applied to multi–dimensional problems having non–homogeneous boundary conditions in one of the coordinate directions, the technique being very much like what we saw in Chapter 4. We shall furthermore demonstrate a generalization of the concept of superposition whereby SOV can be extended to non–homogeneous boundary conditions for more than one coordinate direction. Here, SOV starts to show its limitations in terms of practicality, suggesting the need for more general methods that will yet be introduced.

5.1. The Two–Dimensional Dirichlet Problem

We will start this discussion with the most fundamental two–dimensional problem, $T = T(x, y)$, where the boundary conditions are all of the first type, i.e. Dirichlet (Fig. 5.1). Besides examining the basic configuration, we will contrast two versions of it: one where the length dimensions, L and H, are comparable, and another where $H \gg L$. Note from Fig. 5.1 that for the latter case, we

[5.1] Uni–directional flows, also known as parallel flows, are those in which 2 of the 3 velocity components are zero, meaning the non–linear terms of the Navier–Stokes equations vanish, c.f. Eq. (7.15) on pp. 88. If the pressure gradient is also zero and the flow is driven by boundary motion, then the velocity distribution is governed by Laplace's equation (Wang, 1991). The two–dimensional Couette problem is perhaps the best–known example and solutions for numerous configurations have been derived (e.g. Wendl and Agarwal, 2000). "Potential flow" is the common designation of flows that are both inviscid and irrotational. Here, the primitive variables can be replaced by the stream function and the velocity potential function. It can be shown (e.g. Panton, 1984) that motion is then governed by a pair of Laplace equations for these respective entities.

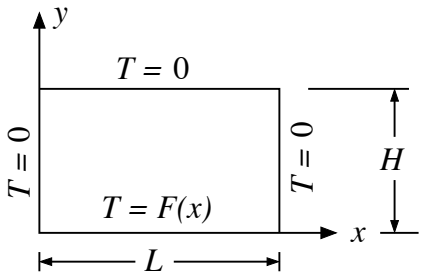


FIGURE 5.1. *Representative two–dimensional steady Dirichlet conduction problem in a rectangle.*

specifically frame the boundary conditions so that conduction is driven primarily along the larger dimension, i.e. in the y direction. In other words, this is one of the exceptions of the discussion we had in §3.1 on pp. 16 regarding the one–dimensional model (c.f. Fig. 3.1).

As mentioned, the two–dimensional problem is governed by Laplace's equation, as described for this case by Eq. (3.1) on pp. 16, and repeated here for convenience:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \,. \tag{5.1}$$

Let us consider the problem shown in Fig. 5.1, i.e. where

$$T(0,y) = T(L,y) = T(x,H) = 0 \qquad \text{and} \qquad T(x,0) = F(x) \,. \tag{5.2}$$

Although Laplace's equation, as exemplified by Eq. (5.1), is a fundamentally different type as compared to the one–dimensional unsteady equation, e.g. Eq. (4.3) on pp. 30,[5.2] the solution procedure is very similar (see §E.1 in appendix E on pp. 151). We find

$$T(x,y) = \frac{2}{L} \sum_{n=1}^{\infty} \left(\frac{1}{\sinh(\zeta_n H)} \int_0^L F(x) \, \sin(\zeta_n x) \, dx \right) \sin(\zeta_n x) \, \sinh\Big[\zeta_n (H-y)\Big] \,, \tag{5.3}$$

where the eigen–values are explicit: $\zeta_n = n\pi/L$. While the solution is formally valid for any boundary specification $F(x)$, it may not be trivial to evaluate the integral if $F(x)$ is sufficiently complex.

As mentioned above, an interesting variation of this problem is obtained by taking $H \gg L$, which we model by changing the relevant boundary condition in Eq. (5.2) to

$$T(x, y \to \infty) = 0 \,. \tag{5.4}$$

In other words, we move the top boundary in Fig. 5.1 such that the domain becomes semi–infinite in the y direction. Recall that we had invoked a similar specification in the Rayleigh problem (§4.6), which we justified by a logic argument. Eq. (5.4) requires no such argument — it is obvious to observe simply that the top boundary, which is homogeneous, is simply moved "very far away".

[5.2] PDEs can be further classified by type, Eq. (5.1) being an *elliptic* PDE and Eq. (4.3) being a *parabolic* one. Such classification has implications in the formal theory of differential equations, but will not contribute significantly to our more practical focus on problem solving. See, e.g. Hellwig (1964), Hildebrand (1976), or Carrier and Pearson (1976).

This variation of the problem is described in appendix E, §E.2 on pp. 153, the solution being

$$T(x,y) \;=\; \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L F(x)\,\sin(\zeta_n x)\,dx\right)\,\sin(\zeta_n x)\,e^{-\zeta_n y}\,, \tag{5.5}$$

where the eigen–values are the same as for the finite version of the problem, $\zeta_n = n\pi/L$.

Note that neither of the solutions in Eqs. (5.3) and (5.5) depends on any material properties, though they are both a function of geometric properties. One might be tempted to conclude that this is a consequence of the fact that Laplace's equation (5.1) also does not contain any material properties, and therefore that this is a general property of steady Dirichlet problems. But, this is not the case. It holds here only because both the Laplace equation *and* the boundary conditions do not depend upon material properties. It is easy to see, for example, that the temperature distribution will involve k if any of the boundary conditions are of the second or third kind (c.f. §2.5 on pp. 13).

We allowed for an arbitrary boundary condition, $T(x,0) = F(x)$, in this problem, however, suppose we focus on a special case of a constant boundary temperature, $T(x,0) = T_0$, for the two configurations. We can perform the boundary integral (see §E.3 on pp. 154), finding

$$T(x,y) \;=\; \frac{4\,T_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n\,\sinh(\zeta_n H)}\,\sin(\zeta_n x)\,\sinh\Big[\zeta_n(H-y)\Big]\,, \tag{5.6}$$

and

$$T(x,y) \;=\; \frac{4\,T_0}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n}\,\sin(\zeta_n x)\,e^{-\zeta_n y}\,, \tag{5.7}$$

for finite and semi–infinite geometries, respectively. An interesting question is how large H must become in order that Eq. (5.6) converges to Eq. (5.7). We answer this question in this particular case by comparing the temperature profile at a specific height, $y = 0.1\,m$, in a billet of width $L = 0.5m$ and having a variable height, H (Fig. 5.2). The bottom boundary is set to $T(x,0) = 30C$.

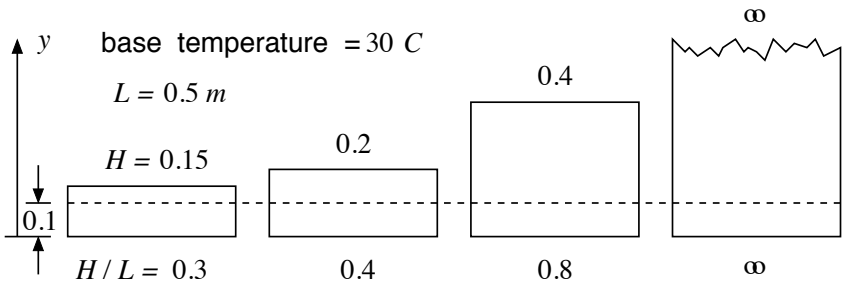


FIGURE 5.2. *Steady Dirichlet solution for 4 different aspect ratios of a billet, including the semi–infinite case (rightmost diagram). Each configuration has a width of $L = 0.5\,m$, a constant base temperature of $T(x,0) = 30\,C$, and homogeneous temperatures on all other boundaries. We compare the temperature distribution for each at a location $y = 0.1$ (the common dashed line).*

Intuitively, we see that the top boundary should have a decreasing influence on $T(x,0.1)$ as it is moved further away. In other words, we would expect the temperature profile to converge to some

particular limiting form as H/L becomes sufficiently large. How large must this be? Fig. 5.3 shows profiles for several H/L ratios, 0.3, 0.4, 0.8, and ∞, for which the profile appears to converge

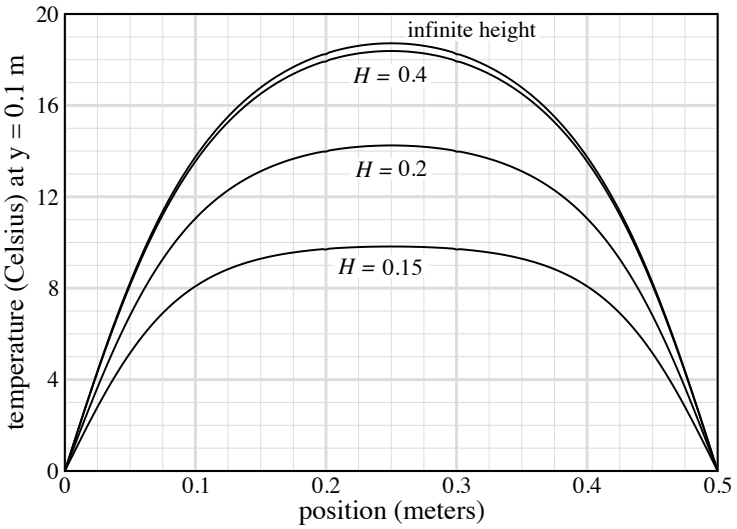


FIGURE 5.3. *Temperature profiles at $y = 0.1\,m$ for parameters given in Fig. 5.2.*

surprisingly quickly, reaching a maximum of around $19\,C$ at midline. All profiles are symmetric about $x = 0.25$, of course, because boundary conditions at $x = 0$ and $x = 0.5$ are the same. The convergence rate is a little less striking if we consider the fact that we are evaluating the profile fairly close to the non–homogeneous boundary at the base. Given the profile at $y = 0.1$, it might make more sense to report the H/y ratios, which are 1.5, 2, 4, and ∞, respectively. We could choose other y locations at which to do this comparison, and the profiles and the convergence would be commensurately different.

One aspect that is qualitatively different for configurations that have multiple space dimensions is that it is more difficult to visualize the resulting solutions. For example, for 1–dimensional unsteady problems, it is a simple matter to plot temperature profiles at various times and this gives a very informative synopsis of the dynamics (c.f. Fig. 4.4 on pp. 32). However, 2–dimensional configurations can still be well–represented in graphical form by contour or surface plots. For example, Fig. 5.4 shows $T(x, y)$ represented as a surface for the finite domain solution, Eq. (5.6), having $T(x, 0) = 30\,C$ and $H = L = 0.5\,m$. Such visualization is obviously very useful for quickly seeing where the interesting trends within the domain are located. For example, it is clear that there is substantial heat transfer in the x direction near the base.

5.2. Some Numerical Issues

There are also issues that relate to the numerics of *evaluating* exact solutions, including those discussed in this section. Although there is no widely–accepted and precise nominal distinction, we emphasize the difference between the numerical methods for managing these fairly straightforward contingencies versus those for solving the governing conservation equations themselves, e.g. finite–volume algorithms for the Navier–Stokes equations (Tannehill et al., 1997), which are enormously more sophisticated. The latter, known by various names, including "computational fluid dynamics"

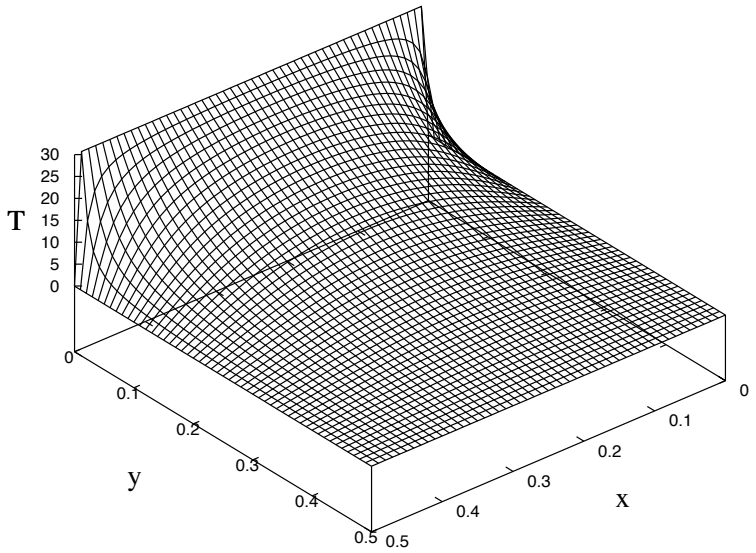


FIGURE 5.4. *$T(x, y)$ represented as a surface for a square domain of edge length $0.5\,m$ having a constant base temperature of $30\,C$ with remaining boundaries held at $0\,C$. Terms were summed at each point until the fractional change for each additional term fell below 10^{-6}.*

and "computational fluid mechanics and heat transfer" (c.f. §1.5 on pp. 5), are founded upon making suitable approximations to the differential terms within the equations and then solving the resulting, invariably large system of algebraic equations numerically. These topics are far beyond our current scope and are not discussed further.

5.2.1. Floating Point Arithmetic. An equation like (5.7) for the semi–infinite domain poses no particular difficulty. The trigonometric "sine" fluctuates between -1 and 1, while the exponential and n^{-1} terms both decay toward 0 (at different rates). In particular, the convergence of this series can be very fast, depending partially upon y, since the exponential can decay very quickly. Consequently, it appears there are no serious numerical issues to consider in evaluating this expression.

Consider now the related solution for the finite domain, Eq. (5.6). The "hyperbolic sine", sinh, is an increasing function and can overflow when the series does not converge fast enough (Forsythe et al., 1977).[5.3] For example, Table 5.1 shows the growth of the sinh term in the denominator of Eq. (5.6) for selected n for the square finite domain, i.e. $H/L = 1$. In fact, Eq. (5.6) requires taking the *ratio* of two sinh terms. If a large number of terms is indeed required for a particular set of parameters, one must use more sophisticated methods. For example, when the arguments become large (because n becomes large in the summation), one possibility is to simply substitute the limiting form $\sinh\varphi \to e^{\varphi}/2$ when $\varphi \gg 1$, as can be seen from the identity in Eq. (3.14) on pp. 21. The advantage is that the ratio of these two problematic large numbers can then

[5.3]Numerical overflow is the condition in which a mathematical number is too large to be represented within the floating–point register of a particular computational device.

TABLE 5.1. Approximate growth of sinh term for selected n in Eq. (5.6) when $H = L$

n	$\sinh(n\pi)$	n	$\sinh(n\pi)$
1	11.54874	121	6.145×10^{164}
21	2.243×10^{28}	141	1.191×10^{192}
41	4.349×10^{55}	161	2.310×10^{219}
61	8.432×10^{82}	181	4.478×10^{246}
81	1.635×10^{110}	201	8.682×10^{273}
101	3.169×10^{137}	221	1.683×10^{301}

be evaluated *algebraically* instead of *numerically* for those terms where numerical overflow would otherwise occur. Specifically,

$$\frac{\sinh\Big[\zeta_n(H-y)\Big]}{\sinh(\zeta_n H)} \;\to\; \frac{e^{\zeta_n(H-y)}}{e^{\zeta_n H}} \;=\; e^{\zeta_n(H-y)-\zeta_n H} \;=\; e^{-\zeta_n y}\,,$$

which is again numerically well–behaved. The astute reader will immediately notice that we've actually done little more than to re–derive the resulting form for the semi–infinite case! Except, here we used a limiting argument of the form that observed $e^{-\varphi} \to 0$ for $\varphi \gg 1$ in the identity in Eq. (3.14). Other cases may not be so straightforward, requiring perhaps more sophisticated asymptotic approximations or extended precision floating–point calculations using special numerical libraries.

An interesting example of the number of terms required for convergence is given by the case we had just shown in Fig. 5.4. That is, Fig. 5.5 plots the number of terms required to evaluate $T(x, y)$ at

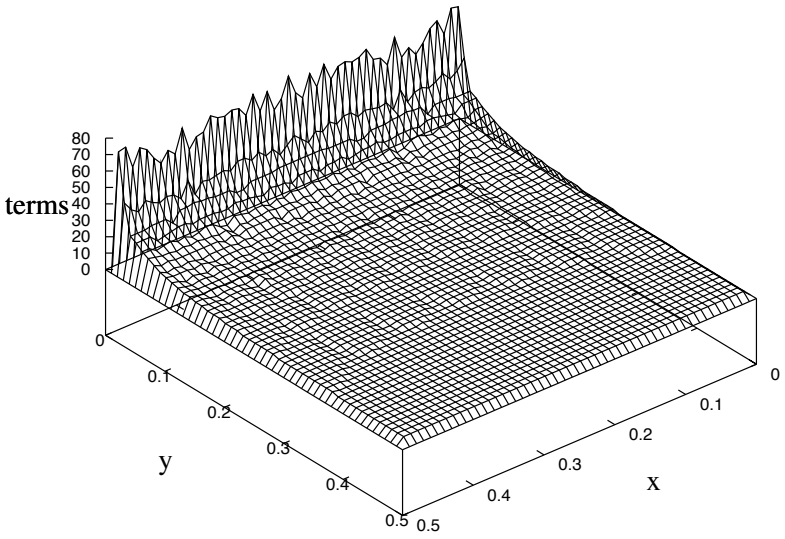


FIGURE 5.5. *Number of terms required to evaluate $T(x, y)$ in Fig. 5.4.*

each (x, y). As we might have expected, it appears that locations having higher gradients generally require more terms. Note also that there are some (x, y) locations near the $y = 0$ boundary requiring more than 70 terms, suggesting sinh values on the order of 10^{100} (Table 5.1). In this particular plot, these points are located along the line $y = 0.01$. Had we attempted to evaluate the solution even closer to the base, say at $y = 0.001$, the number of terms would have been commensurately larger. Finally, the numbers of terms precisely at the boundaries are shown as zero because values of T are known *a priori* at these locations.

5.2.2. Convergence Involving Periodic Functions. We observed above that the trigonometric sine poses no floating–point anomalies, yet it still has its own issues related to its periodicity and how this property can affect convergence, i.e. the point at which one decides that sufficient terms have been summed to give an answer of suitable accuracy. Consider, for example, algorithm 5.1 to evaluate the finite domain solution in Eq. (5.6) that we have been discussing and which is plotted in Fig. 5.4 for a specific parameter set. This algorithm is essentially a simple

Algorithm 5.1 Simple Algorithm for Evaluating Eq. (5.6)

specify convergence threshold, ϵ, suitably near 0
specify point at which to evaluate solution (x, y)
specify dimensions of the domain, H and L
specify constant non–homogeneous boundary temperature, $T(x, 0) = T_0$
set $(n, T, T_{old}) = (1, 0, 0)$
repeat
  $T_{old} = T$
  $T = T + \sin(\zeta_n x) \sinh[\zeta_n(H - y)]/[n \sinh(\zeta_n H)]$
  $n = n + 2$
until $|T - T_{old}| \leq \epsilon$
print T

loop that runs until the contribution of the current term falls below a user–defined convergence threshold of ϵ. This sort of convergence test is quite common,[5.4] but in this case, there is a subtle omission that leads to pre–mature truncation of the series, i.e. a false convergence, at certain (x, y) locations in the domain. To understand the problem and ascertain a solution, we must consider the periodic nature of $\sin(\zeta_n x)$ in Eq. (5.6) more closely.

In essence, if a particular term in the summation happens to coincide with conditions for which sine vanishes because it happens to cross the 0–axis for that particular combination of parameters, i.e. $\sin(\zeta_n x) = 0$, then it will *appear* to the simple ϵ test in algorithm 5.1 that convergence has been obtained, even though it may not have been. See an actual example of this phenomenon in Fig. 5.6 for the case already examined in Fig. 5.4. This plot shows the convergence process for two points that are very close to one another in the domain: $(0.01, 0.01)$ and $(0.02, 0.01)$. Notice that $\sin(\zeta_n x)$ is never zero for $x = 0.01$, so the pre–mature convergence phenomenon never occurs at this point. In essence, the period of this function is such that there is never an eigen–mode for which $\zeta_n x$ is

[5.4] It appears, for example, to have been used by Zill and Cullen (2000), whose Fig. 13.14 shows the jagged surface near the non–homogeneous boundary that is consistent with the type of numerical anomaly we describe in this section.

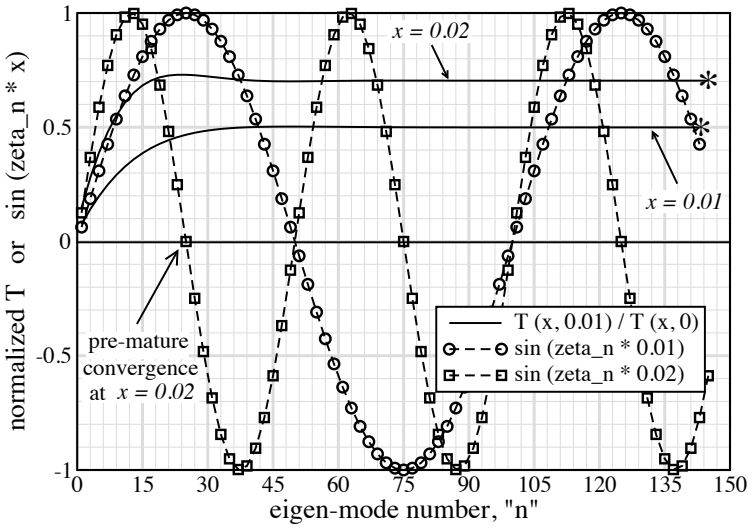


FIGURE 5.6. *Convergence study of* $T(0.01, 0.01)$ *and* $T(0.02, 0.01)$ *for the finite–domain Dirichlet configuration plotted in Fig. 5.4. The dashed curves represent* $\sin(\zeta_n x)$ *with circles and squares denoting the values at the discrete, odd mode numbers, as specified in Eq.* (5.6)*, for* $x = 0.01$ *and* $x = 0.02$*, respectively. Solid curves show the convergence to the two respective temperatures, normalized as* $T(x, 0.01)/T(x, 0)$*, with asterisks indicating true convergence. Potentially pre–mature convergence for* $x = 0.02$ *is noted.*

a multiple of π; Its discrete values always "hop over" the zero point. Consequently, $T(0.01, 0.01)$ converges using algorithm 5.1 without any special modification.

Matters are clearly quite different for $x = 0.02$, where $\sin(\zeta_n x)$ vanishes at $n = 25$ after having summed $(n + 1)/2 = 13$ terms. As written, algorithm 5.1 will halt at this point. Notice that the corresponding temperature, $T(0.02, 0.01)$, has not yet converged, being about 3.5% higher than its correct value. Yet, now that the problem has been recognized, amending the algorithm procedurally is fairly straightforward. We could, for example, simply add a second test which would not allow the algorithm to halt if $|\sin(\zeta_n x)|$ is below a second user–chosen convergence threshold (algorithm 5.2). With such an extra condition, the loop would always be forced through at least one extra iteration,

Algorithm 5.2 Improved Algorithm for Evaluating Eq. (5.6)

```
  :
  :
  repeat
    :
    :
  until |T − T_old| ≤ ε_a and |sin(ζ_n x)| ≥ ε_b
  print T
```

so as never to terminate at a mode for which sine vanishes. Note in Fig. 5.6 that there are 3 such out–of–bounds points that are forcibly skipped for $x = 0.02$ on the way to proper convergence.

Is there are more systematic way to characterize this problem, in particular when it occurs and the degree to which it affects the accuracy of T? The answer is affirmative, of course. We have already observed in vague terms that this phenomenon is a function of periodicity. Fig. 5.6 demonstrates that it is specifically a function of the half–period, i.e. the distance between every successive crossing of the 0–axis. Given the mode number, n, as the independent variable, the half–period of $\sin(n\,\pi\,x/L)$ is[5.5]

$$\frac{\delta}{2} \;=\; \frac{1}{2}\,\frac{2\,\pi}{\pi\,x/L} \;=\; \frac{L}{x}\,.$$

Because n can only take odd integer values in Eq. (5.6), this phenomenon will exist wherever the half–period L/x is an odd integer. For example, for $x = 0.02$ and $L = 0.5$ in the above example, we have $L/x = 0.5/0.02 = 25$ and the sine term is

$$\sin\left(\frac{n\,\pi\,x}{L}\right) \;=\; \sin\left(\frac{n\,\pi}{25}\right)\,,$$

whereby this term vanishes at $n = 25$, 75, and 125, as shown in Fig. 5.6. Note that, while the term *would* also vanish for, e.g. 50 and 100, those numbers are even and are therefore not part of the set of eigen–modes in this particular solution, and therefore not indices in the summation. This observation also explains why the location $x = 0.01$ does *not* experience this numerical anomaly. Because its associated half–period is $L/x = 0.5/0.01 = 50$, there will never be an n that is both a multiple of 50 and odd.

The above discussion actually does not tell the whole story in that there are a additional sets of x–values where the first 0–axis crossings will occur at *multiples* of x that result in exactly a half period. Perhaps the easiest way to visualize this phenomenon is by tabulating eigen–functions along with their interior zeroes and where the consequent critical values of x reside (Table 5.2). Take

TABLE 5.2. Interior zeroes and critical values of x for first several odd modes.

n (odd)	$\sin(n\pi x/L)$	max $x = L$	multiples of π that can give zeroes in the interior $x < L$	critical values of x that result in zero with duplicates canceled
1	$\sin(1\pi x/L)$	π	—	—
3	$\sin(3\pi x/L)$	3π	$\pi, 2\pi$	$\frac{L}{3}, \frac{2\,L}{3}$
5	$\sin(5\pi x/L)$	5π	$\pi, 2\pi, 3\pi, 4\pi$	$\frac{L}{5}, \frac{2\,L}{5}, \frac{3\,L}{5}, \frac{4\,L}{5}$
7	$\sin(7\pi x/L)$	7π	$\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$	$\frac{L}{7}, \frac{2\,L}{7}, \frac{3\,L}{7}, \frac{4\,L}{7}, \frac{5\,L}{7}, \frac{6\,L}{7}$
9	$\sin(9\pi x/L)$	9π	$\pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi$	$\frac{L}{9}, \frac{2\,L}{9}, \cancel{\frac{3\,L}{9}}, \frac{4\,L}{9}, \frac{5\,L}{9}, \cancel{\frac{6\,L}{9}}, \frac{7\,L}{9}, \frac{8\,L}{9}$

$n = 9$, for example. The eigen–function is $\sin(9\pi x/L)$ and the maximum value of the argument is 9π at the boundary, $x = L$. However, the function can be zero in the interior of the problem, $0 < x < L$, if its argument is any lower multiple of π, i.e. $\pi, 2\pi, 3\pi$, etc. This will result in pre–mature convergence at $n = 9$ and the values of x that result in this condition are easily found

[5.5]A function having the property $f(n) = f(n + \delta)$ for all real values of n is *periodic*. The period of $\sin(\varphi n)$ can be demonstrated by a simple logical argument (Combellack, 1962). If δ represents the period, then by definition it must be equal to the smallest value for which $\sin(\varphi n) = \sin[\varphi(n + \delta)] = \sin(\varphi n + \delta\varphi)$. Of course, 2π is the period of the basic sine function, i.e. $\sin\xi = \sin(\xi + 2\pi)$. Given the correspondence between φn and ξ, we conclude $\delta\varphi = 2\pi$, meaning the period is $\delta = 2\pi/\varphi$.

algebraically as $L/9, 2L/9$, etc. Notice that there will be various duplicated values for higher modes, for example $L/3$ and $2L/3$ in $n = 3$ are duplicated for $n = 9$.

The other aspect we can readily deduce is the magnitude of this error, if it is allowed to occur. In particular, series truncation occurs at the first 0–axis crossing, i.e. $n = L/x$, for example as shown in Fig. 5.6 for $n = 25$. Given that a total of $(n+1)/2$ will have been summed, the magnitude of the error decreases with x. Clearly, for example, $x = L/3$ and $x = L/5$ would give the largest errors, with only $(3 + 1)/2 = 2$ and $(5 + 1)/2 = 3$ terms summed, respectively (Table 5.2). For very small x, the minimum number of terms summed may eventually exceed those required for actual convergence, in which case the error would not arise, even under our naïve convergence test in algorithm 5.1.

A few final points are worth mention. First, it should be clear that the phenomenon discussed here may apply to other series and should be considered especially when any periodic or quasi–periodic functions are involved. Such would include, for example, the family of Bessel functions of the first kind (Andrews, 1985). Second, we do not include $L/1$ in any of the above discussion, because this location is specified by the boundary condition. Third, many problems, including the one discussed above are symmetric about a mid–line or a mid–plane,[5.6] and this is reflected in Table 5.2 in the symmetry of values about $x = L/2$.

5.3. Uniqueness of the Dirichlet Solution

A slightly more esoteric, but extremely interesting question regarding differential equations is whether a solution, presuming one can be derived, is actually unique. We have already encountered cases that demonstrate the answer is "not necessarily". For example, using the methods shown in §B.2 on pp. 135, it is easy to show that the system

$$\Psi''(x) + \zeta^2\Psi(x) = 0\,, \qquad \Psi(0) = 0\,, \qquad \Psi(1) = 0$$

has an infinite number of non–trivial solutions of the form $\Psi = C\,\sin(n\pi x)$, where $\zeta = n\pi$, C is a constant, and $n \in \{1, 2, 3, \dots\}$. This simple example is especially intriguing since the problems we have examined thus far for Laplace's equation involve very similar sub–problems of this type. Such questions of uniqueness are an important topic in the broader theory of differential equations (Hellwig, 1964).

Here, we will introduce some concepts related to uniqueness and show that the above solutions for the two–dimensional Dirichlet problem are indeed unique. Such proofs are often based on the idea of presuming at the outset that 2 different solutions actually exist and then demonstrating by some means that these solutions must actually be one and the same — and therefore there really

[5.6] Symmetry about the mid–line $x = L/2$ for Eq. (5.6) is suggested by Fig. 5.4. However, the symmetry question can be proved formally using the "angle–difference" relationship for sine (Beyer, 1984). That is, the problem is symmetric if

$$\sin\left(\frac{n\,\pi\,x}{L}\right) = \sin\left(\frac{n\,\pi\,(L-x)}{L}\right) = \sin\left(n\,\pi - \frac{n\,\pi\,x}{L}\right) = \cancelto{0}{\sin(n\,\pi)}\cos\left(\frac{n\,\pi\,x}{L}\right) - \cos(n\,\pi)\,\sin\left(\frac{n\,\pi\,x}{L}\right)\,,$$

the last expression being a consequence of the angle–difference rule. Note the cancellation for the sine of multiples of π. Since $\cos(n\,\pi) = (-1)^n$, the right hand side simplifies to $-(-1)^n \sin(n\,\pi\,x/L)$. Noting that n is always odd in Eq. (5.6), there are always an even number of negative signs in this expression, which mutually cancel one another, leaving the right–hand–side identically $\sin(n\,\pi\,x/L)$, which proves the equivalence.

is only one unique answer.[5.7] We will frame the uniqueness proposition in the context of the finite Cartesian configuration above, although it will be clear in the detailed derivation in §E.4 pp. 155 that it applies in general to the Dirichlet problem, irrespective of geometry.

The full proof is shown in §E.4 but can be sketched, as follows. Start by assuming that two different solutions exist to this problem, $T_1(x,y)$ and $T_2(x,y)$.[5.8] Each of these satisfies Laplace's equation (5.1), as well as the boundary conditions, Eq. (5.2). Now define $\theta = T_1 - T_2$ and note that the fact that we have defined $T_1 \neq T_2$ also implies $\theta \neq 0$, *if* the 2–solution proposition is true. Note that θ itself is also a solution to a Dirichlet problem. To wit, it satisfies both Laplace's equation

$$\nabla^2\theta \;=\; \nabla^2\Big(T_1 - T_2\Big) \;=\; \nabla^2 T_1 \;-\; \nabla^2 T_2 \;=\; 0 \;-\; 0 \;=\; 0$$

(because T_1 and T_2 each satisfy Laplace's equation) and it satisfies the boundary conditions

$$\begin{aligned}
\theta(0,y) &= T_1(0,y) \;-\; T_2(0,y) \;=\; 0 \;-\; 0 \;=\; 0 \\
\theta(L,y) &= T_1(L,y) \;-\; T_2(L,y) \;=\; 0 \;-\; 0 \;=\; 0 \\
\theta(x,H) &= T_1(x,H) \;-\; T_2(x,H) \;=\; 0 \;-\; 0 \;=\; 0 \\
\theta(x,0) &= T_1(x,0) \;-\; T_2(x,0) \;=\; F(x) \;-\; F(x) \;=\; 0\,.
\end{aligned}$$

The last statement is particularly important, as it fully establishes that $\theta = 0$ around the entire boundary of the problem. The proof then uses Green's theorem to show that $\nabla\theta = 0$ within the domain by virtue of the very fact that $\theta = 0$ identically on the boundary. One can then deduce from $\nabla\theta = 0$ that θ itself must be a constant. The boundary value of $\theta = 0$ is invoked once more: this constant *must* be zero, whereby $\theta = 0$ identically within the domain itself. Finally, since $\theta = T_1 - T_2$, we are left with the conclusion that $T_1 = T_2$, i.e. these two solutions, initially presumed to be different, can only be the same.

5.4. Two–Dimensional Conduction Having Mixed Boundary Conditions

Let us now consider a variation of the two–dimensional problem, $T = T(x,y)$, where the boundary conditions are of a *mixed* type, specifically a simple combination of Dirichlet and Neumann (Fig. 5.7). Note in referring back to Fig. 5.1, that we have also switched the top and bottom boundaries (turning the problem "upside down" as it were) to illustrate a few additional mathematical implications.

If the vertical boundaries are very well insulated, they can be taken as essentially adiabatic, so that heat transfer at $x = 0$ and $x = L$ is zero. Fourier's Law immediately implies the 2 associated boundary conditions are

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} \;=\; \left.\frac{\partial T}{\partial x}\right|_{x=L} \;=\; 0\,,$$

[5.7] This idea can be illustrated by an elementary algebra problem, say $\varphi + 3 = 5$, for which it is readily shown that $\varphi = 2$. However, this does not formally prove that there is only *one* solution for this problem, even though the equation is simple enough to see by inspection that no other value would satisfy it. To prove the proposition, assume there are at least two different solutions, φ and η, so that both $\varphi + 3 = 5$ *and* $\eta + 3 = 5$. However, by the transitive property of equality, $\varphi + 3 = \eta + 3$, from which we immediately conclude $\varphi = \eta$. That is, the two assumed solutions are actually one and the same.

[5.8] One of these could be the solution derived above, Eq. (5.3), but the proof does not actually depend on assuming specific forms of T_1 and T_2 — only that there are 2 such solutions and that they are different.

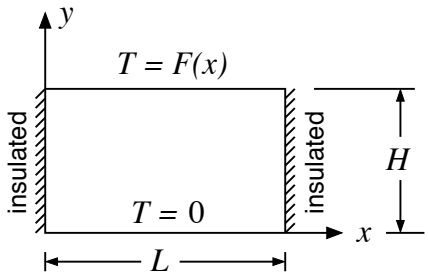


FIGURE 5.7. *Two–dimensional steady conduction in a rectangle having adiabatic side (vertical) boundaries and Dirichlet conditions at the horizontal boundaries.*

which are Neumann boundary conditions (c.f. Eq. (2.23) on pp. 14). We again let conduction be driven by a primary temperature gradient in the y direction, i.e.

$$T(x,H) \;=\; F(x) \qquad \text{and} \qquad T(x,0) \;=\; 0\,,$$

which are Dirichlet boundary conditions, although reversed with respect to the previous problem, as mentioned.

Physically speaking, this configuration seems only marginally different from the one we just examined. The mathematically interesting aspect of this problem is how the Neumann boundary conditions affect the solution. In short, we find that $\zeta_0 = 0$ is a new eigen–value, which rounds–out the set of positive eigen–values, $\zeta_n = n\pi/L, n \in \{1,2,3,\dots\}$ we have found for other problems. Its corresponding eigen–mode is governed by independent ODEs and results in a conspicuously independent addition to the general solution

$$T(x,y) \;=\; C_0\, y \;+\; \sum_{n=1}^{\infty} C_n\, \cos(\zeta_n x)\, \sinh(\zeta_n y) \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L}\,. \tag{5.8}$$

$$C_n \;=\; \frac{2}{L\, \sinh(\zeta_n H)} \int_0^L F(x)\, \cos(\zeta_n x)\, dx\,. \tag{5.9}$$

$$C_0 \;=\; \frac{1}{H\, L} \int_0^L F(x)\, dx \tag{5.10}$$

whose derivation is shown in full in appendix F on pp. 158.

There are several interesting aspects of this solution. First, it is clear that the adiabatic (Neumann) boundary conditions are identically satisfied. The first partial derivative in x is

$$\frac{\partial T}{\partial x} \;=\; -\sum_{n=1}^{\infty} C_n\, \zeta_n\, \sin(\zeta_n x)\, \sinh(\zeta_n y)\,.$$

Evaluating this expression at $x = 0$ and $x = L$ indicates an infinite summation of terms, each multiplied by $\sin 0$ and $\sin(n\pi)$, respectively, the latter vanishing because its argument is always a multiple of π.

Secondly, recall that we examined the case of the constant non–homogeneous boundary temperature for the Dirichlet problem in §5.1, with an interesting instance of $T(x,y)$ plotted in Fig. 5.4.

However, the solution for the special case of constant boundary condition for this problem may be a little surprising. If we take $F(x) = T_H$, we find

$$C_n = \frac{2}{L\,\sinh(\zeta_n H)} \int_0^L T_H\,\cos(\zeta_n x)\,dx = \frac{2\,T_H}{L\,\sinh(\zeta_n H)} \int_0^L \cos(\zeta_n x)\,dx = 0$$

$$C_0 = \frac{1}{H\,L} \int_0^L T_H\,dx = \frac{T_H}{H\,L} \int_0^L dx = \frac{T_H\,(L-0)}{H\,L} = \frac{T_H}{H}\,.$$

Note that we've exploited the orthogonality of the cosine function to determine $C_n = 0$ without actually performing the integral.[5.9] Clearly, the series vanishes, leaving only the "zero–mode" term, with the result

$$T(x,y) \;\to\; T(y) = \frac{T_H}{H}\;y\,,$$

which is nothing more than the elementary solution for 1–dimensional steady conduction.[5.10] We may not have anticipated this result, meaning that the math would have shown us something that our intuition did not detect. In retrospect, however, the reduction of $T(x,y) \to T(y)$ does make sense. The Neumann conditions require independence of x at the vertical boundaries. If the conditions at $y = H$ are themselves independent of x, then there appears to be no mechanism to create any temperature variation in x.

Let us now examine a problem more like what might occur in a real–world application. Fig. 5.8 shows a rectangular conduction domain of dimensions $L \times H = 0.1 \times 0.05$ (meters) having 2 heat

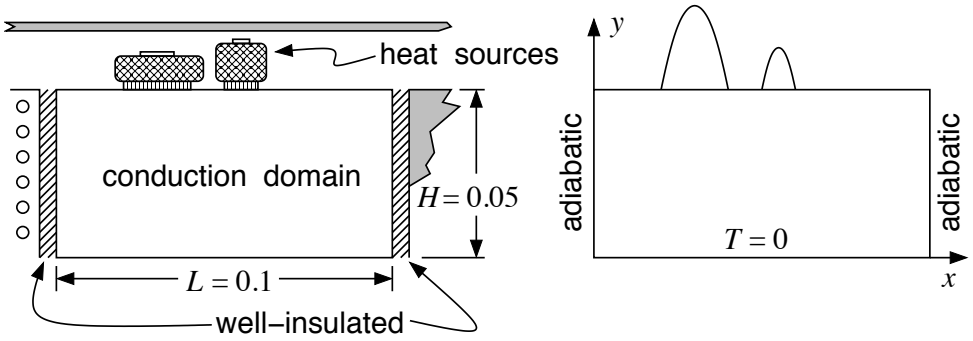


FIGURE 5.8. *Electro–mechanical instrument containing a conduction domain with 2 heat–generating elements mounted on top (left panel) and a corresponding physical model showing the approximate boundary effect of these elements as two half–sine waves (right panel).*

generating elements mounted on the top. If these elements produce heat at a steady rate and if the insulation on the vertical boundaries is very effective, then the solution in Eqs. (5.8) to (5.10) is a good model of this problem, provided we can assign reasonable estimates to the temperature distributions imposed at the top boundary. For this example, let us assume each element creates a sine–type distribution, so that the boundary temperature at $y = H$ is given by the piece–wise

[5.9] See Eq. (F.7) on pp. 161.

[5.10] See e.g. footnote 3.2 on pp. 18 or introductory texts (e.g. Holman, 2010; Bergman et al., 2011).

function

$$F(x) \;=\; \begin{cases} 0 & : \quad 0 \le x < 0.02 \\ 50\,\sin\Big[10\,\pi(5\,x-0.1)\Big] & : \quad 0.02 \le x < 0.04 \\ 0 & : \quad 0.04 \le x < 0.05 \\ 20\,\sin\Big[50\,\pi(2\,x-0.1)\Big] & : \quad 0.05 \le x < 0.06 \\ 0 & : \quad 0.06 \le x \le 0.1\,, \end{cases} \tag{5.11}$$

where the units are in degrees Celsius. Because $F(x)$ is rather more complicated than examples we have used previously, the evaluations of C_0 and the C_n are also more complicated. The work and subsequent results are shown in detail in §F.3 on pp. 162.

A surface plot is useful once again for visualizing the temperature distribution, $T(x, y)$, in the configuration shown in Fig. 5.8 and we write a summation loop, similar to algorithm 5.2. That is, we are conscious that there is a periodic function, here being cosine in Eq. (5.8), and we take steps to prevent the numerical artifact of pre–mature convergence from occurring (c.f. §5.2.2 on pp. 46). We are also careful to assign T at any points on the vertical edges, i.e. at $T(0, y)$ and $T(0.1, y)$, to the same value as its adjacent neighbor in the interior, i.e. $T(\Delta x, y)$ and $T(0.1 - \Delta x, y)$, respectively, where Δx is the discrete resolution of our plot in the x direction. In this way, we accurately represent the adiabatic (Neumann) boundary condition. Fig. 5.9 shows the resulting temperature

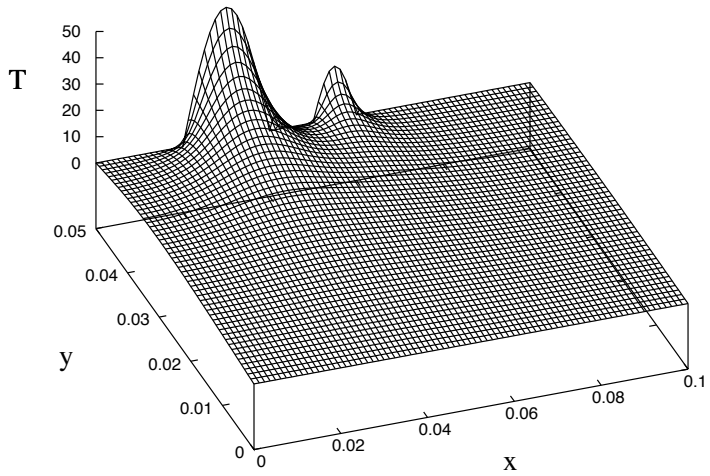


FIGURE 5.9. *$T(x, y)$ plotted as a surface for the mixed boundary condition problem in Fig. 5.8. Here, $\Delta x = 0.00125$ and $\Delta y = 0.001$.*

distribution, $T(x, y)$.[5.11] Notice how the heat flow "spreads out" in the x direction as one moves

[5.11] We do not show the plot of the number of terms necessary for convergence in this case. It looks roughly like its counterpart for the Dirichlet problem in Fig. 5.5 on pp. 45 in that numbers are relatively low in much of the domain, but increase dramatically near the non–homogeneous boundary.

from the top boundary at $y = 0.05$ into the interior. The plot also correctly depicts the adiabatic boundary conditions, $\partial T/\partial x = 0$, as we just specified.

Like the Dirichlet problem in §5.1, the mixed problem, as we have specified it here, does not depend upon the thermal conductivity, k. Conversely, the heat transfer depends upon k via Fourier's Law and, though the effect is somewhat subtle on the surface plot in Fig. 5.9, it should be noted that there is a continuous gradient in the y direction, which might be best discerned by the arcing shape of the edge at $x = 0$. Given all these observations, it seems that a good qualitative summary of this configuration is that heat diffuses outward from the somewhat concentrated heat energy sources at the top boundary, but the adiabatic side boundaries limit the primary conduction to the y direction.

Of course, it is possible to gain even more insight into this problem by developing its mathematical description a little further. For example, it is a fairly straightforward formality at this point to derive $\partial T/\partial y$ from Eq. (5.8), and therefore to write the heat flux in the y direction directly from Fourier's Law as

$$q''_y \;=\; q''_y(x,y) \;=\; -\,k\,\frac{\partial T(x,y)}{\partial y} \;=\; -\,k\left[C_0 \;+\; \sum_{n=1}^{\infty} C_n\,\zeta_n\,\cos(\zeta_n x)\,\cosh(\zeta_n y)\right] \tag{5.12}$$

where again $\zeta_n = n\pi/L$. One could easily plot values as a surface, the matter being clear from Eq. (5.12) that $q''_y \propto k$ and that the real details of the distribution are still encoded by T, which is independent of k for this particular case.[5.12] However, we are chasing a slightly more interesting finding with this line of thought.

Suppose we now think about the domain in Fig. 5.7 (pp. 51) in a control volume sense to examine energy conservation in the finite, macroscopic sense. We can observe that, because the vertical sides are perfectly insulated, the rate at which heat enters the volume from the top boundary at $y = H$ must be equal to the rate at which it leaves the other side at $y = 0$, a situation quite analogous, for example, to the conservation of mass for incompressible flow between two locations in a pipe (Munson et al., 2006). We can pick a horizontal line at some given y, integrating Eq. (5.12) for $x : 0 \to L$ to find the total rate at which heat crosses that line (Fig. 5.10).[5.13] The values should

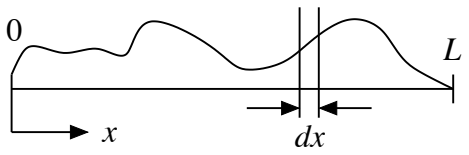


FIGURE 5.10. *Integrating heat flux in Eq.* (5.12) *over an arbitrary line in the domain.*

then be equal when substituting $y = 0$ and $y = H$ in order than energy is conserved for the control

[5.12] In terms of a surface plot like Fig. 5.9, one could think of this observation in the sense that the entire surface would simply "rise or fall" as a whole, according to various values of k, without changing its shape. Conversely, were the temperature problem itself to be a function of k, for example by including a boundary condition of the third kind, then the shape itself would change, too.

[5.13] We are often used to thinking about heat transfer simply as heat flux multiplied by cross–sectional area, quite like Eq. (2.3) on pp. 8, but this model is only valid if the flux is constant within that area. Such is clearly not the case here. Instead, we must integrate. If we reason that $q''_y = dq_y/dA$, then we can determine q_y by integrating dq_y in the form of $q''_y\,dA$.

volume, as a whole.[5.14] Carrying out this integration, we find

$$\begin{aligned} q_y \;=\; \int_0^L dq_y \;&=\; -\underbrace{1\cdot}_{\text{unit depth}} \int_0^L k\left[C_0 \;+\; \sum_{n=1}^{\infty} C_n\,\zeta_n\,\cos(\zeta_n x)\,\cosh(\zeta_n y)\right] dx \\ &=\; -\,k\left[C_0\,x \;+\; \sum_{n=1}^{\infty}\frac{C_n\,\zeta_n}{\zeta_n}\,\sin(\zeta_n x)\,\cosh(\zeta_n y)\right]\Bigg|_0^L \\ &=\; -\,k\left[C_0\left(L-0\right) \;+\; \sum_{n=1}^{\infty} C_n\left(\cancelto{0}{\sin\left[\frac{n\,\pi}{L}\,L\right]} - \sin 0\right)\cosh(\zeta_n y)\right] \\ q_y \;&=\; -\,k\,C_0\,L \end{aligned}$$

in units of J/s, or Watts. The entire series vanishes, leaving the rather surprising result shown here. While the temperature distribution is an infinite series, Eq. (5.8), the heat transfer is evidently this simple expression. Note that the negative sign indicates heat flow in the "$-\,y$" direction.

Several corollaries are immediately evident from this result. First, conservation of energy is clearly satisfied for the control volume, as a whole. The heat transfer across any horizontal line, including the bottom and top boundaries, $y = 0$ and $y = H$, respectively, is $|k\,C_0\,L|$ and there is no heat transfer at the vertical boundaries because they are adiabatic. Obviously, the net rate of heat entering the control volume is equal to the net rate leaving. Secondly, the heat transfer is governed solely by the zero–mode coefficient, C_0, with the other mode coefficients, $\{C_1, C_2, C_3, \dots\}$, being relevant only to the temperature distribution. The practical implication is that one only needs to derive C_0 from Eq. (5.10) if interested in heat transfer for a given problem, skipping evaluation of the C_n from Eq. (5.9), which are generally more complicated.

5.5. Superposition for More Complex Non–Homogeneous Problems

In the above sections in this chapter, we have examined problems where only one of the boundaries is non–homogeneous. In each of those cases, we found the separation of variables method to be applicable, and we ultimately obtained a solution as a series of eigen–modes by the principle of superposition. In this sense, these cases all belong to the same class of problems. If we liked, we could further round–out this discussion by looking at problems where *both* of the boundaries in one of the coordinates, say x, were non–homogeneous. The separation of variables method is still likewise applicable and we would follow the same procedures that we have already described. If, for example, we were to redefine the Dirichlet problem in §5.1 so that the boundaries at $y = 0$ and $y = H$ were both non–homogeneous, we would find that the expansion we derived in §E.1 on pp. 151 would generalize to a series of the form (Zill and Cullen, 2000)

$$T(x,y) \;=\; \sum_{n=1}^{\infty}\Big[C_{n,1}\,\cosh(\zeta_n y) \;+\; C_{n,2}\,\sinh(\zeta_n y)\Big]\sin(\zeta_n x)\,,$$

[5.14] Here, the mathematical context in Fig. 5.10 is integration along a line. However, the physical context is heat conduction in units of J/s, i.e. heat flux (power per unit area) times area. Since this is only a 2–D problem, there is no actual variation in the third dimension, i.e. "into the paper", so we could think of this either as heat transfer rate per unit depth, or in the sense of multiplying the result by a unit depth. The results are the same, though we show the latter procedure here.

where we would then have to solve for two countably infinite sets of eigen–values, $C_{n,1}$ and $C_{n,2}$. Such problems are not fundamentally different from those we have already discussed, so we will not examine them further.

However, what about problems that have non–homogeneous conditions on 2 *different* coordinate boundaries, say at $x = L$ and $y = H$? Here, the basic separation of variables method falters. The explanation is readily found by reviewing any of the relevant problems we have already solved, say again the finite Dirichlet case in §E.1 on pp. 151. In particular, we first obtain the set of eigen–functions by solving the ODE for whichever coordinate has the homogeneous boundary conditions. After solving the ODE for the second coordinate, we use the eigen–functions as a basis for an expansion to determine the set of mode coefficients that will simultaneously cause the non–homogeneous boundary for the second coordinate to be satisfied. Clearly, this basic procedure will not work if *both* coordinates have at least one non–homogeneous boundary condition.

Although the basic SOV method does not work here, we can still further exploit the extremely important property of superposition for linear equations by dividing such problems into a series of simpler ones. The overall solution is then the sum of all the individual solutions (Fig. 5.11). This

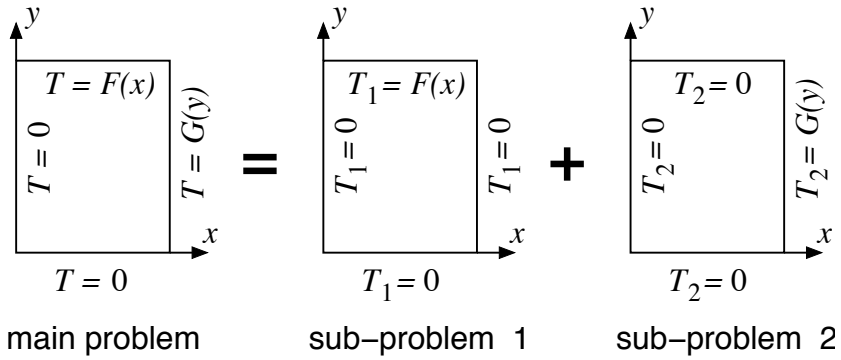


FIGURE 5.11. *The Dirichlet problem having non–homogeneous boundaries at both the $x = L$ and $y = H$ boundaries can be divided into 2 separate problems, each having just 1 of the non–homogeneous conditions.*

strategy obviously uses superposition in a much more sweeping way. The proof of this claim is fairly straightforward, being something along the lines of the difference of solutions we showed in §5.3. The two sub–problems shown in Fig. 5.11 are

$$\nabla^2 T_1 = 0 \quad : \quad T_1 = F(x) \quad \text{at} \quad y = H \quad \text{and} \quad T_1 = 0 \quad \text{elsewhere}$$

$$\nabla^2 T_2 = 0 \quad : \quad T_2 = G(y) \quad \text{at} \quad x = L \quad \text{and} \quad T_2 = 0 \quad \text{elsewhere.}$$

If it is true that $T = T_1 + T_2$, then we should expect this sum to describe the left–hand side of the diagram in Fig. 5.11. For Laplace's equation, we have

$$\nabla^2 T_1 \, + \, \nabla^2 T_2 \, = \, \nabla^2 \Big(T_1 \, + \, T_2 \Big) \, = \, \nabla^2 T \, = \, 0 \, , \qquad \checkmark$$

and for the left, top, right, and bottom boundaries, respectively, we have

$$\begin{aligned} T_1(0,y) + T_2(0,y) &= 0 + 0 = 0 = T(0,y) \qquad \checkmark \\ T_1(x,H) + T_2(x,H) &= F(x) + 0 = F(x) = T(x,H) \qquad \checkmark \\ T_1(L,y) + T_2(L,y) &= 0 + G(y) = G(y) = T(L,y) \qquad \checkmark \\ T_1(x,0) + T_2(x,0) &= 0 + 0 = 0 = T(x,0)\,, \qquad \checkmark \end{aligned}$$

which does indeed describe the main problem in Fig. 5.11. A moment of consideration should convince one that this principle holds in 3 dimensions, as well. Superposition thus allows us to treat a vast number of problems by combining various combinations of simpler ones. It is one of the most powerful and useful concepts we can exploit for linear differential equations!

There are several additional points to mention. First, this concept does *not* hold in general for non–linear problems, i.e. those where either the PDE, the boundary conditions, or both are non–linear.[5.15] When using this method, one should be careful not to inadvertently create *too many* sub–problems, less the solution process become unnecessarily tedious. For example, were the main problem in Fig. 5.11 to have non–homogeneous conditions at all 4 boundaries, we could still limit the division to 2 sub–problems because SOV is still applicable when opposite boundaries are non–homogeneous. Finally, although the combination of SOV and superposition allows us, in principle, to manage these more complicated problems, the basic SOV methods seems to be reaching its limits. There are quite a number of other physical effects, including unsteadiness, heat generation, and non–linearities, that will prompt us to examine more sophisticated mathematical methods in the next chapter.

5.6. Three–Dimensional Steady Conduction

We alluded to applying the separation of variables method in 3–D in the previous section. Although the overall process is largely the same as for 2–D problems, there are some minor procedural differences that should be explained. We will sketch the process for the basic Dirichlet problem having all homogeneous boundary conditions, except at $z = D$, where $T(x,y,D) = F(x,y)$, as shown in Fig. 5.12. The reader may wish for comparison purposes to refer periodically to §B.1 on pp. 134 and §E.1 on pp. 151, where we originally introduced the SOV method in detail for problems in (x,t) and (x,y), respectively. Here, $T = T(x,y,z)$ is the solution we are seeking for the 3-D Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{5.13}$$

and we do so by assuming a separable form for the temperature of $T(x,y,z) = \Psi(x)\,\Gamma(y)\,\Omega(z)$. The main difference now becomes obvious as we attempt to carry–out the actual separation process.

[5.15] Consider, for example, a non–linear operator of the form $\varphi\,\nabla^2\varphi$, where $\varphi = \varphi(x,y)$ is some scalar function. If we presume φ is the sum of the solutions of 2 simpler problems, φ_1 and φ_2, substitution shows the corresponding operator

$$\Big(\varphi_1 + \varphi_2\Big)\,\nabla^2\Big(\varphi_1 + \varphi_2\Big) = \varphi_1\,\nabla^2\varphi_1 + \varphi_1\,\nabla^2\varphi_2 + \varphi_2\,\nabla^2\varphi_1 + \varphi_2\,\nabla^2\varphi_2 \neq \varphi_1\,\nabla^2\varphi_1 + \varphi_2\,\nabla^2\varphi_2\,.$$

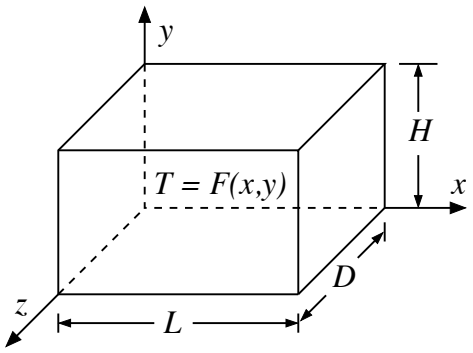


FIGURE 5.12. *Representative three–dimensional steady Dirichlet conduction problem in a parallelpiped.*

Substituting the assumed form of the solution gives $\Psi''\Gamma\Omega+\Psi\Gamma''\Omega+\Psi\Gamma\Omega''=0$, so that dividing by $\Psi\,\Gamma\,\Omega$ yields

$$\frac{\Psi''(x)}{\Psi(x)} \,+\, \frac{\Gamma''(y)}{\Gamma(y)} \,+\, \frac{\Omega''(z)}{\Omega(z)} \,=\, 0\,.$$

In 2–D, whether in (x,t) or (x,y), we were able to separate the two coordinates in a single step, as shown e.g. in §E.1 on pp. 151. Here, we must take 2 steps, since there are 3 independent coordinates, which will lead to 3 ODEs. Let us process Ψ first. Taking this term to the opposite side and again invoking the constancy argument (as explained in §B.1), we find

$$\frac{\Psi''(x)}{\Psi(x)} \,=\, -\,\left[\frac{\Gamma''(y)}{\Gamma(y)} \,+\, \frac{\Omega''(z)}{\Omega(z)}\right] \,=\, -\,\zeta_x^2\,.$$

The problem in Ψ is identical to the one shown in §B.1: the corresponding governing equation being Eq. (B.6) on pp. 135, and boundary conditions being Eqs. (B.7) and (B.8). There, we found the solution to be

$$\Psi_m(x) \,=\, C_{x,m}\,\sin(\zeta_{x,m}x) \qquad\qquad \zeta_{x,m} \,=\, \frac{m\,\pi}{L}\,,$$

where $\zeta_{x,m}$ are the eigen–values *for the x direction* (note the extra x subscript) and $C_{x,m}$ are the yet–to–be–determined coefficients for the countably infinite modes $m\in\{1,2,3,\dots\}$.

We now do a second round of separation. Mindful of the modified notation for the x eigen–values, we can rearrange the above equation as

$$\frac{\Gamma''(y)}{\Gamma(y)} \,=\, -\,\frac{\Omega''(z)}{\Omega(z)} \,+\, \zeta_{x,m}^2 \,=\, -\,\zeta_y^2\,,$$

again invoking the same constancy argument. Except for the different geometric dimension, the problem in Γ is identical to the one we just solved in Ψ. That is, the ODE is the same and the boundary conditions are both homogeneous. Therefore

$$\Gamma_n(y) \,=\, C_{y,n}\,\sin(\zeta_{y,n}y) \qquad\qquad \zeta_{y,n} \,=\, \frac{n\,\pi}{H}\,,$$

where $\zeta_{y,n}$ are the eigen–values *for the y direction* and $C_{y,n}$ are the yet–to–be–determined coefficients for the countably infinite modes $n\in\{1,2,3,\dots\}$.

This leaves the last ODE for the z coordinate as: $\Omega''/\,\Omega=\zeta_{x,m}^2+\zeta_{y,n}^2$, which can be written

$$\Omega''(z) \,-\, \Big(\zeta_{x,m}^2 \,+\, \zeta_{y,n}^2\Big)\Omega(z) \,=\, 0\,.$$

This is again a type of equation with which we have already dealt, c.f. the sub–problem in Γ for the 2–D Dirichlet problem having mixed boundary conditions in §F.1 on pp. 158. In fact, given that $\Omega(0) = 0$ and that we realize $\zeta_{x,m}^2 + \zeta_{y,n}^2$ is also just a constant, it is exactly the same problem as in §F.1. We can show by the same sort of developments that

$$\Omega_{m,n}(z) \;=\; C_{m,n}\,\sinh\left(\sqrt{\zeta_{x,m}^2 \,+\, \zeta_{y,n}^2}\;z\right).$$

As usual, we subsume the previous mode coefficients into the current one and apply superposition, whereby we can write the general solution as the *double* summation

$$T(x,y,z) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}\,\sin(\zeta_{x,m}x)\,\sin(\zeta_{y,n}y)\,\sinh\left(\sqrt{\zeta_{x,m}^2 \,+\, \zeta_{y,n}^2}\;z\right). \tag{5.14}$$

Finally, the mode coefficients are determined in the routine way, i.e. by exploiting orthogonality of the eigen–functions, so that the non–homogeneous boundary condition is satisfied. Except, here we have double integrals because there are 2 infinite sets of eigen–functions. As usual, we evaluate the solution at the boundary

$$T(x,y,D) \;=\; F(x,y) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{m,n}\,\sin(\zeta_{x,m}x)\,\sin(\zeta_{y,n}y)\,\sinh\left(\sqrt{\zeta_{x,m}^2 \,+\, \zeta_{y,n}^2}\;D\right),$$

subsequently operating with *both* eigen–functions to obtain

$$\int_0^H\int_0^L F(x,y)\,\sin(\zeta_{x,m}x)\,\sin(\zeta_{y,n}y)\,dx\,dy \;=$$

$$\int_0^H\int_0^L \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{i,j}\,\sin(\zeta_{x,i}x)\,\sin(\zeta_{y,j}y)\,\sinh\left(\sqrt{\zeta_{x,i}^2 \,+\, \zeta_{y,j}^2}\;D\right)\sin(\zeta_{x,m}x)\,\sin(\zeta_{y,n}y)\,dx\,dy\,.$$

Importantly, the double integral on the right hand side can be written in product form

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} C_{i,j}\,\sinh\left(\sqrt{\zeta_{x,i}^2 \,+\, \zeta_{y,j}^2}\;D\right)\int_0^H \sin(\zeta_{y,n}y)\,\sin(\zeta_{y,j}y)\,dy\int_0^L \sin(\zeta_{x,m}x)\,\sin(\zeta_{x,i}x)\,dx\,, \tag{5.15}$$

whereby the orthogonality property[5.16] indicates that this product vanishes unless both $i = m$ *and* $j = n$. The normalization integrals have the identical form to that shown in Eq. (B.16) on pp. 138, so that some additional development would show

$$\int_0^H\int_0^L F(x,y)\,\sin(\zeta_{x,m}x)\,\sin(\zeta_{y,n}y)\,dx\,dy \;=\; C_{m,n}\cdot\frac{H}{2}\cdot\frac{L}{2}\,\sinh\left(\sqrt{\zeta_{x,m}^2 \,+\, \zeta_{y,n}^2}\;D\right),$$

from which $C_{m,n}$ is readily solved. This completes the general solution for the Dirichlet problem. As usual, specific configurations would necessitate specification of the boundary condition, $F(x,y)$, subsequent evaluation of $C_{m,n}$, followed finally by algorithmic implementation to carry–out the actual summation.

[5.16] See e.g. discussion in §B.3 on pp. 137, especially footnote B.5 on pp. 137.

5.7. Additional Remarks

We have examined several multi–dimensional, steady heat conduction problems and, in the process, have solidified the application of the separation of variables method. It is clear that there are numerous additional possible configurations, based upon the combinations of dimensionality and boundary conditions. Their solution procedures are largely methodological variations of the same themes shown here and, of course, one starts to notice the pattern essentially of assembling "components" of various ODEs to construct a general solution for a given configuration. Many of these cases would further require framing the general solution as a superposition of 2 or even 3 sub–problems of the kind shown in Fig. 5.11. The possibilities become enormously larger when various different coordinate systems are also considered (Hildebrand, 1976). The separation of variables method starts to show its practical limits for many of these problems, and indeed, there are numerous others that are not separable at all. For those, we shall have to examine more sophisticated approaches.

5.8. Exercises

5.1 For the 2–D steady conduction configuration in the domain $0 \leq x \leq L$ and $0 \leq y \leq H$, we examined the so–called *Neumann problem*, where the boundaries at $x = 0$ and $x = L$ are adiabatic, $T|_{y=0} = 0$ and $T|_{y=H} = F(x)$. The solution is

$$T(x,y) \;=\; C_0\, y \;+\; \sum_{n=1}^{\infty} C_n\, \cos(\zeta_n x)\, \sinh(\zeta_n y) \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L}$$

$$C_0 \;=\; \frac{1}{H\,L}\int_0^L F(x)\,dx \qquad\qquad C_n \;=\; \frac{2}{L\,\sinh(\zeta_n H)}\int_0^L F(x)\,\cos(\zeta_n x)\,dx\;.$$

Consider now the specific instantiation of this problem: $L = H = 1$ and $F(x) = x$. Derive the exact solution for this problem, including determination of the full set of mode coefficients, eigen–values, etc.

5.2 Referring to exercise 5.1: Render the solution either as a surface plot or a contour plot using your favorite method, e.g. writing a program in any scripted or compiled language, using MatLab, etc. Give enough documentation so that a colleague could duplicate the work, e.g. a program listing with comments. Make sure to properly handle numerical problems, e.g. pre–mature convergence due to the cyclical function and numerical overflow.

5.3 The separation of variables (SOV) method routinely leads to solutions in the form of infinite series. Consider, for example, the following solution for $T(x,y)$ in a 2–D domain of $0 \le x \le 1$ and $0 \le y < \infty$ for steady heat conduction, as obtained by SOV

$$\begin{aligned} T(x,y) &= \frac{4\,T_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{e^{-n\pi y}\,\sin(n\pi x)}{n} \\ &= \frac{4\,T_0}{\pi} \left(\frac{e^{-\pi y}\,\sin(\pi x)}{1} + \frac{e^{-3\pi y}\,\sin(3\pi x)}{3} + \frac{e^{-5\pi y}\,\sin(5\pi x)}{5} + \cdots \right) . \end{aligned}$$

We can see by inspection that this series will converge very quickly if y is large because the exponential term decays rapidly. Conversely, for very small y it can converge quite slowly. It is often possible to sum a series analytically, reducing it to a single exact term, and such would be extremely useful for slow–convergence cases. Demonstrate that this particular solution can be analytically summed as

$$T(x,y) \;=\; \frac{2\,T_0}{\pi}\,\arctan\left[\frac{\sin(\pi x)}{\sinh(\pi y)} \right] .$$

Suggested approach: This problem can be solved by converting to the complex domain. Per Euler's formula, $e^{i\beta} = \cos\beta + i\sin\beta$, where $i = \sqrt{-1}$, note that $\sin(n\pi x) = \Im\left(e^{in\pi x}\right)$. Use this conversion to re–write the series as an expansion of the complex variable z in the form $T = \sum \Im(z^n)/n$ and compare this to the expansion

$$\ln\left(\frac{1+z}{1-z} \right) \;=\; 2\left(\frac{z^1}{1} + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \cdots \right) ,$$

the trick here being to recognize the proper specification of z. Convert the log–based result back to the real domain using the log formula of the polar representation on the complex plane

$$\ln z' \;=\; \ln\left(r\,e^{i\theta} \right) \;=\; \ln r \;+\; i\,\theta\,,$$

i.e. where z' is another complex variable and the physical solution is the *imaginary part* of that result. The hyperbolic identity $\sinh\beta = (e^{\beta} - e^{-\beta})/2$ will be helpful for the very last step.

5.4 The 2-D steady Dirichlet conduction problem in $0 \le x \le L$ and $0 \le y < \infty$ having a constant boundary temperature of T_0 at $y = 0$ and homogeneous boundary conditions elsewhere is

$$T(x,y) \;=\; \frac{4\,T_0}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\, \sin(\zeta_n x)\, e^{-\zeta_n y}\,,$$

where $\zeta_n = n\,\pi/L$ are the explicit eigenvalues. If the material's thermal conductivity is k, determine the total heat conduction along the boundary $x = L$, simplifying to the greatest degree possible. You may assume the material has a depth b "into the paper" in which there is no variation in T. Also, determine the total heat conduction along $y = 0$, showing that it is precisely double the amount for $x = L$, as we would expect for a semi–infinite, symmetric configuration. *Hint:* In both results, you should expect a series expansion in n^{-1}. This series is not trivial to evaluate, so you may leave it as is.

CHAPTER 6

Eigen–function Integral Transforms for Diffusion Equations

In §5.5 on pp. 55 we discussed the limitations of the basic separation of variables (SOV) method for conduction problems. In many cases, both the solution process and the solution itself will start to become impractical. Here, we discuss a more general method based on using the eigen–function itself as part of an integral transform that will allow systematic solving of non–homogeneous problems. The *modus operandi* of this technique is to remove one or more of the second–derivative (diffusion) terms from the governing equation so as to reduce the PDE to an ODE.[6.1] Consider again the example of the unsteady 1–D Dirichlet problem, first discussed in §4.3 on pp. 30. The second–derivative term can be removed, rendering a first–order ODE in time, which is readily solved for the transformed temperature. The transformed equation is almost always much more straightforward to solve. The result is then inverted, using an inverse transform, to recover the physical solution.

Integral transforms using the spatial eigen–function are particularly powerful for heat conduction problems. While it is true that there are many general methods for solving linear systems, they are often accompanied by procedural difficulties. For instance, the Laplace transform can be used to remove the partial derivative with respect to time. However, the inversion step is generally difficult, especially if the transformed result cannot readily be re–arranged as a series of basic, easily–invertible forms. The methodology we discuss here has no such problem. Inversion is straightforward because the the inversion rule is available at the outset of the problem.

The integral transform technique using the eigen–function as the transform kernel is actually fundamentally rooted in the separation of variables technique. The transform and its inverse are derived by considering the general representation of the solution as an eigen–function expansion, i.e. a Fourier series. Indeed, the method makes direct use of eigen–functions, eigen–values, and normalization integrals developed from SOV, and, in some sense, it could even be thought of as an extension of the SOV method.

Application for heat conduction problems seems to date from the work of Doetsch (1936). It has been heavily developed since the 1960s (e.g. Ölçer, 1964, 1965, 1969; Mikhailov, 1968) and is well–represented in modern texts (e.g. Özişik, 1980; Cotta, 1993). Here, we will start with a gentle introduction by way of some examples, then develop the more formal theory. The method will be of particular use in solving the generalized Couette problem in §8.3 starting on pp. 100.

6.1. Revisiting the Dirichlet Problem: Developing an Integral Transform Pair

We first examined the 1–D unsteady conduction problem having Dirichlet boundary conditions in §4.3 on pp. 30. Recall, in particular, the process of expanding the initial condition in §B.3

[6.1] Recall that this is the same *modus* as the similarity transform we examined for the Rayleigh problem in §4.6 on pp. 36.

on pp. 137 in order to determine the mode coefficients, the form being given by Eq. (B.12) and repeated here for convenience

$$T(x,0) \;=\; F(x) \;=\; \sum_{n=1}^{\infty} C_n \,\sin(\zeta_n x)\,.$$

Imagine now instead of expanding merely the initial condition $T(x,0)$ in terms of the eigen–functions, that we expand the entire solution in this manner[6.2]

$$T(x,t) \;=\; \sum_{n=1}^{\infty} C_n(t) \,\sin(\zeta_n x)\,. \tag{6.1}$$

We could then proceed in the usual fashion of exploiting the orthogonality property of the eigen–function, here $\sin(\zeta_n x)$, to determine $C_n(t)$.[6.3] That is, we multiply by the eigen–function and integrate over the domain

$$\begin{aligned}\int_0^L T(x,t)\,\sin(\zeta_n x)\,dx \;&=\; \sum_{m=1}^{\infty} C_m(t) \int_0^L \sin(\zeta_m x)\,\sin(\zeta_n x)\,dx \\ &=\; C_n(t)\,N(\zeta_n)\,,\end{aligned}$$

where $N(\zeta_n)$ is the usual normalization integral.[6.4] This result implies that the solution in Eq. (6.1) can also be written in the form

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \left[\frac{1}{N(\zeta_n)} \int_0^L T(x,t)\,\sin(\zeta_n x)\,dx\right] \sin\left(\zeta_n x\right)\,. \tag{6.2}$$

Admittedly, Eq. (6.2) has a certain "circular" or self–referential quality in that it states the solution in terms of itself. This might initially appear to be of extraordinarily limited usefulness. However, suppose we define a transformed temperature, $\overline{T}$, as the integral in this expression. We could then write $\overline{T}$ in terms of T, and *vice versa* as

$$\overline{T}(\zeta_n,t) \;=\; \int_0^L T(x,t)\,\sin(\zeta_n x)\,dx \tag{6.3}$$

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \frac{\overline{T}(\zeta_n,t)}{N(\zeta_n)}\,\sin\left(\zeta_n x\right)\,, \tag{6.4}$$

the former being the *integral transform* and the latter being the *inversion*.

Transform–inversion pairs, such as Eqs. (6.3) and (6.4), are the nucleus of this method. In essence, a problem is first transformed to the $\overline{T}$ domain, where, as we will see, certain extremely important simplifications become possible. The solution of $\overline{T}$ itself is then obtained. Finally,

[6.2] Of course, we could immediately compare this expression to Eq. (B.11) on pp. 136 and Eq. (B.15) on pp. 138, which together imply

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \left[\left(\frac{2}{L}\int_0^L F(x)\,\sin(\zeta_n x)\,dx\right)\,e^{-\alpha\zeta_n^2 t}\right] \sin\left(\zeta_n x\right)\,,$$

after which we would quickly conclude that $C_n(t)$ above is equal to the term in square brackets here. While true, this observation does not contribute to the development of the integral transform pair that we are attempting to illustrate here.

[6.3] For this particular example, the 1–D unsteady Dirichlet problem, this process is shown in §B.3 on pp. 137.

[6.4] The norm in this particular case is $L/2$, as shown by Eq. (B.16) on pp. 138.

inversion is used to recover the physical solution for T. As we alluded to above, this last step is straightforward because the inversion law is already known.

Let us return to the Dirichlet example to finish illustrating this procedure. We note the following critical aspect of the method: *We already know the form of the eigen–function and eigen–values at the outset of the problem.*[6.5] This insight is the intimate connection back to the SOV method. Recall that the eigen–function and eigen–values are a product of the auxiliary eigen–problem that arose in SOV by the very invocation of the separation assumption. For the Dirichlet problem, this was specified by Eqs. (B.6) through (B.8) on pp. 135

$$\Psi_n''(x) \,+\, \zeta_n^2 \Psi_n(x) \,=\, 0 \tag{6.5}$$

$$\Psi_n(0) \,=\, 0 \qquad\qquad \Psi_n(L) \,=\, 0\,, \tag{6.6}$$

which are repeated here for convenience and whose solution is $\Psi_n = C_n \sin(\zeta_n x)$, where the eigen–values are explicitly defined as $\zeta_n = n\pi/L$.[6.6] The appropriate corresponding auxiliary problem will always furnish all eigen–related information, i.e. eigen–values, eigen–function, and normalization integral, and the auxiliary problem is always derived from a homogenized version of the physical problem. We will discuss these aspects further below.

6.2. Solving the 1–D Unsteady Dirichlet Problem using the Integral Transform

With the integral transform pairs in Eqs. (6.3) and (6.4) now in hand, let us actually solve the 1–D unsteady Dirichlet problem for $T = T(x,t)$ from its beginning once again, except now using the integral transform method. We re–state the problem here for convenience as[6.7]

$$\frac{\partial T}{\partial t} \,=\, \alpha\, \frac{\partial^2 T}{\partial x^2} \qquad\qquad T(0,t) \,=\, T(L,t) \,=\, 0 \qquad\qquad T(x,0) \,=\, F(x)\,.$$

The first step is eigen–function multiplication and integration of the PDE, as mentioned above

$$\int_0^L \frac{\partial T}{\partial t}\, \sin(\zeta_n x)\, dx \,=\, \alpha \int_0^L \frac{\partial^2 T}{\partial x^2}\, \sin(\zeta_n x)\, dx\,. \tag{6.7}$$

Note here that we have already removed the constant diffusivity, α, to the outside of the right–hand integral. With respect to the time–derivative term on the left, because integration is only a function of x with constant limits and differentiation is only with respect to t, we can reverse their order, ultimately observing

$$\int_0^L \frac{\partial T}{\partial t}\, \sin(\zeta_n x)\, dx \,=\, \frac{\partial}{\partial t} \int_0^L T\, \sin(\zeta_n x)\, dx \,=\, \frac{d\overline{T}}{dt}\,, \tag{6.8}$$

the last step being a consequence of employing the integral transform in Eq. (6.3). Note how we have also changed the partial derivative sign to a regular derivative sign because $\overline{T} = \overline{T}(\zeta_n, t)$ in Eq. (6.3), i.e. it is *not* any longer a function of x, but rather only of t and the (constant) eigen–values.

[6.5] This is perhaps obvious, as we are already showing the eigen–function in the transform itself in Eq. (6.3). However, the point is crucial and worth emphasizing because it holds in the general case.

[6.6] This solution is the basis of the particular transform–inverse pair in Eqs. (6.3) and (6.4), which is the appropriate pair for the 1–D Dirichlet problem.

[6.7] See Eq. (4.3) on pp. 30 and appendix B on pp. 134.

For the space–derivative term on the right, we will do integration by parts (IBP) twice,[6.8] whereby

$$\begin{aligned}
\int_0^L \frac{\partial^2 T}{\partial x^2}\,\sin(\zeta_n x)\,dx &= \left(\sin(\zeta_n x)\,\frac{\partial T}{\partial x}\right)\Big|_0^L - \int_0^L \frac{\partial T}{\partial x}\,\zeta_n\,\cos(\zeta_n x)\,dx \qquad \boxed{\text{1st IBP}} \\
\boxed{\text{2nd IBP}} \qquad &= \left(\sin(\zeta_n x)\,\frac{\partial T}{\partial x}\right)\Big|_0^L - \zeta_n\left[\left(\cos(\zeta_n x)\,T\right)\Big|_0^L - \int_0^L T\,(-\zeta_n)\sin(\zeta_n x)\,dx\right] \\
&= \left(\sin(\zeta_n x)\,\frac{\partial T}{\partial x} - \zeta_n\cos(\zeta_n x)\,T\right)\Big|_0^L - \zeta_n^2\int_0^L T\,\sin(\zeta_n x)\,dx\,. \\
&= \left(\cancelto{0}{\sin(n\pi)}\,\frac{\partial T}{\partial x}\Big|_{x=L} - \zeta_n\cancelto{0}{\cos(n\pi)\,T}\Big|_{x=L}\right) \qquad \boxed{\text{substitute } \zeta_n = n\pi/L} \\
&\quad - \left(\cancelto{0}{\sin(0)\cdot}\,\frac{\partial T}{\partial x}\Big|_{x=0} - \zeta_n\cancelto{0}{\cos(0)\cdot T}\Big|_{x=0}\right) - \zeta_n^2\,\overline{T}\,.
\end{aligned} \tag{6.9}$$

Let us discuss the actual results of this operation, along with some important observations that we will later generalize. First, it clear that the integration by parts steps bring the boundary conditions on x explicitly into the transform and that they are *evaluated* in the transform domain, i.e. T at both $x = 0$ and $x = L$. Although they vanish here because the particular version of this Dirichlet problem has homogeneous boundary conditions, we should suspect that, in general, this property will allow us to treat non–homogeneous boundary conditions in the transform domain. That is, their effects will be accounted for in the ODE (easier) part of the problem. This will prove to be a tremendous simplification of the problem!

The second aspect is that the eigen–functions are also evaluated at the boundary, and these also vanish in this particular case by virtue of Eqs. (6.5) and (6.6), i.e. the sine function is zero on both boundaries. This may not be the case in general and we will find this term becomes part of a more general statement at the boundary if the boundary conditions are of the most general (Robbins) type, as illustrated in the succeeding example.

Lastly, the integral transform definition itself is again applied to the very last term, yielding $\overline{T}$ in Eq. (6.9), but without any derivative. In other words, the operation has the overall effect of removing the second–derivative in transform space, rendering a simple algebraic term in its place!

[6.8] The basic form of integration by parts (Courant and John, 1965) is

$$\int u\,dv \;=\; (u\,v)\Big| \;-\; \int v\,du\,.$$

In the first and second rounds, respectively, we let

$$u \;=\; \sin(\zeta_n x) \quad \text{and} \quad dv \;=\; \frac{\partial^2 T}{\partial x^2}\,dx \qquad \text{so that} \qquad du \;=\; \zeta_n\,\cos(\zeta_n x)\,dx \quad \text{and} \quad v \;=\; \frac{\partial T}{\partial x}$$

$$u \;=\; \cos(\zeta_n x) \quad \text{and} \quad dv \;=\; \frac{\partial T}{\partial x}\,dx \qquad \text{so that} \qquad du \;=\; -\,\zeta_n\,\sin(\zeta_n x)\,dx \quad \text{and} \quad v \;=\; T\,.$$

We can re–assemble the governing equation in transform space by substituting Eqs. (6.8) and (6.9) into Eq. (6.7), whereby

$$\frac{d\overline{T}}{dt} \;=\; -\,\alpha\,\zeta_n^2\,\overline{T}\,,$$

which is now an ODE. Its solution is[6.9]

$$\overline{T} \;=\; C_t\,e^{-\alpha\zeta_n^2 t}\,,$$

where C_t is yet to be determined.

We should pause a moment here to ponder how C_t is assigned. This equation and its solution are in the *transform domain*, so we must also evaluate C_t from the initial conditions of the physical problem, but also in the *transform domain*. This task is simple enough, since we already have the transform law in Eq. (6.3), which for the initial condition becomes

$$\overline{T}(\zeta_n,0) \;=\; \overline{F}(\zeta_n) \;=\; \int_0^L F(x)\,\sin(\zeta_n x)\,dx\,.$$

Substitution in the above shows

$$\overline{T}(\zeta_n,0) \;=\; \overline{F}(\zeta_n) \;=\; C_t\,e^{-\alpha\zeta_n^2\cdot 0}\,,$$

from which we conclude that $C_t = \overline{F}$ and that the solution in the transform domain is

$$\overline{T}(\zeta_n,t) \;=\; \overline{F}(\zeta_n)\,e^{-\alpha\zeta_n^2 t} \;=\; e^{-\alpha\zeta_n^2 t}\int_0^L F(x)\,\sin(\zeta_n x)\,dx\,. \tag{6.10}$$

Inversion to obtain the actual physical solution is straightforward. Applying Eq. (6.4), we find

$$\begin{aligned} T(x,t) \;&=\; \sum_{n=1}^{\infty}\frac{1}{N(\zeta_n)}\,\overline{F}(\zeta_n)\,e^{-\alpha\zeta_n^2 t}\,\sin(\zeta_n x) \\ &=\; \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L F(x)\,\sin(\zeta_n x)\,dx\right)\sin(\zeta_n x)\;e^{-\alpha\zeta_n^2 t}\,. \end{aligned}$$

where $\zeta_n = n\pi/L$, from above. Note that we have used the fact that the norm is $N = L/2$, as already noted in footnote 6.4 on pp. 64 and Eq. (B.16) on pp. 138.

6.3. Basis of Integral Transform Method in Separation of Variables

We have obviously obtained the exact same solution with the integral transform method as we did when using SOV, c.f. footnote 6.2 on pp. 64. It is worth a *post hoc* examination of exactly how this solution was developed and its basis in and relationship to the SOV method.

The latter was first demonstrated in detail in §B.1 on pp. 134 for the 1–D unsteady Dirichlet problem. Essentially, the general solution for $T(x,t)$ was separated into 2 uni–variate functions $\Psi(x)$ and $\Gamma(t)$, the first of which proved to be an eigen–value problem, yielding a set of eigen–values and corresponding eigen–functions. We then expanded the initial condition in Fourier sine

[6.9] This equation can be written as $\overline{T}' + \alpha\,\zeta_n^2\,\overline{T} = 0$. It can be solved by the method of the auxiliary equation (Ross, 1965), which in this case is $\varphi + \alpha\zeta_n^2 = 0$, so that $\varphi = -\alpha\zeta_n^2$ is the only root and the solution is its exponential. See also Eq. (B.9) on pp. 136, where we used this same observation in solving the original Dirichlet problem using separation of variables.

series, c.f. Eq. (B.12) on pp. 137, so that the initial condition would also be satisfied. The norm appeared as a necessary by–product.

The integral transform method is also based on a Fourier expansion, c.f. Eq. (6.1) on pp. 64. Stated in a simplistic way, the difference is that the "mode coefficient" contains not only the norm and boundary function integral, but also the functionality that accounts for time. This allows for the definition of the integral transform pair, Eqs. (6.3) and (6.4) in this particular example. The boundary conditions of the problem were evaluated explicitly in transform space, leading up to a readily–solvable ODE. As we shall see below, it is at this stage that many more complicated phenomena are manageable using the integral method. In a sense, this is where it will offer real mathematical leverage. Finally, the solution itself is obtained through straightforward inversion, whose rule is known at the outset because of the way the transform pair itself was developed.

6.4. 1–D Unsteady Conduction with Heat Generation and Robbins Conditions

The unsteady 1–D Dirichlet example that we just examined (repeated, actually) serves as a very good illustration of some of the basic properties and procedures of the eigen–function integral transform method. However, it does not actually demonstrate that the method really furnishes any new and useful capability beyond what SOV already does routinely. Let us take a step further toward both generality and abstraction with another example. Consider a 1–D unsteady conduction problem that adds two non–trivial complications, Robbins boundary conditions at both faces and internal heat generation (Fig. 6.1). The mathematical statement of the problem consists of a

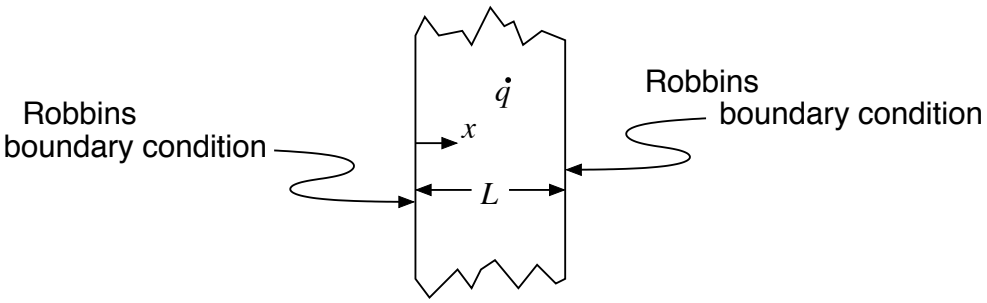


FIGURE 6.1. *One–dimensional transient problem having heat generation and boundary conditions of the third kind.*

slightly more generalized field equation, as well as convective boundary conditions on both faces[6.10]

$$\frac{1}{\alpha}\,\frac{\partial T}{\partial t} = \frac{\dot{q}}{k} + \frac{\partial^2 T}{\partial x^2} \tag{6.11}$$

$$h_0\,T_\infty = h_0\,T\Big|_{x=0} - k\,\frac{\partial T}{\partial x}\Big|_{x=0} \tag{6.12}$$

$$h_L\,T_\infty = h_L\,T\Big|_{x=L} + k\,\frac{\partial T}{\partial x}\Big|_{x=L} \tag{6.13}$$

$$T(x,0) = F(x)\,, \tag{6.14}$$

[6.10]Eq. (6.11) is a special case of Eq. (2.17) on pp. 13 for conduction only the x direction. Eqs. (6.12) and (6.13) can be derived directly from the standard Robbins form in Eq. (2.25) on pp. 15.

where $\dot{q} = \dot{q}(x)$ is a heat generation source term and h_0 and h_L are the convection coefficients at the left and right faces, respectively. For the moment, we will assume that these coefficients are constants. The problem is again to solve for $T = T(x,t)$.

We asserted above that we will always be able to characterize the eigen–related entities *a priori*, though we have not yet proved this statement. In the previous example, we had already solved the problem using SOV, so the eigen entities were available. In this example, we will instead use the eigen–function and eigen–values, $\Psi_n(x)$ and ζ_n respectively, purely in their symbolic form without yet invoking them specifically. The purpose of this abstraction is to show that the reduction of the second–derivative term to an algebraic one in Eq. (6.9) was not just a fortuitous coincidence of the *specific* eigen–function, $\sin(\zeta_n x)$, for that problem,[6.11] but is rather a basic consequence of the auxiliary Helmholtz condition in Eq. (6.5) on pp. 65.[6.12] Consequently, we will represent the integral transform pair in the slightly more general form

$$\overline{T}(\zeta_n, t) \;=\; \int_0^L T(x,t)\, \Psi_n(x)\, dx \tag{6.15}$$

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \frac{\overline{T}(\zeta_n, t)}{N(\zeta_n)}\, \Psi_n(x)\,, \tag{6.16}$$

where $\Psi_n = \Psi_n(x)$ and ζ_n are the appropriate eigen–function and eigen–values, respectively. Moreover, the homogenized boundary conditions for the auxiliary problem are obtained directly from the physical boundary conditions. They are

$$h_0\, \Psi_n\Big|_{x=0} \;-\; k\; \frac{d\Psi_n}{dx}\bigg|_{x=0} \;=\; 0 \tag{6.17}$$

$$h_L\, \Psi_n\Big|_{x=L} \;+\; k\; \frac{d\Psi_n}{dx}\bigg|_{x=L} \;=\; 0\,. \tag{6.18}$$

The time–derivative, i.e. the left had side of Eq. (6.11), transforms in the same way as for the Dirichlet example above from Eq. (6.8) on pp. 65 to $d\overline{T}/dt$. However, the right hand side is now a little more involved. We must do the proper transform for both the heat generation term and the diffusion (second–derivative) term. For the latter, we again do 2 rounds of integration by parts (IBP), once again picking the same assignments as described in footnote 6.8 on pp. 66. We find

$$\begin{aligned}
\int_0^L \frac{\partial^2 T}{\partial x^2}\, \Psi_n(x)\, dx \;&=\; \left(\Psi_n\, \frac{\partial T}{\partial x}\right)\bigg|_0^L \;-\; \int_0^L \frac{\partial T}{\partial x}\, \frac{d\Psi_n}{dx}\, dx \qquad \boxed{\text{1st IBP}} \\
\boxed{\text{2nd IBP}} \qquad &=\; \left(\Psi_n\, \frac{\partial T}{\partial x}\right)\bigg|_0^L \;-\; \left[\left(T\, \frac{d\Psi_n}{dx}\right)\bigg|_0^L \;-\; \int_0^L T\, \frac{d^2\Psi_n}{dx^2}\, dx\right] \\
&=\; \left(\Psi_n\, \frac{\partial T}{\partial x} \;-\; T\, \frac{d\Psi_n}{dx}\right)\bigg|_0^L \;+\; \int_0^L T\, \frac{d^2\Psi_n}{dx^2}\, dx\,.
\end{aligned} \tag{6.19}$$

[6.11] In Eq. (6.9), the two–step integration by parts allowed for direct application of the integral transform in the last step because the sine function is proportional to its own second–derivative, being "$-\,\zeta_n^2\, \sin(\zeta_n x)$".

[6.12] While all 1–D Cartesian problems are governed by the Helmholtz–type ODE in Eq. (6.5), the boundary conditions will vary. For the 1–D Dirichlet problem, they are given by Eq. (6.6), which leads to the particular eigen–function $\Psi_n(x) = \sin(\zeta_n x)$ and $\zeta_n = n\pi/L$.

We are now in at least a partial position to show both how non–homogeneous effects are readily accounted for in the transform domain and how the reduction of the second–derivative term holds in general and was not simply a fluke in the previous example. These aspects all stem from exploiting the auxiliary eigen–value problem. Specifically, we use the fact that $\Psi_n'' = -\,\zeta_n^2 \Psi_n$ from Eq. (6.5) on pp. 65 and the observation that the boundary conditions can be re–arranged algebraically as

$$\left.\frac{d\Psi_n}{dx}\right|_{x=0} \;=\; \left.\frac{h_0}{k}\,\Psi_n\right|_{x=0}$$

$$\left.\frac{d\Psi_n}{dx}\right|_{x=L} \;=\; -\left.\frac{h_L}{k}\,\Psi_n\right|_{x=L} .$$

We can then develop the form in Eq. (6.19) somewhat further, as follows.

$$\begin{aligned}
\int_0^L \frac{\partial^2 T}{\partial x^2}\,\Psi_n(x)\,dx \;&=\; \left(\Psi_n\,\frac{\partial T}{\partial x} \;-\; T\,\frac{d\Psi_n}{dx}\right)\Bigg|_{x=L} - \left(\Psi_n\,\frac{\partial T}{\partial x} \;-\; T\,\frac{d\Psi_n}{dx}\right)\Bigg|_{x=0} \\
&\qquad + \int_0^L T\left(-\,\zeta_n^2 \Psi_n\right) dx \\
&=\; \left(\Psi_n\,\frac{\partial T}{\partial x} \;+\; \Psi_n\,T\,\frac{h_L}{k}\right)\Bigg|_{x=L} - \left(\Psi_n\,\frac{\partial T}{\partial x} \;-\; \Psi_n\,T\,\frac{h_0}{k}\right)\Bigg|_{x=0} \\
&\qquad - \zeta_n^2 \int_0^L T\,\Psi_n\,dx \\
&=\; \left[\frac{\Psi_n}{k}\left(k\,\frac{\partial T}{\partial x} \;+\; h_L\,T\right)\right]\Bigg|_{x=L} + \left[\frac{\Psi_n}{k}\left(h_0\,T \;-\; k\,\frac{\partial T}{\partial x}\right)\right]\Bigg|_{x=0} \\
&\qquad - \zeta_n^2 \int_0^L T(x,t)\,\Psi_n(x)\,dx \\
&=\; \frac{h_L\,T_\infty}{k}\,\Psi_n(x)\Big|_{x=L} + \frac{h_0\,T_\infty}{k}\,\Psi_n(x)\Big|_{x=0} - \zeta_n^2\,\overline{T}(\zeta_n,t)\,. \qquad (6.20)
\end{aligned}$$

Note that the two boundary evaluations are obtained directly from Eqs. (6.12) and (6.13) on pp. 68. That is, the non–homogeneous boundary conditions are accounted for within the transform domain. It may perhaps be an unconscious habit by this point to presume that the Ψ_n cancel at the boundaries, but we remind the reader that this is not the case here because the physical boundary conditions are non–homogeneous — the Ψ_n will take on sets of non–zero values that make sure that those conditions are satisfied. Also, the second–derivative term again collapses by using the integral transform definition in Eq. (6.15).

It now remains for the heat generation term to be properly transformed. This goes according to the transform definition, as well, giving

$$\int_0^L \frac{\dot{q}(x)}{k}\,\Psi_n(x)\,dx \;=\; \frac{\overline{\dot{q}}}{k}\,. \qquad (6.21)$$

We can now assemble all the components of the transformed problem to obtain the ODE in transform space. Specifically, we substitute the various transformed terms for their corresponding

physical terms, including Eqs.(6.20) and (6.21) into Eq. (6.11) on pp. 68, finding

$$\frac{1}{\alpha}\,\frac{d\overline{T}}{dt} \;=\; \frac{\overline{\dot{q}}}{k} \;+\; \frac{h_0\,T_\infty}{k}\,\Psi_n(0) \;+\; \frac{h_L\,T_\infty}{k}\,\Psi_n(L) \;-\; \zeta_n^2\,\overline{T}(\zeta_n,t)\,. \tag{6.22}$$

Before proceeding further, let us make a few comments regarding Eq. (6.22). This equation is indeed an ODE, since $\overline{T}$ is not a function of x. The first 3 terms on the right hand side represent, respectively, the heat generation term, which makes Eq. (6.11) non–homogeneous, and the non–homogeneous boundary conditions at $x = 0$ and $x = L$, the components of the boundary conditions in Eqs. (6.12) and (6.13). All the non–homogeneous components of the physical problem in (x, t) are handled in the ODE! However, this equation is itself non–homogeneous and it does not appear that we can readily make a clever change of variables to render it homogeneous.[6.13]

In order to solve Eq. (6.22), let us make a few careful observations of its mathematical dependencies. Many of the parameters in this equation are constant: α, k, h_0, h_L, and T_∞. Strictly speaking, ζ_n is also constant, though there are a countably infinite number of these corresponding to the set of eigen–modes. We have also been a little too informal with the eigen–function notation here. Formally, $\Psi_n = \Psi_n(\zeta_n, x)$, so that what really appears in Eq. (6.22) is actually $\Psi_n(\zeta_n, 0)$ and $\Psi_n(\zeta_n, L)$. That is, the Ψ is also a function of the eigen–values.[6.14] This observation also implies $\overline{\dot{q}} = \overline{\dot{q}}(\zeta_n)$, as specified by Eq. (6.21). What these observations indicate is that every term in Eq. (6.22) that does not involve $\overline{T}$ is *at most* a function of ζ_n. Therefore, let us define the formality

$$G(\zeta_n) \;=\; \frac{\alpha}{k}\,\Big[\,\overline{\dot{q}}(\zeta_n) \;+\; h_0\,T_\infty\,\Psi_n(\zeta_n,0) \;+\; h_L\,T_\infty\,\Psi_n(\zeta_n,L)\Big] \tag{6.23}$$

so that we can re–arrange Eq. (6.22) into the canonical form

$$\frac{d\overline{T}}{dt} \;+\; \alpha\,\zeta_n^2\,\overline{T}(\zeta_n,t) \;=\; G(\zeta_n)\,. \tag{6.24}$$

Note from the initial condition of the physical problem in Eq. (6.14) on pp. 68 that the initial condition for the transformed problem is derived directly from Eq. (6.15)

$$\overline{T}(\zeta_n,0) \;=\; \int_0^L T(x,0)\,\Psi_n(x)\,dx \;=\; \int_0^L F(x)\,\Psi_n(x)\,dx \;=\; \overline{F}(\zeta_n)\,. \tag{6.25}$$

The ODE is first order and can be conveniently solved by the method of the integrating factor (Martin and Reissner, 1956; Nelson et al., 1960), here $e^{\alpha\zeta_n^2 t}$ being the appropriate factor.[6.15] This

[6.13] Recall, for example, that we were able to change variables in the model of the generalized fin, converting non–homogeneous Eq. (3.11) on pp. 20 to homogeneous Eq. (3.12).

[6.14] We have seen in numerous cases that Ψ is a function of both x and ζ_n, for example in §E.1 on pp. 151 where we found $\Psi_n = C_n \sin(\zeta_n x)$. In Eq. (6.22), these are *evaluated* at the boundaries, so they are really only functions of ζ_n in this particular context.

[6.15] The homogeneous form of Eq. (6.24) is $\overline{T}' + \alpha\,\zeta_n^2\,\overline{T} = 0$, which can be used to infer the integrating factor that will make the equation *exact*. (We found a similar equation for the Dirichlet problem, c.f. footnote 6.9 on pp 67, but did not need the integrating factor approach because the problem itself is homogeneous.) Multiplying by $dt/\overline{T}$ and developing the equation shows

$$\frac{d\overline{T}}{\overline{T}} \;+\; \alpha\,\zeta_n^2\,dt \;=\; 0 \qquad\qquad \ln\overline{T} \;+\; \int \alpha\,\zeta_n^2\,dt \;=\; C_1 \qquad\qquad e^{\ln\overline{T}\,+\,\int\alpha\,\zeta_n^2\,dt} \;=\; e^{\ln\overline{T}}\,e^{\alpha\,\zeta_n^2\int dt} \;=\; e^{C_1}\,,$$

whereby we can infer the integrating factor from the form

$$\overline{T}\;\underbrace{e^{\alpha\,\zeta_n^2 t}}_{\text{factor}} \;=\; C_2\,.$$

can be done in a few straightforward steps: multiply Eq. (6.24) by the integrating factor, integrate it, exploiting the fact that the left hand side is now *exact*, re–arrange to obtain $\overline{T} = \overline{T}(\zeta_n, t)$ on the left, and finally apply the initial condition. These steps are shown as

$$\begin{aligned}
\int_0^t \Big(\underbrace{e^{\alpha\,\zeta_n^2 t'}\,\frac{d\overline{T}}{dt'} \;+\; e^{\alpha\,\zeta_n^2 t'}\,\alpha\,\zeta_n^2\,\overline{T}(\zeta_n, t')}_{\text{exactly integrable}} \Big)\, dt' \;&=\; \int_0^t e^{\alpha\,\zeta_n^2 t'}\,G(\zeta_n)\,dt' \\
\Big(\overline{T}(\zeta_n, t')\,e^{\alpha\,\zeta_n^2 t'}\Big)\Big|_0^t \;&=\; G(\zeta_n)\,\frac{1}{\alpha\,\zeta_n^2}\,e^{\alpha\,\zeta_n^2 t'}\Big|_0^t \\
\overline{T}(\zeta_n, t)\,e^{\alpha\,\zeta_n^2 t} \;-\; \underbrace{\overline{T}(\zeta_n, 0)}_{\overline{F}(\zeta_n)}\,\cancelto{1}{e^{\alpha\,\zeta_n^2 0}} \;&=\; \frac{G(\zeta_n)}{\alpha\,\zeta_n^2}\Big(e^{\alpha\,\zeta_n^2 t} \;-\; \cancelto{1}{e^{\alpha\,\zeta_n^2 0}}\Big) \\
\overline{T}(\zeta_n, t)\,e^{\alpha\,\zeta_n^2 t} \;&=\; \frac{G(\zeta_n)}{\alpha\,\zeta_n^2}\Big(e^{\alpha\,\zeta_n^2 t} \;-\; 1\Big) \;+\; \overline{F}(\zeta_n) \\
\overline{T}(\zeta_n, t) \;&=\; \frac{G(\zeta_n)}{\alpha\,\zeta_n^2}\Big[1 \;-\; e^{-\alpha\,\zeta_n^2 t}\Big] \;+\; \overline{F}(\zeta_n)\,e^{-\alpha\,\zeta_n^2 t}\,.
\end{aligned}$$

Before we proceed further, let us make a few comments about the result thus far. First, it is easy to check that this expression is indeed the solution to Eq. (6.24) by direct substitution. Second, compare this result to that obtained for the simpler homogeneous Dirichlet problem of the previous example in Eq. (6.10) on pp. 67. Its solution consisted only of the second term on the right shown here. In essence, all the non–homogeneous effects at the boundaries and from heat generation have been neatly bundled into the first term, specifically $G(\zeta_n)$. Third, we can get a very good quantitative feel for how the conduction dynamics will evolve by looking simply at the time–related terms here. At $t = 0$, the contribution to $\overline{T}$ will be strictly from $\overline{F}$, the initial condition, since its exponential will be 1 and the term in square brackets for the non–homogeneous effects will be zero. As time proceeds, these contributions will start to "trade places" as it were, with the dynamics eventually being dominated by the non–homogeneous conditions.

We can now take advantage of the inverse transform to complete the problem in a straightforward fashion. Invoking Eq. (6.16), we find

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \frac{\Psi_n(\zeta_n, x)}{N(\zeta_n)} \;\left[\frac{G(\zeta_n)}{\alpha\,\zeta_n^2}\Big(1 \;-\; e^{-\alpha\,\zeta_n^2 t}\Big) \;+\; \overline{F}(\zeta_n)\,e^{-\alpha\,\zeta_n^2 t}\right] \tag{6.26}$$

as the solution to the physical problem, where various non–homogeneous contributions are contained in $G(\zeta_n)$ in Eq. (6.23). Of course, this is only the symbolic solution in abstract form. Though it may still give a feeling of being incomplete, application to a specific problem is now a mathematical formality, though admittedly not without some work. One would still have to execute the usual steps of applying the actual eigen–function, computing the eigen–values and the norm, integrating the initial condition, as well as applying the non–homogeneous parameters. We will defer these steps until an example later in the chapter which yet generalizes this problem a bit more.

6.5. General Theory of the Eigen–Function Integral Transform*

The examples we just examined give a glimpse of the significant mathematical utility provided by this method for problems having diffusion–type derivatives. Here, we will develop the method in general terms in order to formally demonstrate its applicability to more complicated problems. Most advanced texts on this topic show a presentation roughly of what we are about to develop here (e.g. Özişik, 1980; Cotta, 1993), although we will give a bit more detail that will undoubtedly clarify certain aspects left unexplained in those treatments.

Consider the general conduction problem having a temperature $T = T(\mathbf{r}, t)$, where $\mathbf{r}$ are orthogonal coordinates, for example $\mathbf{r} = (x, y, z)$ in the rectangular system. The temperature distribution is governed by a diffusion equation of the type shown in Eq. (2.17) on pp. 13, here written in general vector form

$$\frac{1}{\alpha}\,\frac{\partial T(\mathbf{r},t)}{\partial t} \;=\; \frac{\dot{q}(\mathbf{r},t)}{k} \;+\; \nabla^2 T(\mathbf{r},t) \tag{6.27}$$

and boundary conditions of the general Robbins kind in non–homogeneous form. On the j–th boundary, this expression is

$$k\,\frac{\partial T}{\partial n_j} \;+\; h_j\,T \;=\; f_j(\mathbf{r}_j, t)\,, \tag{6.28}$$

where $\partial/\partial n_j$ is the derivative in the *outward* normal direction to that boundary, h_j represents convective conditions at this boundary, and f_j is a specific boundary function at j which itself can depend upon time, t, and the remaining independent coordinates at that boundary, $\mathbf{r}_j$. For example, at a boundary of a constant value of x, say $x = L$, the normal derivative is $\partial/\partial n_j = \partial/\partial x$ and there are two remaining independent coordinates, whereby $\mathbf{r}_j = (y, z)$. Finally, there is general initial condition

$$T(\mathbf{r}, t) \;=\; F(\mathbf{r})\,. \tag{6.29}$$

The steps for the general problem are the same as those we observed for the two examples above, although there are some additional generalities we must derive.

First, we must develop the appropriate integral transform pair, which, like in the above examples depends upon an auxiliary eigen–value problem in the form of a Helmholtz–type equation

$$\nabla^2 \Psi_n \;+\; \zeta_n^2\,\Psi_n \;=\; 0 \tag{6.30}$$

$$k\,\frac{\partial \Psi_n}{\partial n_j} \;+\; h_j\,\Psi_n \;=\; 0\,, \tag{6.31}$$

where $\Psi_n = \Psi_n(\zeta_n, \mathbf{r}_\zeta)$ and $\mathbf{r}_\zeta$ are all the coordinates having an associated eigen–function.[6.16] The auxiliary problem in Ψ_n can readily be produced by applying SOV to the *homogeneous* version of the physical problem in Eqs. (6.27) and (6.28).

Let us digress into a bit of detail not found in the standard derivations (e.g. Özişik, 1980; Cotta, 1993) that will be particularly helpful in understanding the notational implications of the next several developments. We have previously used the symbol "Ψ_n" as the eigen–function for a *specific* coordinate direction, usually x. When there were more than 1 set of eigen–functions, e.g. as in §5.6 on pp. 57 for the steady 3–D conduction problem, where we had another set of eigen–functions in y, we used a second, altogether different eigen–function symbol, Γ, in that

[6.16] Do not confuse the eigen–mode (index) n with the normal direction to the local boundary n_j.

particular instance. Here, we are treating *all* the coordinates in "one fell swoop", so to speak. The eigen–function $\Psi_n(\zeta_n, \mathbf{r}_\zeta)$ is actually the *product* of all the relevant, single coordinate–specific eigen–functions. Again taking the steady 3–D problem in §5.6 as an example, this function would be the product $\sin(\zeta_{x,m}x) \cdot \sin(\zeta_{y,n}y)$, as shown in Eq. (5.14) on pp. 59, and $\mathbf{r}_\zeta = (x, y)$.

This observation now seems to raise a new problem, which is that there are 2 eigen–mode indices, m and n, because the 3–D steady solution is actually represented as a double summation. Here, we are only showing a single index, n, on our generalized "product" eigen–function, $\Psi_n(\zeta_n, \mathbf{r}_\zeta)$. How can this be? We reconcile this seeming contradiction using the simple mathematical concept of *correspondence* between members of infinite sets.[6.17] That is, we can think of each value of the "general" single index, n, as corresponding to a pair of the "coordinate–specific" indices:

"general" single index:	1	2	3	4	5	6	7	8	$\cdots$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	
"coordinate–specific" index pair:	$(1,1)$	$(2,1)$	$(1,2)$	$(1,3)$	$(2,2)$	$(3,1)$	$(4,1)$	$(3,2)$	$\cdots$

This concept is surely much better visualized using the "diagonalization" representation, where we march through the converse diagonals of the matrix of pairs of coordinate–specific indices, assigning a "general" index to each coordinate–specific pair, the latter shown as a subscript for each pair.[6.18]

$$\begin{bmatrix}
(1,1)_1 & & (1,2)_3 & \rightarrow & (1,3)_4 & & (1,4)_{10} & \rightarrow & (1,5)_{11} \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow & \\
(2,1)_2 & & (2,2)_5 & & (2,3)_9 & & (2,4)_{12} & & . \\
& \swarrow & & \nearrow & & \swarrow & & & \\
(3,1)_6 & & (3,2)_8 & & (3,3)_{13} & & . & & . \\
\downarrow & \nearrow & & \swarrow & & & & & \\
(4,1)_7 & & (4,2)_{14} & & . & & . & & . \\
& \swarrow & & & & & & & \\
(5,1)_{15} & & . & & . & & . & & . \\
\downarrow & & & & & & & &
\end{bmatrix}$$

[6.17] The concept of correspondence is one of the main tools in set theory to understand the cardinalities of sets. It can be used, for example, to show perhaps somewhat surprisingly that the cardinality of the set of natural numbers is equal to the cardinality of the set of all the *pairs* of natural numbers using the "diagonalization" representation of the matrix above (Hahn, 1956). That is, each number can be shown to be in one–to–one correspondence with each *pair* of numbers. Note that we do not proceed strictly in the horizontal or vertical directions, i.e. $(1,1), (1,2), (1,3), \ldots$ or $(1,1), (2,1), (3,1), \ldots$, respectively, because those series are infinite in one of the indices at a constant value of 1 for the other. We would never "get to the next series", so to speak. The concept has more practical implications too, for example in summation of multiple series, as discussed in §H.3 starting on pp. 178.

[6.18] This concept holds for triple eigen–functions, as well, for example

$$(1,1,1)_1, (2,1,1)_2, (1,2,1)_3, (1,1,2)_4, (1,1,3)_5, \ldots$$

although this is very difficult to represent visually as is done in the above matrix.

Note the further significance of this representation in the context of the orthogonality property of *multiple*, i.e. products of eigen–functions. That is, the orthogonality property can be cast directly in terms of the single general index. As an example, we point again to the steady 3–D conduction problem, specifically Eq. (5.15) on pp. 59, where we observed that all terms vanish except those where equivalence was realized for *both* eigen–values simultaneously. In other words, both eigen–values are "bound together" under the single, general eigen–value, so the orthogonality property will still hold for equivalence of just the general, single index alone.[6.19] That is, we have

$$\int_{\mathbf{r}} \Psi_m(\zeta_m, \mathbf{r}_\zeta)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r} \;=\; \begin{cases} 0 & \text{for} \quad m \neq n \\ N(\zeta_n) & \text{for} \quad m = n. \end{cases} \tag{6.32}$$

where $N(\zeta_n)$ is the usual normalization integral (norm)

$$N(\zeta_n) \;=\; \int_{\mathbf{r}_\zeta} \Psi_n^2(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r}_\zeta \,,$$

where we again observe that this expression may represent more than one coordinate direction, e.g. $\mathbf{r}_\zeta = (x, y)$ implies this is a double integral having $d\mathbf{r}_\zeta = dx\, dy$ for the 3–D steady conduction problem.

Returning now to the development of the solution, the functional expansion, first proposed for the basic 1–D Dirichlet problem in Eq. (6.1), can then be written

$$T(\mathbf{r}, t) \;=\; \sum_{n=1}^{\infty} C_n(t)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta) \,,$$

whence the orthogonality procedure is undertaken

$$\begin{aligned} \int_{\mathbf{r}_\zeta} T(\mathbf{r}, t)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r}_\zeta \;&=\; \sum_{m=1}^{\infty} C_m(t) \int_{\mathbf{r}_\zeta} \Psi_m(\zeta_m, \mathbf{r}_\zeta)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r}_\zeta \\ &=\; C_n(t)\, N(\zeta_n) \,. \end{aligned}$$

Consequently, we can again form our seemingly circular expression

$$T(\mathbf{r}, t) \;=\; \sum_{n=1}^{\infty} \left[\frac{1}{N(\zeta_n)} \int_{\mathbf{r}_\zeta} T(\mathbf{r}, t)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r}_\zeta \right] \Psi_n(\zeta_n, \mathbf{r}_\zeta) \,,$$

by which we really mean to separate this expression into a transform and an inversion pair, given respectively by

$$\overline{T}(\zeta_n, t) \;=\; \int_{\mathbf{r}_\zeta} T(\mathbf{r}, t)\, \Psi_n(\zeta_n, \mathbf{r}_\zeta)\, d\mathbf{r}_\zeta \tag{6.33}$$

$$T(\mathbf{r}, t) \;=\; \sum_{n=1}^{\infty} \frac{\overline{T}(\zeta_n, t)}{N(\zeta_n)}\, \Psi_n(\zeta_n, \mathbf{r}_\zeta) \,. \tag{6.34}$$

These are the general representations of the transform–inversion pairs that will be used. Note again that the single series form accounts for multiple spatial dimensions according to the "correspondence" principle above.

[6.19] The general orthogonality property is proved in §6.6 below.

Next, we operate on both sides of the governing (physical) Eq. (6.27) using the eigen–function product

$$\frac{1}{\alpha}\,\frac{\partial}{\partial t}\int_{\mathbf{r}_\zeta} T(\mathbf{r},t)\,\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,d\mathbf{r}_\zeta \;=\; \frac{1}{k}\int_{\mathbf{r}_\zeta}\dot{q}(\mathbf{r},t)\,\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,d\mathbf{r}_\zeta \;+\; \int_{\mathbf{r}_\zeta}\nabla^2 T(\mathbf{r},t)\,\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,d\mathbf{r}_\zeta\,,$$

which we write in shorthand as

$$\frac{1}{\alpha}\,\frac{d\overline{T}}{dt} \;=\; \frac{\overline{\dot{q}}}{k} \;+\; \int_{\mathbf{r}_\zeta}\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,\nabla^2 T(\mathbf{r},t)\,d\mathbf{r}_\zeta\,, \tag{6.35}$$

where we have again changed partial derivative in t to an ordinary one following the observation that the transformed variable is not a function of any of the spatial dimensions. As with the examples above the task is to use this form to eliminate the second derivatives, represented now in the general case by the Laplacian, ∇^2.

Let us now invoke a relationship that exists between the scalar functions Ψ_n and T by virtue of *Green's second theorem* with an eye toward decomposing the integral in Eq. (6.35). Written formally in 3–dimensions, this is[6.20]

$$\iiint\left(\Psi_n\,\nabla^2 T \;-\; T\,\nabla^2\Psi_n\right)dv \;=\; \iint\left(\Psi_n\,\frac{\partial T}{\partial n} \;-\; T\,\frac{\partial\Psi_n}{\partial n}\right)dA\,, \tag{6.36}$$

however we shall change the notation to correspond to our "general single index" convention discussed above. Re–arranging the terms, we find

$$\underbrace{\int_{\mathbf{r}_\zeta}\Psi_n\,\nabla^2 T\,d\mathbf{r}_\zeta}_{\text{field term}} \;=\; \underbrace{\int_{\mathbf{r}_\zeta} T\,\nabla^2\Psi_n\,d\mathbf{r}_\zeta}_{\text{field term}} \;+\; \underbrace{\sum_j\int_{\mathbf{r}_j}\left(\Psi_n\,\frac{\partial T}{\partial n_j} \;-\; T\,\frac{\partial\Psi_n}{\partial n_j}\right)d\mathbf{r}_j}_{\text{boundary term}}\,, \tag{6.37}$$

where $d\mathbf{r}_j$ is the appropriate differential expression for boundary j and the summation indicates that the integral is taken over all boundaries.[6.21]

The terms on the right–hand side of Eq. (6.37) are then further developed. The field term is immediately simplified by invoking the auxiliary problem in Eq. (6.30) on pp. 73, specifically the form $\nabla^2\Psi_n \;=\; -\,\zeta_n^2\,\Psi_n$, which can be substituted to give

$$\int_{\mathbf{r}_\zeta} T\,\nabla^2\Psi_n\,d\mathbf{r}_\zeta \;=\; -\int_{\mathbf{r}_\zeta} T\,\zeta_n^2\,\Psi_n\,d\mathbf{r}_\zeta \;=\; -\,\zeta_n^2\int_{\mathbf{r}_\zeta} T\,\Psi_n\,d\mathbf{r}_\zeta \;=\; -\,\zeta_n^2\,\overline{T}\,.$$

Note that the last term results from direct application of the integral transform in Eq. (6.33) on pp. 75. Conversely, the boundary term in Eq. (6.37) is handled by complementation of the boundary conditions for the physical and auxiliary problems. Specifically, multiply Eq. (6.28) on pp. 73 by

[6.20] Green's theorem can be derived from the Divergence theorem as shown in appendix G, in particular as Eq. (G.3) on pp. 166. Our specific application of the theorem here relies on assigning the scalar functions as $\varphi_1 = \Psi_n$ and $\varphi_2 = T$.

[6.21] It is common practice to introduce $\sum$ into this expression (e.g. Özişik, 1980) basically as an indicator that we will be applying the method for physical configurations that can be naturally described by orthogonal coordinate systems, e.g. (x, y, z), where there will be discrete boundaries at constant values of each coordinate, e.g. "the boundary at $x = L$".

Ψ_n, multiply Eq. (6.31) on pp. 73 by T, and subtract the results, finding

$$\begin{aligned} k\,\Psi_n\,\frac{\partial T}{\partial n_j} \;+\; h_j\,\Psi_n\,T \;&=\; \Psi_n\,f_j(\mathbf{r}_j,t) \\ -\left(k\,T\,\frac{\partial \Psi}{\partial n_j} \;+\; h_j\,T\,\Psi\right) \;&=\; -\,0 \\ \hline k\left(\Psi_n\,\frac{\partial T}{\partial n_j} \;-\; T\,\frac{\partial \Psi}{\partial n_j}\right) \;&=\; \Psi_n\,f_j(\mathbf{r}_j,t)\,. \end{aligned}$$

Once we divide through by k, it is clear that we can substitute $\Psi_n\,f_j/k$ in for the boundary term in Eq. (6.37). Consequently, using the above two developments, Eq. (6.37) can be recast as

$$\int_{\mathbf{r}_\zeta} \Psi_n\,\nabla^2 T\,d\mathbf{r}_\zeta \;=\; -\,\zeta_n^2\,\overline{T} \;+\; \sum_j \int_{\mathbf{r}_j} \frac{\Psi_n\,f_j(\mathbf{r}_j,t)}{k}\,d\mathbf{r}_j\,. \tag{6.38}$$

Finally, let us now reconstruct the full transformed Eq. (6.35) being careful to restore all the functional notation

$$\frac{1}{\alpha}\,\frac{d\overline{T}(\zeta_n,t)}{dt} \;=\; \frac{\overline{\dot{q}}(\zeta_n,t)}{k} \;-\; \zeta_n^2\,\overline{T}(\zeta_n,t) \;+\; \sum_j \int_{\mathbf{r}_j} \frac{\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,f_j(\mathbf{r}_j,t)}{k}\,d\mathbf{r}_j\,. \tag{6.39}$$

Clearly, each term is a function of, at most, t and ζ_n and the PDE in Eq. (6.27) on pp. 73 has now been reduced to an ODE. Moreover, the boundary conditions, which are non–homogeneous in the general case, Eq. (6.28), have already been incorporated into this result. The only substantive step that remains is to solve Eq. (6.39), subject to the transformed initial condition

$$\overline{T}(\zeta_n,0) \;=\; \int_0^L T(x,0)\,\Psi_n(x)\,dx \;=\; \int_0^L F(x)\,\Psi_n(x)\,dx \;=\; \overline{F}(\zeta_n)\,. \tag{6.40}$$

Similar to what we did in Eq. (6.23) on pp. 71 for the preceding example problem, let us gather all the non–homogeneous terms in Eq. (6.40) into a single "source term" and re–arrange the expression as

$$\frac{d\overline{T}(\zeta_n,t)}{dt} \;+\; \alpha\,\zeta_n^2\,\overline{T}(\zeta_n,t) \;=\; G(\zeta_n,t)\,, \tag{6.41}$$

where the source term is

$$G(\zeta_n,t) \;=\; \frac{\alpha}{k}\left[\overline{\dot{q}}(\zeta_n,t) \;+\; \sum_j \int_{\mathbf{r}_j} \Psi_n(\zeta_n,\mathbf{r}_\zeta)\,f_j(\mathbf{r}_j,t)\,d\mathbf{r}_j\right]\,. \tag{6.42}$$

Careful inspection will indicate two relevant points. First, the *form* of Eq. (6.41) is the same as that of Eq. (6.24) on pp. 71 for the Robbins example. This implies that the same integrating factor, $e^{\alpha\zeta_n^2 t}$, will be applicable to the solution process.[6.22] Second, we allowed for time–dependent boundary conditions in the original problem statement, Eq. (6.28) on pp. 73, so we will not be able to carry–out the actual integration below without fully specifying $G(\zeta_n,t)$.

[6.22] This integrating factor was derived in footnote 6.15 on pp. 71.

Let us now proceed to solve Eq. (6.41). Once again, we will annotate the integrating factor solution procedure. We have[6.23]

$$\int_0^t \Big(\underbrace{e^{\alpha\,\zeta_n^2 t'}\,\frac{d\overline{T}}{dt'} \;+\; e^{\alpha\,\zeta_n^2 t'}\,\alpha\,\zeta_n^2\,\overline{T}(\zeta_n,t')}_{\text{exactly integrable}} \Big)\,dt' \;=\; \int_0^t e^{\alpha\,\zeta_n^2 t'}\,G(\zeta_n,t)\,dt'$$

$$\Big(\overline{T}(\zeta_n,t')\,e^{\alpha\,\zeta_n^2 t'}\Big)\Big|_0^t \;=$$

$$\overline{T}(\zeta_n,t)\,e^{\alpha\,\zeta_n^2 t} \;-\; \underbrace{\overline{T}(\zeta_n,0)}_{\overline{F}(\zeta_n)}\,\cancelto{1}{e^{\alpha\,\zeta_n^2\cdot 0}} =$$

$$\overline{T}(\zeta_n,t)\,e^{\alpha\,\zeta_n^2 t} \;=\; \int_0^t e^{\alpha\,\zeta_n^2 t'}\,G(\zeta_n,t)\,dt' \;+\; \overline{F}(\zeta_n)$$

$$\overline{T}(\zeta_n,t) \;=\; e^{-\alpha\,\zeta_n^2 t}\,\left(\int_0^t e^{\alpha\,\zeta_n^2 t'}\,G(\zeta_n,t)\,dt' \;+\; \overline{F}(\zeta_n)\right).$$

The last step is now the straightforward inversion of $\overline{T}$ to obtain the actual solution in physical space. Applying Eq. (6.34), we find

$$T(\mathbf{r},t) \;=\; \sum_{n=1}^{\infty}\,\frac{1}{N(\zeta_n)}\,\Psi_n(\zeta_n,\mathbf{r}_\zeta)\,e^{-\alpha\,\zeta_n^2 t}\,\left(\int_0^t e^{\alpha\,\zeta_n^2 t'}\,G(\zeta_n,t)\,dt' \;+\; \overline{F}(\zeta_n)\right), \tag{6.43}$$

where again $\overline{F}(\zeta_n)$ and $G(\zeta_n,t)$ are defined by Eqs. (6.40) and (6.42), respectively.

6.6. General Theory of Orthogonality*

Footnotes B.5 on pp. 137 and C.2 on pp. 142 demonstrate the orthogonality property for a few specific instances of eigen–functions. However, to complete our discussion of the general form of the integral transform method in the last section, we must also show that the orthogonality property of the eigen–functions holds under general terms, as well. In particular, let us prove that the cancellation property indicated in Eq. (6.32) on pp. 75 applies for 2 different eigen–modes, i.e. when $m \neq n$.

Consider eigen–functions for 2 different eigen–modes, i.e. $\Psi_m(\zeta_m,\mathbf{r}_\zeta)$ and $\Psi_n(\zeta_n,\mathbf{r}_\zeta)$. Each of these satisfies its own Helmholtz–type auxiliary relation of the type shown in Eq. (6.30) on pp. 73

$$\nabla^2\Psi_m \;+\; \zeta_m^2\,\Psi_m \;=\; 0 \qquad \text{and} \qquad \nabla^2\Psi_n \;+\; \zeta_n^2\,\Psi_n \;=\; 0\,,$$

where $m \neq n$ and therefore $\zeta_m \neq \zeta_n$. Multiply the left equation by Ψ_n and the right equation by Ψ_m, then subtract the results

$$\Psi_n\,\nabla^2\Psi_m \;+\; \zeta_m^2\,\Psi_n\,\Psi_m \;=\; 0$$

$$\Psi_m\,\nabla^2\Psi_n \;+\; \zeta_n^2\,\Psi_m\,\Psi_n \;=\; 0$$

$$\Psi_n\,\nabla^2\Psi_m \;-\; \Psi_m\,\nabla^2\Psi_n \;+\; \left(\zeta_m^2 \;-\; \zeta_n^2\right)\Psi_m\,\Psi_n \;=\; 0\,.$$

[6.23] Compare the result here to exercise 1.2 on pp. 7.

Move the Laplacian terms to the right. Then, integrate over the domain, for example as shown in Eq. (6.32) on pp. 75, after which we apply Green's second integral theorem[6.24]

$$\begin{aligned}
\left(\zeta_m^2 \,-\, \zeta_n^2\right) \Psi_m \, \Psi_n \;&=\; \Psi_m \, \nabla^2 \Psi_n \,-\, \Psi_n \, \nabla^2 \Psi_m \\
\left(\zeta_m^2 \,-\, \zeta_n^2\right) \int_{\mathbf{r}} \Psi_m(\zeta_m, \mathbf{r}_\zeta) \, \Psi_n(\zeta_n, \mathbf{r}_\zeta) \, d\mathbf{r} \;&=\; \int_{\mathbf{r}} \left(\Psi_m \, \nabla^2 \Psi_n \,-\, \Psi_n \, \nabla^2 \Psi_m\right) d\mathbf{r} \\
&=\; \int_S \left(\Psi_m \, \frac{\partial \Psi_n}{\partial \hat{n}} \,-\, \Psi_n \, \frac{\partial \Psi_m}{\partial \hat{n}} \right) dS \,, \qquad (6.44)
\end{aligned}$$

where S is the surface (boundary) of the domain and where we are again careful to realize the distinction between the potentially confusing notation of the eigen–mode (index) n and the normal direction to the local boundary $\hat{n}$.

Now, we note that each eigen–problem is governed by homogeneous boundary conditions of the type shown in Eq. (6.31) on pp. 73

$$k \, \frac{\partial \Psi_m}{\partial \hat{n}} \,+\, h \, \Psi_m \,=\, 0 \qquad \text{and} \qquad k \, \frac{\partial \Psi_n}{\partial \hat{n}} \,+\, h \, \Psi_n \,=\, 0$$

and that again multiplying these equations by Ψ_n and Ψ_m, respectively, and subsequently subtracting the results will show that

$$\Psi_m \, \frac{\partial \Psi_n}{\partial \hat{n}} \,-\, \Psi_n \, \frac{\partial \Psi_m}{\partial \hat{n}} \,=\, 0 \,.$$

Consequently, the right hand side of Eq. (6.44) vanishes, leaving

$$\left(\zeta_m^2 \,-\, \zeta_n^2\right) \int_{\mathbf{r}} \Psi_m(\zeta_m, \mathbf{r}_\zeta) \, \Psi_n(\zeta_n, \mathbf{r}_\zeta) \, d\mathbf{r} \,=\, 0 \,,$$

and, because we explicitly invoked the condition $\zeta_m \neq \zeta_n$, it must be the case that the integral itself vanishes, thereby proving the cancellation property.

6.7. Exercises

6.1 The integral transform solution has been formally derived under "general boundary conditions" of the 3–rd kind, for example

$$k \, \frac{\partial T}{\partial n} \,+\, h \, T \,=\, f \,,$$

where n is the outward normal direction to the boundary. The Dirichlet boundary condition of the 1–st kind can be obtained simply by setting $k = 0$. However, this requires a modification in the term $G(\zeta_n, t)$ because k is in the denominator. In particular, show that the problem can be resolved by replacing Ψ_m/k with $-h^{-1} \cdot \partial \Psi_m / \partial n$.

6.2 Given the auxiliary eigen–problem $\Psi_n''(x) + \zeta_n^2 \Psi_n(x) \,=\, 0$ having boundary conditions $\Psi_n(0) = 0$ and

$$\frac{d \, \Psi_n}{dx} \,+\, h \, \Psi_n \,=\, 0$$

at $x = L$, show that eigen–values, ζ_n, are the positive roots of $\zeta_n \cot\left(\zeta_n L\right) = -h$.

[6.24] See appendix G, especially Eq. (G.3) on pp. 166.

CHAPTER 7

Introduction to Convection

UP UNTIL NOW, we have studied the mechanism of conduction, which is the special case of heat transfer through a medium where there is no bulk medium motion. We were not especially specific about what the medium consisted of, though we suspect by the few examples we furnished, most will have taken it to be solid matter. The medium can just as well be liquid or gas, provided those fluids are at rest. Once they are moving, heat energy is also macroscopically transported by that very motion. This more general mode of heat transfer is called *convection*. Here, we shall introduce some of the foundational ideas for convection, including the now–much–larger set of governing equations (and various special cases) and we will also place the concept of the convection coefficient on firmer theoretical ground.

What we will quickly see in these developments is that convection is a fundamentally more complicated proposition than conduction. A few specific exceptions notwithstanding, the latter is governed by *linear* differential equations, for which there is an enormous body of mathematical theory that can be brought to bear for specific problems. We have examined some of these in the previous chapters. Conversely, convection depends on fluid motion, which — a few specific exceptions notwithstanding — is governed by *non–linear* equations. There is still no general mathematical theory or framework for non–linear differential equations, meaning one must typically fall back on *ad hoc* methods or approximations (Hildebrand, 1976). The literature in this area is immense (e.g. van Dyke, 1964; Bender and Orszag, 1999; Nayfeh, 2004) and we will only have the luxury of sampling a few small bits. In what may be somewhat of a surprise, many of these methods are little, if at all more complicated than those for linear equations, yet they are capable of very accurate results. We will concentrate on these approaches.

7.1. Conservation Laws for Energy, Mass, and Momentum

Convection is governed by 3 of the most basic conservation laws for physical systems[7.1]

- conservation of mass (continuity)
- conservation of momentum (Newton's Second Law of motion)
- conservation of energy (the First Law of thermodynamics).

It will be clear that these represent a direct generalization of the purely conductive mechanism of heat transfer for the allowance of motion and that the conduction equation itself could be immediately recovered as a special case by simply substituting zero for every appearance of a velocity component. There are a number of complementary ways in which these equations can be derived, for example using Reynolds' Transport Theorem (Currie, 1993), from physical arguments (White,

[7.1] We limit our discussion here to non–mixed, non–reactive fluids. Were we to consider the heat transfer aspects of more complicated systems for factors such as these, we would have to account for conservation of other entities, for example conservation of chemical species, the laws of chemical reactions, etc.

1974), invoking traditional differential analysis (Burmeister, 1983; Bejan, 1984; Munson et al., 2006), etc. We developed the conduction equation using standard analysis of a differential element in §2.3. Here, we will merely sketch the derivation of the general equations for incompressible flow using a combination of *ad hoc* arguments and appeals to to more fundamental sources that delve into their mathematical and physical foundations (White, 1974; Panton, 1984). The reader is referred to those texts for full details.

7.1.1. The Continuity Equation. The main restriction we will place on our study of convection is that the fluid, and thus the underlying flow is *incompressible*, meaning the density, ρ, is constant. As a thermodynamic variable, the density varies according to pressure and temperature as (Panton, 1984)[7.2]

$$\frac{d\,\rho}{\rho} \;=\; \omega\,dP \;-\; \beta\,dT\,,$$

where the bulk expansion coefficient, β, and the compressibility coefficient, ω, are fluid properties given respectively by

$$\beta(P,T) \;=\; -\,\frac{1}{\rho}\;\left.\frac{\partial\rho}{\partial T}\right|_P \qquad\qquad \omega(P,T) \;=\; \frac{1}{\rho}\;\left.\frac{\partial\rho}{\partial P}\right|_T \tag{7.1}$$

and, for a truly incompressible fluid, $\omega = \beta = 0$. For our purposes of convection heat transfer, we can always take liquids as being incompressible, as well as gases that are moving at low to moderate speeds.

The continuity equation is a statement of conservation of mass. One of the implications of an incompressible fluid is that velocity is an exact proxy for the rate of mass flow because the fluid can neither expand nor contract. Consequently, we can derive the incompressible form of the continuity equation by a straightforward, *ad hoc* argument that references the conservation principles developed in §2.3. Specifically, we re–label the heat transfer notation in Fig. 2.3 on pp. 11 to instead represent velocity components (and, by proxy, mass transfer rates). For example, the heat transfer at the left face of the elemental control volume in the x direction, q_l, now becomes the velocity at that face, u_l, and so forth.[7.3] Moreover, since the fluid is incompressible, the total rate of flow into the element must be equal to the total rate of flow out, otherwise expansion or compression would be occurring. This implies

$$u_l\,\delta y\,\delta z \;+\; v_b\,\delta x\,\delta z \;+\; w_f\,\delta x\,\delta y \;=\; u_r\,\delta y\,\delta z \;+\; v_t\,\delta x\,\delta z \;+\; w_n\,\delta x\,\delta y\,,$$

[7.2] Note that this expression is nothing more than the differential of $\rho = \rho(P,T)$, i.e. written properly according to the Chain Rule as

$$d\rho \;=\; \frac{\partial\rho}{\partial P}\,dP \;+\; \frac{\partial\rho}{\partial T}\,dT$$

when the definitions of β and ω in Eq. (7.1) are substituted.

[7.3] As in §2.3, the re–labeled velocity components are expansions in Taylor series, omitting all higher–order terms:

$$u_l \;=\; u \;-\; \frac{\partial u}{\partial x}\,\frac{\delta x}{2} \qquad \text{and} \qquad u_r \;=\; u \;+\; \frac{\partial u}{\partial x}\,\frac{\delta x}{2}$$

$$v_b \;=\; v \;-\; \frac{\partial v}{\partial y}\,\frac{\delta y}{2} \qquad \text{and} \qquad v_t \;=\; v \;+\; \frac{\partial v}{\partial y}\,\frac{\delta y}{2}$$

$$w_f \;=\; w \;-\; \frac{\partial w}{\partial z}\,\frac{\delta z}{2} \qquad \text{and} \qquad w_n \;=\; w \;+\; \frac{\partial w}{\partial z}\,\frac{\delta z}{2}$$

which, after substituting expressions for u_l and the other quantities and doing a little algebra, leads to the conservation equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \,. \tag{7.2}$$

Of course, this formula is one specific representation in rectangular coordinates of the general expression[7.4]

$$\nabla \cdot \mathbf{V} = 0 \,. \tag{7.3}$$

Interestingly, this equation describes both steady and unsteady flows. There is no $\partial/\partial t$ term for incompressible flow as there would be for unsteady flows for compressible fluids (Burmeister, 1983). This observation is consistent with the view of incompressibility as being a phenomenon of total rate of mass influx being equal to the total rate of outflux. This "quasi–steady–state" interpretation implies that there is neither any "rate of increase" of mass within the element, which would indicate compression, nor "rate of decrease", which would be expansion, regardless of whether the actual flow is steady or not. The velocity distributions even for unsteady flows adjust immediately such that Eq. (7.3) is always satisfied.

7.1.2. The Energy Equation. We developed what we have been calling the conduction equation in §2.3. Specifically, Eq. (2.16) on pp. 12 describes conduction heat transfer under fairly general conditions and we examined a number of configurations governed by various special cases of this expression, some of which are listed in §2.4. The general equation was derived for a fixed, but arbitrary (x, y, z) point in space, according to the notion that there is no motion of the medium itself. Convection basically relaxes this restriction of an element restricted to a fixed point to allow for motion.

Recall that fluid motion is routinely described from the *Eulerian* perspective (Currie, 1993). That is, physical entities are specified by the independent variables for position, (x, y, z), as measured with respect to some coordinate system, and for time, t. For example, a fluid particle moves $d\mathbf{r}$ from position $\mathbf{r}$ to position $\mathbf{r} + d\mathbf{r}$ in an instant of time (Fig. 7.1). In other words, its (x, y, z) location is a function of time due to its motion. This simple observation has far–reaching mathematical implications. For example, if we go back and consider the differential element in Fig. 2.3 (pp. 11) to be a "particle" within the *fixed* (motionless) medium, as we originally did, its functional description of temperature is

$$T = T(x, y, z, t) \,,$$

again noting that the particle's (x, y, z) location does not change. However, this same particle when *moving* has a temperature described functionally as

$$T = T\big(x(t),\, y(t),\, z(t),\, t\big) \,.$$

[7.4] The "del operator" in rectangular coordinates is

$$\nabla = \hat{\imath}\,\frac{\partial}{\partial x} + \hat{\jmath}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial k} \,,$$

where the corresponding velocity vector is $\mathbf{V} = u\,\hat{\imath} + v\,\hat{\jmath} + w\,\hat{k}$. Numerous publications list the vector operators in common coordinate systems, primarily cylindrical and spherical coordinates (Kays and Crawford, 1980; Panton, 1984). Perhaps the most comprehensive catalog in this regard is by Moon and Spencer (1961), who furnish the properties for 40 different coordinate systems. See footnote 7.12 on pp. 88 for additional commentary.

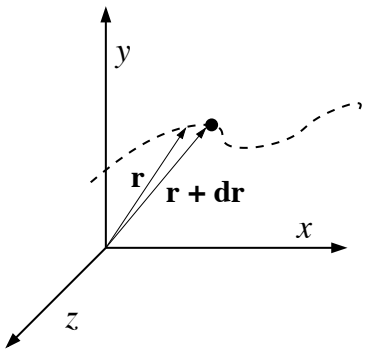


FIGURE 7.1. *Motion of a fluid particle along a pathline as describe by a radius vector* $\mathbf{r}$ *emanating from the origin of an inertial coordinate system.*

According to our original derivation in §2.3, we require the time–rate–of–change of the temperature, which for the former expression for fixed media was found to be simply $\partial T/\partial t$. For the case of motion, it is clear that we must apply the Chain Rule, the result being what is usually called either the total derivative or the material derivative (using D as the associated notation)

$$\begin{aligned}\frac{D\,T}{D\,t} &= \frac{\partial\,T}{\partial\,t} + \frac{\partial\,T}{\partial\,x}\,\frac{d\,x}{d\,t} + \frac{\partial\,T}{\partial\,y}\,\frac{d\,y}{d\,t} + \frac{\partial\,T}{\partial\,z}\,\frac{d\,z}{d\,t} \\ &= \underbrace{\frac{\partial\,T}{\partial\,t}}_{\text{local}} + \underbrace{u\,\frac{\partial\,T}{\partial\,x} + v\,\frac{\partial\,T}{\partial\,y} + w\,\frac{\partial\,T}{\partial\,z}}_{\text{convective}}\,,\end{aligned}$$

where we have used the observation that the velocity components, (u, v, w), are simply the time rates–of–change of their respective coordinates. Evidently, motion generalizes the "local" time rate–of–change term with several new terms that essentially comprise the additional "convective" time rate–of–change. These account for change realized by the mechanism of convective transport. Although we wrote this expression once again in rectangular coordinates, a moment's inspection should indicate that this concept can be generalized for any entity and written in any coordinate system as

$$\frac{D\,(\ \ )}{D\,t} = \frac{\partial\,(\ \ )}{\partial\,t} + \mathbf{V}\cdot\nabla(\ \ ) \tag{7.4}$$

where the appropriate forms are again substituted for $\mathbf{V}$ and ∇.

One more aspect will sufficiently generalize the energy equation for our purposes, and that is the description of how mechanical energy is "degraded" into heat energy. Here, we defer to White (1974) and Panton (1984), who show that the so–called "viscous dissipation function", Φ, describing this phenomenon is[7.5]

$$\Phi = \boldsymbol{\tau} : \nabla\mathbf{V}\,,$$

[7.5] Vector notation seems to get a little unwieldy at this point. Essentially, $\boldsymbol{\tau}$ and $\nabla\mathbf{V}$ are matrices and the colon notation is sometimes used to indicate the "inner product" of two matrices (Eves, 1966; Panton, 1984). The same entity is perhaps more clearly expressed in index notation

$$\Phi = \sum_i \sum_j \tau_{ij}\,\frac{\partial\,u_i}{\partial\,x_j} \qquad \text{where} \qquad \tau_{ij} = \mu\left(\frac{\partial\,u_i}{\partial\,x_j} + \frac{\partial\,u_j}{\partial\,x_i}\right)$$

where $\boldsymbol{\tau}$ is the viscous stress tensor.

Adding these two generalizations to the conduction equation, Eq. (2.16) on pp. 12, we find that the energy equation for "simple" convection can be written as

$$\rho\, c_p \left[\frac{\partial T(\mathbf{r},t)}{\partial t} + \underbrace{\mathbf{V}\cdot\nabla T}_{\substack{\text{convective}\\ \text{acceleration}}} \right] = \dot{q}(\mathbf{r},t) + \nabla\cdot\Big[k(T)\,\nabla T(\mathbf{r},t)\Big] + \underbrace{\boldsymbol{\tau}:\nabla\mathbf{V}}_{\substack{\text{viscous}\\ \text{dissipation}}}, \tag{7.5}$$

where the two new sets of terms introduced by the allowance of fluid motion are labeled.

As an example, let us write this general representation out in long–hand in the rectangular coordinate system, (x,y,z), where the velocity vector is $\mathbf{V} = u\,\hat{i} + v\,\hat{j} + w\,\hat{k}$ and where we assume a constant conductivity, k, so that this property can be moved outside of the dilation ("del dot") term. The velocity gradient and viscous stress tensors are, respectively,

$$\nabla\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \qquad \boldsymbol{\tau} = \begin{bmatrix} \mu\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \\ \mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \mu\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial y}\right) & \mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) \\ \mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \mu\left(\frac{\partial w}{\partial z}+\frac{\partial w}{\partial z}\right) \end{bmatrix}$$

the latter of which represents an incompressible fluid and is clearly symmetric. Here, μ is the so–called *dynamic viscosity*. Taking their inner product, along with the other terms, we find

$$\begin{aligned} \rho\, c_p \left[\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} + w\,\frac{\partial T}{\partial z} \right] &= \dot{q} + k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] \\ &\quad + 2\,\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2 \right] \\ &\quad + \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 \right. \\ &\quad \left. + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 \right], \end{aligned} \tag{7.6}$$

noting that Φ is always positive, since all its terms are quadratic. This observation is consistent with the second–law of thermodynamics, as Φ represents a *source* of heat energy as resulting from irreversible conversion of mechanical energy.

7.1.3. The Momentum Equations. The basis of the conservation of momentum is Newton's second law, roughly understood in an elementary context as the net force being equal to mass times acceleration. More formally, it is

$$\text{sum of the forces} = \text{rate of change of momentum},$$

is the viscous stress tensor for a linear, i.e. Newtonian incompressible fluid, and μ is the dynamic viscosity of the fluid. Strictly speaking, the summation signs are not entirely necessary here, since there is an implied summation convention for repeated indices.

where the momentum, $m\mathbf{V}$, can change because of change in speed, change in direction, or both. Here, we will always take m as being fixed,[7.6] so that, in the context of the previous section, we can immediately write the right hand side of the equation as $\delta x\, \delta y\, \delta z \cdot \rho$ (being mass) multiplied by the total derivative, $D\mathbf{V}/Dt$.[7.7] The other side of the equation is arguably the more challenging here, because we have to account for shear, pressure, and gravitational (or other force–related) contributions. Let us consider only 1 of the 3 coordinate directions, say x, to illustrate matters, the remaining 2 being procedurally identical.

Consider the element in Fig. 7.2 having the forces acting on the 6 surfaces labeled. Surface forces

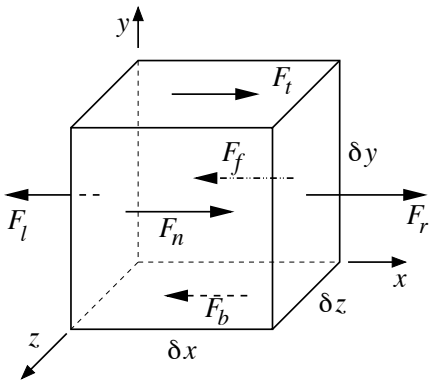


FIGURE 7.2. *Differential element of fluid with all surface forces in the x direction shown: normal forces are present on the right (r) and left (l) faces, while shear forces act on the top (t), bottom (b), near (n) and far (f) faces. In general, there can be a body force such as a component of gravity also acting in the positive x direction (not shown). Corresponding forces in the y and z directions are likewise not shown.*

arise from surface stresses acting on specific areas of the element. These forces can be subdivided into normal contributions, i.e. F_l and F_r on the left and right faces, respectively, and those which act tangentially, i.e. F_b and F_t on the bottom and top faces and F_n and F_f on the near and far faces, respectively. We introduce the following nomenclature for these stresses:

- σ_{xx} — This is a stress which acts on a face normal to the x direction (this is the first x) in the x direction. Forces F_l and F_r in Fig. 7.2 represent σ_{xx}.
- τ_{yx} — This describes a stress which acts on a face normal to the y direction in the x direction, e.g. F_t and F_b in Fig. 7.2.
- τ_{zx} — This is a stress which acts on a face normal to the z direction in the x direction, e.g. F_n and F_f in Fig. 7.2.

Note that the first subscript gives the face which the stress acts on, while the second gives the direction[7.8].

[7.6] There are problems where mass is not constant, for example in propulsion where fuel is consumed and the overall mass of a rocket is decreasing.

[7.7] Note that $D\mathbf{V}/Dt$ is a vector quantity, meaning

$$\frac{Du}{Dt}\,\hat{\imath} + \frac{Dv}{Dt}\,\hat{\jmath} + \frac{Dw}{Dt}\,\hat{k}\,,$$

and that these components belong respectively to 3 corresponding momentum equations in the (x, y, z) directions.

[7.8] A convenient memory aid is "*f*ire *d*epartment" for the order of *f*ace and *d*irection.

To quantify the forces shown in Fig. 7.2, we resort once again to Taylor series expansions of the relevant variables from the center of the differential volume to the boundaries. That is, we assume that σ_{xx}, τ_{yx}, and τ_{zx} are defined in the center of the differential control volume. According to the standard 1–term truncated series (omitting higher order terms), we find

$$\begin{aligned} F_r &= \left(\sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x}\,\frac{\delta x}{2}\right)\delta y\,\delta z & F_l &= \left(\sigma_{xx} - \frac{\partial \sigma_{xx}}{\partial x}\,\frac{\delta x}{2}\right)\delta y\,\delta z \\ F_t &= \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\,\frac{\delta y}{2}\right)\delta x\,\delta z & F_b &= \left(\tau_{yx} - \frac{\partial \tau_{yx}}{\partial y}\,\frac{\delta y}{2}\right)\delta x\,\delta z \\ F_n &= \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\,\frac{\delta z}{2}\right)\delta x\,\delta y & F_f &= \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\,\frac{\delta z}{2}\right)\delta x\,\delta y\,. \end{aligned}$$

At this point, let us add the contribution of a body force, which will usually be gravity[7.9]. Here, the force is the product of mass and acceleration (gravity), which yields $\rho \cdot \delta x\,\delta y\,\delta z \cdot g_x$, where g_x is the component of the gravity vector in the x direction.

Now, the net force is simply the sum of these contributions

$$\begin{aligned} \sum F_x &= F_r - F_l + F_t - F_b + F_n - F_f + \rho\, g_x\,\delta x\,\delta y\,\delta z \\ &= \left(\rho\, g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z\,, \end{aligned}$$

and, in light of the above discussion, including footnote 7.7 and the formula for the total derivative in Eq. (7.4) on pp. 83, we can immediately write

$$\delta x\,\delta y\,\delta z\,\rho\,\frac{Du}{Dt} = \left(\rho\, g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z\,.$$

Canceling the elemental volume $\delta x \cdot \delta y \cdot \delta z$ and expanding the expression for the total derivative, we can write the complete equation for the x direction as

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\right) = \rho\, g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\,. \tag{7.7}$$

By similar procedures, we obtain the y and z direction equations, respectively, as

$$\rho\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\right) = \rho\, g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{7.8}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\right) = \rho\, g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\,. \tag{7.9}$$

Eqs. (7.7) through (7.9) are completely general in the sense that they do not yet make any modeling assumptions for the stress terms. In fact, they contain too many unknowns to solve, i.e. 3 velocity components and additional stress components.

What are still missing are *constitutive relationships* between velocity components and stresses.[7.10] Essentially, these are the analogs for momentum of what Fourier's Law is for conduction (c.f. Eq. (2.1) on pp. 8), and, like that relationship, these "laws" are actually empirical and describe

[7.9] Although gravity is typically associated with the y direction, this is actually only a consequence of choosing coordinates in a certain way. In general, the gravity vector does not have to be aligned with any single coordinate direction and we assume here that there is a component g_x.

[7.10] Constitutive equations actually relate velocity gradients to stresses, not velocity components directly.

certain specific classes of fluids. Here, we will limit our discussion to linear, i.e. so–called *Newtonian fluids*, where stresses are linearly related to deformation rates. We already reported the shear stresses (c.f. footnote 7.5 on pp. 83), i.e.

(7.10)

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

and the normal stresses are (Bejan, 1984):

$$\sigma_{xx} = -P + 2\,\mu\,\frac{\partial u}{\partial x} \qquad \sigma_{yy} = -P + 2\,\mu\,\frac{\partial v}{\partial y} \qquad \sigma_{zz} = -P + 2\,\mu\,\frac{\partial w}{\partial z}\,. \tag{7.11}$$

We can now substitute these constitutive relations into Eqs. (7.10) and (7.11) into the equations we derived for the conservation of momentum, i.e. Eqs. (7.7) through (7.9). Several terms in each equation regroup into the form of the continuity equation, Eq. (7.2), and therefore can be dropped.[7.11] We are left with the following final form of the x momentum equation, and by similar procedures, the corresponding y and z momentum equations as well

$$\rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z}\right) = \rho\,g_x - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{7.12}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z}\right) = \rho\,g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \tag{7.13}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z}\right) = \rho\,g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right). \tag{7.14}$$

[7.11] For example, in the x direction, omitting the body force, we find the "right hand side" (RHS) of Eq. (7.7) to be

$$\begin{aligned} RHS &= \frac{\partial}{\partial x}\left(-P + 2\,\mu\,\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right], \\ &= -\frac{\partial P}{\partial x} + 2\,\mu\,\frac{\partial^2 u}{\partial x^2} + \mu\,\frac{\partial^2 u}{\partial y^2} + \mu\,\frac{\partial^2 v}{\partial y \partial x} + \mu\,\frac{\partial^2 w}{\partial z \partial x} + \mu\,\frac{\partial^2 u}{\partial z^2}\,. \end{aligned}$$

Note that we have assumed μ is constant, so that it can be taken outside of the derivatives. Presuming the order of differentiation is irrelevant, that is for example

$$\frac{\partial^2}{\partial y\,\partial x} = \frac{\partial^2}{\partial x\,\partial y}\,,$$

we can then write

$$\begin{aligned} RHS &= -\frac{\partial P}{\partial x} + 2\,\mu\,\frac{\partial^2 u}{\partial x^2} + \mu\,\frac{\partial^2 u}{\partial y^2} + \mu\,\frac{\partial^2 v}{\partial x \partial y} + \mu\,\frac{\partial^2 w}{\partial x \partial z} + \mu\,\frac{\partial^2 u}{\partial z^2} \\ &= -\frac{\partial P}{\partial x} + \mu\,\frac{\partial^2 u}{\partial x^2} + \mu\,\frac{\partial^2 u}{\partial y^2} + \mu\,\frac{\partial^2 u}{\partial z^2} + \mu\,\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right). \end{aligned}$$

We see that the last term in brackets *is* the continuity equation, which equals zero according to Eq. (7.2) on pp. 82. Therefore, this group of terms drops out entirely.

Note that we can combine Eqs. (7.12) through (7.14) into the compact vector form[7.12]

$$\rho \left(\frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = \rho \, \mathbf{g} - \nabla P + \mu \, \nabla^2 \mathbf{V} \, . \tag{7.15}$$

Taken with the statement of conservation of mass, which is Eq. (7.3) on pp. 82, Eqs. (7.15) are collectively known as the *Navier–Stokes equations* for incompressible flow. When, in addition, speaking of the energy equation as part of this group, we will refer to this set of laws collectively as the "convection equations".

7.2. General Remarks on the Convection Equations

Closer inspection of Eqs. (7.12) through (7.14) immediately reveals why convection heat transfer is mathematically much more complicated than conduction. The "convective terms" on the left–hand side of these 3 equations are all non–linear, meaning in this case that there are 9 instances where velocity components are multiplied by their own derivatives. There is no general mathematical theory or framework for non–linear differential equations that corresponds, for example, to the separation of variables method for linear equations (Hildebrand, 1976), though there are various *ad hoc* methods and approximations that are useful for specific classes of problems.[7.13]

As written,[7.14] conservation of energy, Eq. (7.5) on pp. 84, depends upon, but is entirely *uncoupled* from the conservation laws of mass and momentum, Eq. (7.3) on pp. 82 and Eqs. (7.12) through (7.14), respectively. More specifically, mass and momentum comprise 4 equations in 4 unknowns: pressure distribution, P, and the velocity field, $\mathbf{V}$, which contains the 3 velocity components (u, v, w). These 4 equations can be solved with total disregard for conservation of energy. Subsequently, $\mathbf{V}$ can be inserted in the specified manner into the conservation of energy statement, which is then solved for temperature, T.[7.15] In this specific sense, "convection" itself is non–linear, though the specific component of the energy equation is not.

There are also various exceptions to this strict uncoupling. For example, if temperature differences are sufficiently large such that viscosity is no longer a constant, but rather $\mu = \mu(T)$, then both the energy and the momentum equations and functions of both $\mathbf{V}$ and T. Another important case is that of free convection, where a non–constant density is reconciled with the overall incompressible flow assumption via the *Boussinesq approximation*. Here, T appears in the momentum equation by virtue of modeling the bulk expansion coefficient, β in Eq. (7.1) on pp. 81, by an explicit finite difference. It should be clear that such contingencies render an already–challenging problem even more so.

[7.12] Let us expand briefly on a point made in footnote 7.4 on pp. 82. Eq. (7.15) is shown in vector operator form, so in a subtle sense, it is more general than Eqs. (7.12) through (7.14) that are written specifically in rectangular coordinates. Similar generality applies to continuity, Eq. (7.3) vs. Eq. (7.2), and conservation of energy, Eq. (7.5) vs. Eq. (7.6). Vector operators for numerous coordinate systems are given by Moon and Spencer (1961).

[7.13] In fact, the Navier–Stokes problem itself is particularly deep and is regarded as one of the outstanding unsolved problems in mathematics. It is one of the 7 so–called "Millennium Prize" problems (Dickson, 2000), offering a US $1 million award for rigorous demonstration of certain smoothness and boundedness properties in its 3–D form.

[7.14] We remind the reader that this discussion applies strictly to equations as shown here, i.e. in their incompressible form.

[7.15] The phenomenon we are describing is perhaps most obvious in Eq. (7.6) on pp. 84. The convective acceleration terms on the left–hand side each consist of a product of a *known* velocity component and a first partial derivative of T. There are no terms in this equation that are products of T and any of its own derivatives, so the "convective terms" in this instance are linear, i.e. first–derivatives of T, though with non–constant coefficients.

7.3. Dimensionless Representation of the Convection Equations

We have previously examined the so–called Robbins configuration for conduction in §4.4 and §4.5 based on the dimensionless method of framing the problem. Though dimensionless analysis furnishes a number of kinds of insights and advantages for scientific and engineering work (c.f. footnote 4.6 on pp. 34), its main benefit in that particular instance was the reduction from 8 independent parameters to 3 dimensionless ones, those being dimensionless coordinates x and t and the Biot number.[7.16] In other words, the non–dimensionalizing process offered the maximal compression of information to specify a particular instance of the problem. This is obviously of enormous value in obtaining maximal generality (and thus *utility*) of solution. The essential step was to first non–dimensionalize the governing equation and the boundary and initial conditions, the results being Eqs.(4.9) through (4.11), and then to proceed in the usual mathematical fashion of developing the solution.

As observed above, convection is a generalization of conduction that accounts for fluid motion and the governing equations are commensurately more complicated. Here, it will be of even greater significance to be able to frame problems in a non–dimensional form on certain occasions. In this section, we will briefly sketch the non–dimensionalizing process for the mass, momentum, and energy equations of convection heat transfer based essentially upon the procedures already shown in §4.4. We will also review the resulting dimensionless numbers that coalesce from this procedure.

Here, we will assume the availability of relevant length and velocity scales of L and u_∞, respectively, i.e.

$$x^* \,=\, \frac{x}{L} \qquad y^* \,=\, \frac{y}{L} \qquad z^* \,=\, \frac{z}{L} \qquad u^* \,=\, \frac{u}{u_\infty} \qquad v^* \,=\, \frac{v}{u_\infty} \qquad w^* \,=\, \frac{w}{u_\infty}\,,$$

where the "starred" symbol (*) is the conventional notation for the dimensionless analog of a dimensional quantity. Recall in §4.4 "time" was scaled according to the rate of heat diffusion, c.f. Eq. (4.6) on pp. 34, consistent with the diffusion–only mechanism of conduction and lack of a bulk–motion velocity. The fact that there is now a non–zero reference velocity implies scaling time as

$$t^* \,=\, \frac{u_\infty}{L}\,t\,.$$

Finally, we will assume a reference pressure P_∞ and scale pressure difference according to a multiple of the dynamic pressure

$$P^* \,=\, \frac{P \,-\, P_\infty}{\rho\, u_\infty^2}\,.$$

We note that these choices represent the somewhat generic way of non–dimensionalizing the convection equations and that specific types of problems could necessitate somewhat different scaling.

The actual non–dimensionalizing procedure is quite straightforward, following basically along the lines of the approach shown in §4.4. Specifically, use Chain Rule to transform derivatives to the "starred" system, e.g.

$$\frac{\partial}{\partial t} \,=\, \frac{\partial}{\partial t^*}\,\frac{\partial t^*}{\partial t} \,=\, \frac{u_\infty}{L}\,\frac{\partial}{\partial t^*} \qquad \text{and} \qquad \frac{\partial}{\partial x} \,=\, \frac{\partial}{\partial x^*}\,\frac{\partial x^*}{\partial x} \,=\, \frac{1}{L}\,\frac{\partial}{\partial x^*}\,,$$

etc., then use additional algebra to re–arrange terms and discern relevant dimensionless numbers that characterize the system. Conservation of mass has no parameters and, consequently, realizes

[7.16] See Eq. (4.12) on pp. 35.

the same form as its dimensional ancestor in Eq. (7.3) on pp. 82

$$\nabla^* \cdot \mathbf{V}^* \;=\; 0\,. \tag{7.16}$$

Applying dimensionless derivatives for the momentum Eqs. (7.15) implies

$$\rho\,\frac{u_\infty^2}{L}\,\left(\frac{\partial \mathbf{V}^*}{\partial t^*} \;+\; \mathbf{V}^*\cdot\nabla^*\mathbf{V}^*\right) \;=\; \rho\,\mathbf{g} \;-\; \frac{1}{L}\,\rho\,u_\infty^2\,\nabla^* P^* \;+\; \frac{u_\infty}{L^2}\,\mu\,\nabla^{*2}\mathbf{V}^*\,.$$

which can be re–arranged to the form

$$\frac{\partial \mathbf{V}^*}{\partial t^*} \;+\; \mathbf{V}^*\cdot\nabla^*\mathbf{V}^* \;=\; \frac{L}{u_\infty^2}\,\mathbf{g} \;-\; \nabla^* P^* \;+\; \frac{1}{Re}\,\nabla^{*2}\mathbf{V}^*\,, \tag{7.17}$$

where

$$Re \;=\; \frac{\rho\,u_\infty\,L}{\mu} \;=\; \frac{u_\infty\,L}{\nu} \;=\; \frac{\text{inertial forces}}{\text{viscous forces}} \tag{7.18}$$

is the *Reynolds number*. The physical implication of Re is the relative magnitude of inertial vs. viscous forces, which is a direct consequence of its scaling of the viscous (momentum diffusion) terms, $\nabla^{*2}\mathbf{V}^*$, relative to the un–modified inertial terms on the left–hand side of Eq. (7.17). Viscous effects are *dissipative*, meaning small values of Re imply large damping of any flow disturbances, the result being a tendency to limit and/or degrade instabilities that would lead to a transition from laminar to turbulent flow. Conversely, high values of Re imply little damping and correspondingly larger tendencies toward turbulent flow.

Observe that we have left the gravitational term in a somewhat unorthodox form, $L\,\mathbf{g}/u_\infty^2$. Gravitational effects are important primarily in flows where a liquid free surface is relevant, e.g. so–called "open channel" flows, which are driven by strictly by gravity. In such cases, one coordinate is typically chosen to coincide with the gravitational vector. For example, if alignment with y is selected, then $\mathbf{g} = (0, g, 0)$. The unorthodox term is then readily recognized as $1/Fr^2$, where $Fr = u_\infty/\sqrt{L\,g}$ is the *Froude number*, which indicates the ratio of inertial forces to gravitational forces. Open–channel flows are routinely highly turbulent because their liquid mediums (usually water) have high density, c.f. Eq. (7.18). Such problems are beyond the scope of our basic treatment here, so we will not take the gravitational term as being critical for further discussions.

Finally, let us apply the dimensionless analysis to conservation of energy, Eq. (7.5), where we will take the conductivity, k, as being a constant and further assume that temperature can be non–dimensionalized as

$$T^* \;=\; \frac{T - T_s}{T_\infty - T_s}\,,$$

where T_∞ and T_s are constant reference temperatures for the fluid and a bounding surface, respectively. We will also make the change of variables $\boldsymbol{\tau} = \mu\,\tilde{\boldsymbol{\tau}}$ to draw–out the dynamic viscosity explicitly in the equation. We find

$$\rho\,c_p\,\frac{(T_\infty - T_s)\,u_\infty}{L}\,\left[\frac{\partial T^*}{\partial t^*} \;+\; \mathbf{V}^*\cdot\nabla^* T^*\right] \;=\; \dot{q} \;+\; k\,\frac{T_\infty - T_s}{L^2}\,\nabla^{*2}T^* \;+\; \mu\,\left(\frac{u_\infty}{L}\right)^2\,\tilde{\boldsymbol{\tau}}^* : \nabla^*\mathbf{V}^*\,,$$

which can be re–arranged as

$$\frac{\partial T^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* T^* = \frac{\dot{q}\, L}{\rho\, c_p\, (T_\infty - T_s)\, u_\infty} + \frac{k\, (T_\infty - T_s)\, L}{L^2\, \rho\, c_p\, (T_\infty - T_s)\, u_\infty} \nabla^{*2} T^* + \frac{\mu\, u_\infty^2\, L}{L^2\, \rho\, c_p\, (T_\infty - T_s)\, u_\infty} \tilde{\boldsymbol{\tau}}^* : \nabla^* \mathbf{V}^* ,$$

and subsequently written in dimensionless form as[7.17]

$$\frac{\partial T^*}{\partial t^*} + \mathbf{V}^* \cdot \nabla^* T^* = \dot{q}^* + \frac{1}{Re \cdot Pr} \nabla^{*2} T^* + \frac{Br}{Re \cdot Pr} \tilde{\boldsymbol{\tau}}^* : \nabla^* \mathbf{V}^* . \tag{7.19}$$

There are a number of interesting aspects of this equation.

First, we notice a dimensionless energy generation term, $\dot{q}^*$. In a way, this is an obsolete hold–over (for our purposes) from *pure* conduction, where we had allowed for various mechanisms of energy generation. For convection, we will take viscous dissipation as the sole generation mechanism, where it even exists at all (discussed below). Therefore, we will essentially take $\dot{q}^* = 0$ from this point on, although we recognize that there are more "exotic" problems where this term would be relevant.

There are 2 new dimensionless numbers here, too. First is the *Prandtl number*,

$$Pr = \frac{\nu}{\alpha} = \frac{\text{momentum diffusivity}}{\text{thermal diffusivity}} , \tag{7.20}$$

which is the ratio of the kinematic viscosity, also (appropriately) called the *momentum diffusivity*, and the thermal diffusivity. It is a dimensionless measure of the relative rates at which momentum and heat diffuse. For example, gases have Pr around 1, meaning momentum and heat diffuse roughly at the same rate, while oils and liquid metals have high and low Pr, respectively. This will have implications in different kinds of flows, for example those having a boundary layer, that we will discuss later. Notice that Pr is comprised solely of material properties, so it is actually a property of the fluid itself, unlike, for example, the Reynolds number, which is comprised of both fluid–specific and problem–specific variables.

The second new dimensionless number in Eq. (7.19) is the *Brinkman number*,[7.18]

$$Br = \frac{\mu\, u_\infty^2}{k\, (T_\infty - T_s)} = \frac{\text{heat generation}}{\text{heat transfer}} , \tag{7.21}$$

[7.17] The coefficient of the Laplacian of temperature is developed as

$$\frac{k\, (T_\infty - T_s)\, L}{L^2\, \rho\, c_p\, (T_\infty - T_s)\, u_\infty} = \frac{k}{L\, \rho\, c_p\, u_\infty} = \frac{\alpha}{L\, u_\infty} = \frac{\alpha}{L\, u_\infty} \cdot \frac{\nu}{\nu} = \frac{\nu}{L\, u_\infty} \cdot \frac{\alpha}{\nu} = \frac{1}{Re \cdot Pr}$$

while the coefficient of the dissipation term is processed as

$$\frac{\mu\, u_\infty^2\, L}{L^2\, \rho\, c_p\, (T_\infty - T_s)\, u_\infty} = \frac{\mu\, u_\infty^2}{k\, (T_\infty - T_s)} \cdot \frac{k}{\rho\, c_p} \cdot \frac{1}{u_\infty\, L} = \frac{\mu\, u_\infty^2}{k\, (T_\infty - T_s)} \cdot \frac{\alpha}{u_\infty\, L} \cdot \frac{\nu}{\nu} = \frac{Br}{Re \cdot Pr} \cdot$$

[7.18] Some texts prefer the Brinkman number (Mills, 1999), while others (Özişik, 1985; Bergman et al., 2011) discuss the scaling of dissipation according to the Eckert number

$$Ec = \frac{u_\infty^2}{c_p\, (T_\infty - T_s)} ,$$

from which it is straightforward to demonstrate the relationship $Br = Ec \cdot Pr$.

which is a measure of the amount of heat generation via the mechanism of viscous dissipation relative to the rate of heat transfer. Clearly, all 3 parameters are relevant to heat generation in the most general case, so that low Re and Pr, and high Br will render dissipation an important contribution.

7.4. The No–Slip Viscous Boundary Condition and the Nusselt Number

So far we have considered the convection coefficient, h, only to the extent that it has helped frame certain types of boundary conditions for conduction problems.[7.19] To a large extent, and especially for applications, the goal of convection analysis is actually the determination of h itself because the heat transfer is then immediately available from Newton's Law of Cooling, Eq. (2.24) on pp. 14. There is an important dimensionless interpretation of h when taken in the context of viscous flow.

Recall that viscous flows are governed at boundaries by the so–called "no–slip" boundary condition.[7.20] Specifically, for stationary boundaries, the adjacent fluid is at rest, the implication being that heat transfer at that particular location is strictly by conduction (Fig. 7.3). Of course,

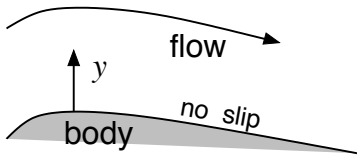


FIGURE 7.3. *In a viscous flow, the fluid velocity at the boundary matches the velocity of the boundary itself, i.e. there is no "slipping". If the boundary is at rest, the fluid at that location is at rest, as well, so that heat transfer exclusively at that location is strictly via the mechanism of conduction, i.e. diffusion.*

convection becomes important any distance away from the boundary.

Given these observations, we can conclude from Fig. 7.3 that, at the boundary location $y = 0$, *but on the fluid side*,[7.21] the heat transfer resulting from any temperature difference between flow and the body is due to conduction only. Furthermore, this heat transfer must be equal to that by the larger mechanism of convection for the flow because there are no heat storage or generation mechanisms acting at the boundary. In effect, conduction within the fluid at the boundary equals convection within the fluid, i.e.

$$-k \left.\frac{\partial T}{\partial y}\right|_{y=0} = h \left(T|_{y=0} - T_\infty \right), \tag{7.22}$$

assuming a fluid freestream temperature of T_∞. Notice the superficial resemblance to the conduction boundary condition of the 3rd kind, Eq. (2.25) on pp. 15. However, these equations are physically

[7.19] We have considered both the general "boundary condition of the third kind" in §2.5, c.f. Eq. 2.25 on pp. 15 and a convection convection as the "built–in" boundary condition in deriving the fin equation in §3.2 on pp. 18.

[7.20] The no–slip boundary condition is not obvious and, in fact, has a complicated history in its own right (Panton, 1984). However, it has been shown empirically to be a very good description of of the behavior of fluids at a boundary and critical to the proper modeling of the motion of viscous fluids, in general (Schlichting, 1979).

[7.21] In the context of calculus, we might frame this as a "one–sided limit", i.e. the limit as we approach $y = 0$ from the positive side, $y \to 0^+$, but still within the restriction of the continuum assumption, of course!

different in an important way — the Fourier term for Eq. (2.25) is on the "body" side, while here it represents the "fluid" side ($y \to 0^-$ vs. $y \to 0^+$, respectively, in the context of footnote 7.21). On a more practical level, the thermal conductivity, k, appearing in Eq. (2.25) is that of the solid body, while here it is of the flowing fluid.

We can now apply the dimensional analysis using the definitions from §7.3 above, specifically, spatial rates–of–change are scaled by length L and temperature is scaled using T_∞ and the boundary temperature

$$\frac{\partial}{\partial y} \;=\; \frac{1}{L}\,\frac{\partial}{\partial y^*} \qquad \text{and} \qquad T^* \;=\; \frac{T - T|_{y=0}}{T_\infty - T|_{y=0}}\,,$$

meaning that the boundary equation above is

$$-\,\frac{k\,(T_\infty - T|_{y=0}\,)}{L}\,\frac{\partial T^*}{\partial y^*}\,\bigg|_{y^*=0} \;=\; h\,(T|_{y=0} - T_\infty)\,.$$

A little algebra shows

$$Nu \;=\; \frac{h\,L}{k} \;=\; \frac{\partial T^*}{\partial y^*}\,\bigg|_{y^*=0} \;=\; \frac{\text{convection heat transfer}}{\text{conduction heat transfer}}\,, \tag{7.23}$$

which is known as the *Nusselt number*. Formally, it is clear that the Nusselt number is the *dimensionless* temperature gradient at the boundary. Moreover, knowledge of the Nusselt number implies h, so that it is not much of an exaggeration to say that the Nusselt number is the desired solution, in a practical sense, to the convection problem. The Nusselt number can also be interpreted as the ratio of convection to conduction, in a sense the "amplification" of heat transfer furnished by convection over just conduction alone, assuming $Nu > 1$. Specifically, if spatial rates–of–change scale with L, then $\partial T/\partial y \sim \Delta T/L$, where $\Delta T = T_\infty - T|_{y=0}$ and

$$Nu \;\sim\; \frac{h\,\Delta T}{\left(\dfrac{k\,\Delta T}{L}\right)}\,.$$

Again, it is important to recognize that the thermal conductivity, k, in Eq. (7.23) is that of the moving fluid.

We will drop the "star" notation (*) for discussions of convection in succeeding chapters for convenience of presentation, reverting to it only briefly as necessary to avoid ambiguity.

CHAPTER 8

Internal Convection: Laminar Flow in Ducts

As was emphasized in §7.2, the admission of fluid motion essentially pushes the dynamics beyond the linear realm into the non–linear. Here, the collection of problems that can be solved by purely analytical means is limited and one must typically resort to numerical and experimental methods to make headway on "real world" problems that have any real complications in terms of geometry or prescribed flow conditions (c.f. Table 1.1 on pp. 6).[8.1] However, there are several classes of flows that can be solved exactly, as well as a broader collection that can be solved approximately, e.g. using integral methods. While their practical use is somewhat limited, these are still enormously useful both in conceptual and mathematical contexts.

8.1. Laminar Fully–Developed Flow: Parallel Motion

A rudimentary phylogeny of incompressible flows from the perspective of "mathematical solveability" is given in Fig. 8.1. Both the Navier–Stokes equations, Eq. (7.15) on pp. 88, and the

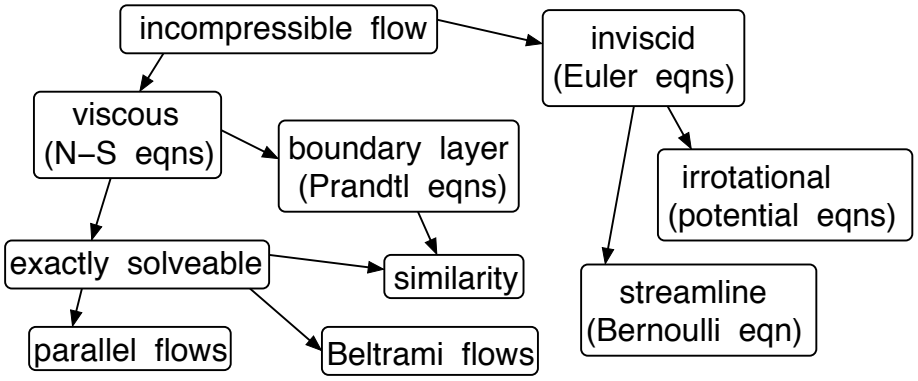


Figure 8.1. *Phylogeny of incompressible flows, roughly according to the conventions prescribed by Wang (1989, 1991)*

so–called Euler equations for inviscid flow are non–linear systems.[8.2] For our purposes in studying convection, we will concentrate on 2 scenarios: flows of the boundary–layer type, which are also still

[8.1] We again remind the reader that we are speaking of *viscous* flows in this statement, not inviscid or further idealized cases, e.g. potential flows and those described by the Bernoulli equation (Fig. 8.1), for which there is an enormous literature on exact solutions. Introductory texts, such as Munson et al. (2006), are a good starting point for these configurations.

[8.2] The Euler equations can be derived by applying the *inviscid* constitutive relationships to the general stress–based momentum conservation statements in Eqs. (7.7) through (7.9) on pp. 86. The constitutive relationships for inviscid flow are that all normal stresses are equal to the pressure and all shear stresses vanish (Munson et al., 2006). We will not study this system in any particular depth.

non–linear (next chapter), and a subset of exactly–solvable viscous flows that are actually linear: parallel flows.[8.3]

Parallel flows are those in which 2 of the 3 velocity components vanish, so that, without loss of generality $\mathbf{V} = (u, 0, 0)$. This restriction has several far–reaching implications. First, consider conservation of mass in the form of Eq. (7.2) on pp. 82. For $v = w = 0$, this equation simplifies to

$$\frac{\partial u}{\partial x} \;=\; 0\,,$$

which says that the velocity profile is not changing in the flow direction, x, i.e. the flow is *fully developed*.[8.4] We immediately infer that the parallel flow assumption will be most applicable to internal duct flows (flows primarily along a particular axial direction), where the duct is *long* (flow is fully–developed).

A second implication is that the substantive portions of the y and z momentum equations, i.e. Eqs. (7.13) and (7.14) on pp. 87, vanish. In particular, the pressure gradient terms are the only ones remaining (again presuming we are omitting the body force term related to gravity, as discussed in §7.2), meaning that these equations are reduced to the trivial statements that pressure in their respective coordinate directions is constant.

Third, the one remaining momentum equation in x is *linearized*, since $\partial u/\partial x = v = w = 0$ means that the non–linear convective terms all cancel

$$\frac{\partial u}{\partial t} \;+\; \cancelto{0}{u\frac{\partial u}{\partial x}} \;+\; \cancelto{0}{v\frac{\partial u}{\partial y}} \;+\; \cancelto{0}{w\frac{\partial u}{\partial z}} \;=\; -\,\frac{\partial P}{\partial x} \;+\; \frac{1}{Re}\left(\cancelto{0}{\frac{\partial^2 u}{\partial x^2}} \;+\; \frac{\partial^2 u}{\partial y^2} \;+\; \frac{\partial^2 u}{\partial z^2}\right),$$

leaving

$$\frac{\partial u}{\partial t} \;=\; -\,\frac{\partial P}{\partial x} \;+\; \frac{1}{Re}\left(\frac{\partial^2 u}{\partial y^2} \;+\; \frac{\partial^2 u}{\partial z^2}\right) \tag{8.1}$$

as the governing equation for momentum conservation for parallel flow.[8.5]

There is a subtle fourth point regarding turbulence, which arises by amplification of disturbances when viscous effects become sufficiently small relative to inertial effects. Turbulence cannot occur in flows governed by Eq. (8.1) because the inertial terms have vanished — this expression fundamentally describes laminar flows. However, this is most emphatically *not* to say that turbulence cannot arise in primarily axial flows — it certainly can! Those instances are governed by more complicated equations and are, strictly speaking, not parallel flows because fluctuating components of velocity exist in the y and z directions.

[8.3] The reader should have a good idea of at least the concept of similarity solutions from §4.6 on pp. 36. Conversely, Beltrami flows seem anecdotally to be appreciably less familiar. Given vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{V}$, Beltrami flows are those which satisfy $\nabla \times (\boldsymbol{\omega} \times \mathbf{V}) = 0$. The subsequent linearization renders the no–slip boundary condition difficult to satisfy, so physical manifestations of such flows are difficult to obtain (Wang, 1989, 1990, 1991).

[8.4] "Fully developed" and $\partial/\partial x = 0$ are almost, but not quite synonymous because *the latter does not apply to pressure*. In other words, for flows whose motion is driven by a pressure gradient it *must* be the case that $\partial P/\partial x \neq 0$, otherwise the fluid would be at rest, and such flows can still be fully developed in terms of velocity profile.

[8.5] Eq. (8.1) is obviously reported in its dimensionless context and is valid for Cartesian (rectangular) coordinates. The diffusion term on the right can be written in terms of the Laplacian, $\nabla^2 u$, where the del operator can be cast in various other coordinate systems. See e.g. footnote 7.4 on pp. 82.

Finally, according to the same arguments above, the energy equation simplifies to

$$\frac{\partial T}{\partial t} + u\,\frac{\partial T}{\partial x} = \frac{1}{Re\cdot Pr}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{Br}{Re\cdot Pr}\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]. \tag{8.2}$$

These equations can be applied to various parallel flow configurations.

8.2. The Planar Couette Problem

The planar Couette problem is defined as the steady flow between 2 infinite parallel plates separated by a distance L, where one of the plates moves in its own plane at a constant velocity u_0. It is one of the most basic viscous flow models (Özişik, 1985; Anderson, 1991; Mills, 1999), as well as being a good model for certain physical devices such as a low–clearance journal bearing having no vertical loading (Fig. 8.2). According to the no–slip boundary condition discussed in §7.4, the

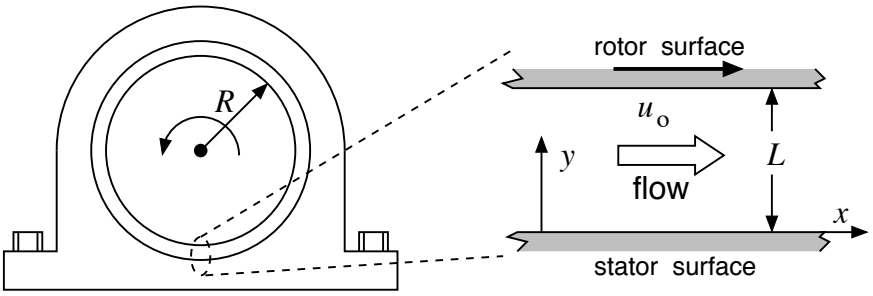


FIGURE 8.2. *Parallel plane Couette flow as a model for a concentric–cylinder (rotor–stator) journal bearing having a very thin gap, i.e. where $R \gg L$. Here, a cylindrical configuration (left panel) is well–approximated in rectangular geometry (right panel).*

velocity at $y = L$ is equal to the speed of the top plate, while the velocity at $y = 0$ is 0, since that plate is at rest. However, we will examine this configuration from the dimensionless perspective, where L and u_0 are the length and velocity scales, respectively, i.e. $y^* = y/L$ and $u^* = u/u_0$. Dispensing with the "star" notation (*) for convenience, we have boundary conditions for velocity of

$$u\Big|_{y=0} = 0 \qquad \text{and} \qquad u\Big|_{y=1} = 1$$

Motion is driven by external sources and there is no significant pressure gradient. Moreover, we will assume, for the time being, that the fluid domain extends significantly "into and out of the paper" in the z direction, so that boundaries are far away and gradients in this direction are thus negligible. Finally, let us assume the flow has been established so that steady–state operation may be assumed. Under these conditions, conservation of momentum in Eq. (8.1) reduces to $d^2u/dy^2 = 0$ and the velocity profile under the above boundary conditions is[8.6]

$$u = y\,. \tag{8.3}$$

[8.6]In Eq. (8.1), $u = u(y, z, t)$, but given independence from z and t and the absence of pressure gradient, $u = u(y)$, so that the equation simplifies to $d^2u/dy^2 = 0$. It is readily integrated as $u = C_1\,y + C_2$, where $C_1 = 1$ and $C_2 = 0$ are also readily determined from the boundary conditions. We remind the reader that this solution is dimensionless, i.e. we mean $u^* = y^*$, from which the actual (dimensional) distribution $u = u_0\,y/L$ is easily recovered.

Conservation of energy in Eq. (8.2) also simplifies. There is no prescribed axial temperature gradient in x, which is consistent with the axi–symmetric nature of the bearing in Fig. 8.2, i.e. that there is no rotational dependence. Taken with the above observations of independence with respect to t and z, the temperature problem reduces to $T = T(y)$ and Eq. (8.2) simplifies to

$$\frac{d^2T}{dy^2} \;=\; -\,Br\,,$$

where we have used the fact that $du/dy = 1$ from the solution of the momentum problem.

Before integrating this equation to obtain T, let us briefly discuss the boundary conditions that would be appropriate for the stationary and sliding surfaces. In an actual bearing problem, as in Fig. 8.2 for example, the conditions at $y = 0$ and $y = L$ would not be prescribed *per se*, but rather would be incidental to the larger thermodynamics of the bearing as a whole. In particular, we might assume that the frictional heat that is generated would be transferred both into the spinning shaft and the bearing block. Given the establishment of a steady–state, the shaft might reach some elevated, but as yet unknown uniform temperature throughout its cross–section. Heat conducted through the bearing block would probably be governed ultimately at its exterior surface by a Robbins boundary condition of the 3rd kind, perhaps reflecting some sort of forced cooling of the block. From an engineering standpoint, this problem would require simultaneous solution of T throughout the various different parts of the device.

We shall solve only the more "academic" problem limited strictly to the flow. The above discussion suggests that we should prescribe boundary conditions of the first kind, specific temperatures, at both $y = 0$ and $y = L$. Take these as T_0 and T_L, respectively. Switching temporarily back to the "star" notation (*) to define dimensionless temperature, we refer back to §7.3, which suggests

$$T^* \;=\; \frac{T(y) - T_0}{T_L - T_0}\,,$$

so that $T^* = 0$ at the bottom (stator) and $T^* = 1$ at the top (rotor).[8.7] Now, dropping the "star" notation (*) again, the boundary conditions for temperature are

$$T\Big|_{y=0} = 0 \qquad \text{and} \qquad T\Big|_{y=1} = 1$$

whereby the solution is found to be[8.8]

$$T \;=\; -\,\frac{Br}{2}\,y^2 \;+\; \left(1 \;+\; \frac{Br}{2}\right)\,y\,. \tag{8.4}$$

The velocity profile in Eq. (8.3) is linear, having no parameters, but the temperature profile is parabolic and depends upon the Brinkman number (Fig. 8.3). For $Br = 0$, heat transfer is by pure conduction and the profile is linear, $T = y$, from Eq. (8.4). Increasing Br represents progressively larger viscous dissipation in the flow, which affects not only the fluid temperature, but also heat transfer at the boundaries.

[8.7] Implicitly, we are assuming $T_L > T_0$, without loss of generality. This also implies that the definition of the Brinkman number in Eq. (7.21) on pp. 91 for this problem is $Br = (\mu\, u_\infty^2)/(k\,[T_L - T_0])$.

[8.8] The governing equation is integrated twice as $T = -\,Br\,y^2/2 + C_1\,y + C_2$ and a little algebra shows $C_2 = 0$ and $C_1 = 1 + Br/2$.

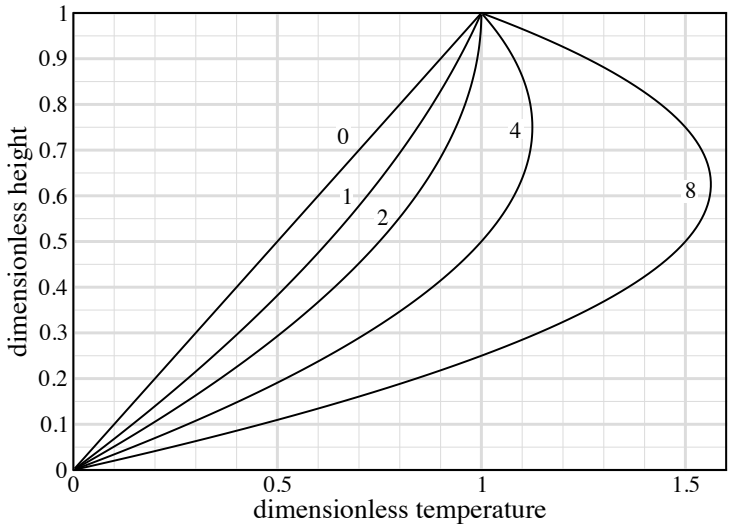


FIGURE 8.3. *Temperature distribution as a function of the Brinkman number in 1–D Couette flow.*

Interestingly, because the convective terms vanish for this problem in the energy equation, heat transfer in the vertical direction can be calculated directly from Fourier's Law. It is relatively straightforward to show

$$q'' \;=\; -\,\frac{k\,(T_L - T_0)}{L}\,\frac{dT}{dy}\,,$$

where dT/dy is dimensionless. The salient point here is that the *sign* of the derivative determines the direction of heat transfer.[8.9] As we just observed, in pure conduction, $T = y$, so $dT/dy = 1$ and $q'' = -k(T_L - T_0)/L$, i.e. a constant rate of heat transfer in the negative y direction, from the rotor to the stator. Also, the maximum (dimensionless) temperature is $T = 1$ at the top plane. The situation can be stated more deftly in terms of the Nusselt number, which can obtained almost effortlessly here because we cast the problem in dimensionless terms at the outset. Recalling from Eq. (7.23) on pp. 93 that the Nusselt number is the dimensionless temperature gradient at a boundary, we can readily see at the bottom plane, for example, that

$$Nu \;=\; \left.\frac{dT}{dy}\right|_{y=0} \;=\; \left.\left(-\,Br\cdot y \;+\; 1 \;+\; \frac{Br}{2}\right)\right|_{y=0} \;=\; 1 \;+\; \frac{Br}{2}\,. \tag{8.5}$$

The heat transfer to the journal housing is linear in Br and for the special case of $Br = 1$, we have $Nu = 1$, consistent with the discussion above.

Fig. 8.3 clearly shows that non–zero Brinkman numbers raise the fluid temperature, and a sufficiently large Br actually results in fluid temperatures that exceed T at the top boundary. Here, the slope flips sign and heat transfer is in the positive y direction, i.e. *to* the rotor, while still maintaining its movement in the negative y direction at the bottom. In essence, there is a net generation of thermal energy and transfer to *both* boundaries. Given the exact solution, we can exactly quantify the terms of this phenomenon. Clearly, the maximum temperature within the

[8.9] Since $T_L > T_0$, we note that the quantity $k(T_L - T_0)/L$ is also greater than zero.

dimensionless domain of $0 \leq y \leq 1$ must occur where $dT/dy = 0$. We can once again take the derivative of the solution in Eq. (8.4), as we just did to calculate Nu. Setting it to zero indicates

$$\frac{dT}{dy} \;=\; -\,Br\cdot y \;+\; 1 \;+\; \frac{Br}{2} \;=\; 0$$

is the relevant equation and its solution

$$y_{max} \;=\; \frac{1}{Br} \;+\; \frac{1}{2} \tag{8.6}$$

determines where the maximum, if any, resides. Observing once again that the physical domain of this problem is limited to $y \leq 1$, this equation is only satisfied for $Br \geq 2$. The physical interpretation is as follows

- $Br < 2$: Fluid in the domain does not exceed the maximum $T = 1$ at the top wall.
- $Br = 2$: A maximum exists and happens to be equal to $T = 1$ and this maximum occurs *at* the top wall. Since, by definition, $dT/dy = 0$ here, there is no heat transfer either way at this location. Tendency of upward heat transfer from the generation/dissipation effect is exactly balanced by the tendency for downward heat transfer due to the temperature difference between the two boundaries. The top boundary is effectively adiabatic.
- $Br > 2$: A maximum exists and it resides within the fluid domain according to Eq. (8.6) and this maximum exceeds $T = 1$. Its magnitude is readily calculated by substituting Eq. (8.6) into Eq. (8.4).[8.10] The y location of maximum temperature and its value are shown in Fig. 8.4, as is Nu at the bottom wall. As $Br \to \infty$, the location of the maximum asymptotically approaches mid–channel, $y = 1/2$, according to Eq. (8.6), but the maximum temperature continues to grow linearly.

It is important to keep in mind that the entire modeling framework we have assumed thus far for convection is based upon material properties holding constant. Although high Brinkman numbers can certainly be obtained, the resulting range of temperatures would be very large, so property variation would become important for most fluids, as would the tendency to de–stabilize into more sophisticated fluid motions (Joseph, 1964, 1965). These are more complicated dynamics and outside our current scope. Consequently, the trends shown in Fig. 8.4 for the limit of large Br are not physically obtainable for many real fluids.

[8.10] The maximum temperature, T_{max}, thus obtained is

$$\begin{aligned}
T_{max} &= -\frac{Br}{2}\left(\frac{1}{Br}+\frac{1}{2}\right)^2 + \left(1+\frac{Br}{2}\right)\left(\frac{1}{Br}+\frac{1}{2}\right) \\
&= -\frac{Br}{2}\left(\frac{1}{Br^2}+\frac{1}{Br}+\frac{1}{4}\right) + \frac{1}{Br}+\frac{1}{2}+\frac{1}{2}+\frac{Br}{4} \\
&= -\frac{1}{2\,Br} - \frac{1}{2} - \frac{Br}{8} + \frac{1}{Br} + 1 + \frac{Br}{4} \\
T_{max} &= \frac{Br}{8} + \frac{1}{2} + \frac{1}{2\,Br}\,.
\end{aligned}$$

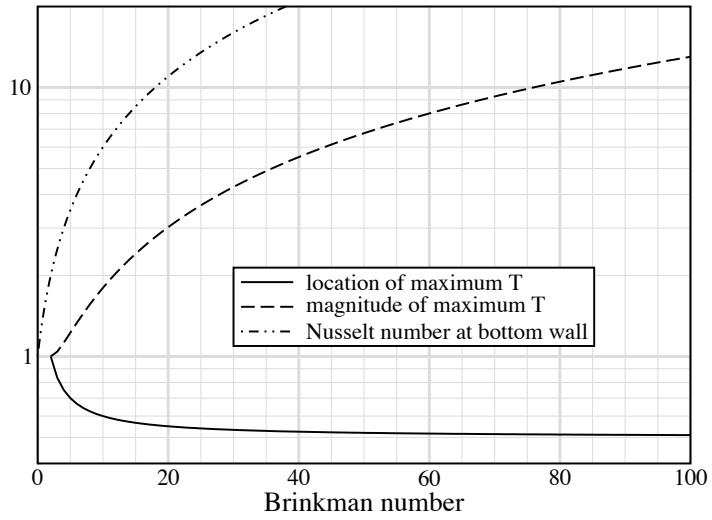


FIGURE 8.4. *Maximum temperature and its location in 1–D Couette flow and Nusselt number at the bottom wall.*

8.3. The General Couette Problem

The planar Couette configuration in Fig. 8.2 depends only on one independent coordinate. Consequently, it is a rather straightforward problem, by which we mean that it can be solved by direct integration. The physical interpretation of that model is that there is no variation in the direction "into and out of the paper", or equivalently, that there are no boundaries in that direction. The device's depth dimension is infinite. Here, we will add the physically–relevant restriction of end–boundaries, so that the device is finite, i.e. physically realizable, as shown in Fig. 8.5. Once

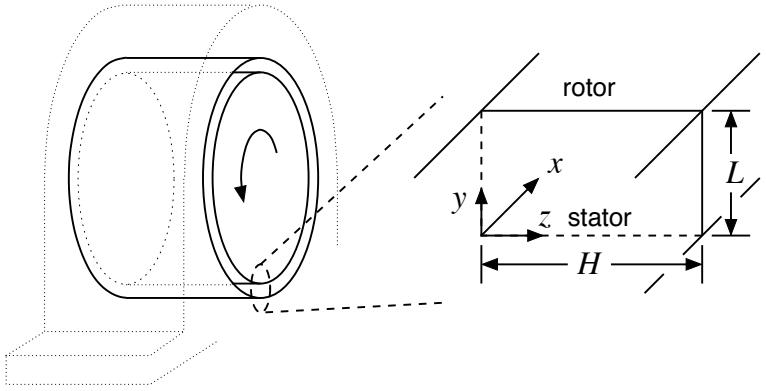


FIGURE 8.5. *Parallel Couette flow in the rectangular cross–section as a model for a finite–width concentric–cylinder (rotor–stator) journal bearing having a very thin gap. As in Fig. 8.2 on pp. 96, the cylindrical configuration (left panel) is well–approximated in rectangular geometry (right panel). However, this model is more physically realistic than the planar version in Fig. 8.2 because it specifies end–surfaces that contain the fluid.*

again the flow is in the x direction, but is now bounded by physical surfaces in both the y and z coordinates. That is, it takes place in the cross–sectional domain $0 \le y \le L$ and $0 \le z \le H$.

Let us first be very systematic in defining the problem, especially in terms of its dimensionless aspect. If we maintain the dimensionless basis of the planar problem in y from the previous section, then the domain in z is non–dimensionalized as

y coordinate	z coordinate
$0 \le y \le L$	$0 \le z \le H$
$0 \le \dfrac{y}{L} \le 1$	$0 \le \dfrac{z}{L} \le \dfrac{H}{L}$

according to our design in §7.3 on pp. 89. Therefore, let us define the *aspect ratio* of the domain as $\phi = H/L$, so that if we again drop the "star" notation (*), the dimensionless problem domain is

$$0 \le y \le 1 \qquad \text{and} \qquad 0 \le z \le \phi \,,$$

as shown in Fig. 8.6. As with the planar problem, we will take the flow here to be steady–state

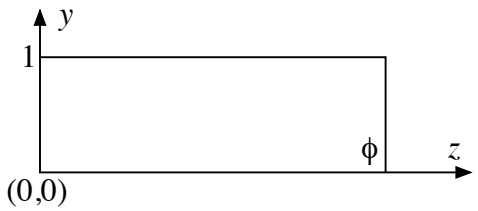


FIGURE 8.6. *Dimensionless domain for the finite–width Couette problem. Flow is the x direction, i.e. "into the paper".*

and have no gradients in the flow direction, i.e. in x, including a pressure gradient, which is again consistent with the axi–symmetric nature of the bearing, i.e. that there is no rotational dependence. Conservation laws for momentum and energy in Eq. (8.1) and Eq. (8.2) then simplify, respectively, to[8.11]

$$\frac{\partial^2 u}{\partial y^2} \; + \; \frac{\partial^2 u}{\partial z^2} \; = \; 0 \tag{8.7}$$

$$\frac{\partial^2 T}{\partial y^2} \; + \; \frac{\partial^2 T}{\partial z^2} \; = \; - \, Br \left[\left(\frac{\partial u}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial z} \right)^2 \right] , \tag{8.8}$$

where $u = u(y, x)$ and $T = T(y, z)$. Boundary conditions on velocity are much the same as for the planar problem, with the exception of adding the end–boundaries. As this is viscous flow, all

[8.11] Eq. (8.7) is, by now, immediately recognized as a Laplace equation. Likewise, Eq. (8.8) has the Laplacian operator on the left, but has terms on the right–hand–side, as well. In the order of operations of solving this system, these terms are *known* because the problem in u is addressed first. Consequently, the right–hand–side is taken as a source term, in which case expressions of this form are known as *Poisson equations*.

boundaries are "no–slip", implying

$$u\Big|_{y=0} = u\Big|_{z=0} = u\Big|_{z=\phi} = 0 \qquad \text{and} \qquad u\Big|_{y=1} = 1\,, \tag{8.9}$$

the last equation at $y = 1$ representing the motion "into the paper" of the rotating shaft.

For the energy equation, we will make a slight change, a simplification actually, because this problem is already quite a bit more complex than its planar cousin in the previous section.[8.12] Specifically, instead of different boundary temperatures for the rotor and stator (and perhaps even the additional end–boundaries here), let us take all surfaces in Fig. 8.6 to be at the same constant temperature, T_0. The viscous dissipation is then the only mechanism "driving" the temperature problem. This implies a slight modification of the dimensionless temperature and the Brinkman number from the planar Couette problem. Switching momentarily to the "star" notation (*), we define[8.13]

$$T^* = \frac{T(y,z) - T_0}{T_0} \qquad\qquad Br = \frac{\mu\, u_0^2}{k\, T_0}\,.$$

Consequently, the boundary conditions for temperature are homogeneous of the first kind

$$T^*\Big|_{y=0} = T^*\Big|_{z=0} = T^*\Big|_{z=\phi} = T^*\Big|_{y=1} = 0\,. \tag{8.10}$$

Of course, the problem for $u(y,z)$ could readily be managed using SOV, although the one for $T(y,z)$ is clearly much more difficult. In the interest of continuity, we shall address both using the eigen–function integral transform method introduced in chapter 6.[8.14] A detailed analysis is given in appendix H starting on pp. 167, with the solution for $u(y,z)$ shown in Eq. (H.5) on pp. 169 and the solution for $T(y,z)$ given by Eq. (H.15) on pp. 178.[8.15]

The immediate question revolves around how the idealized (planar) Couette problem differs quantitatively from the finite configuration of this section, but the answer actually depends to some degree on the point–of–view from which the question is being asked. For instance, if one is concerned strictly with the temperature distribution at the centerline, the issue is really framed as finding how far away the side walls of the device must be before their effect is negligibly small at the center. This question is readily answered by merely plotting $T(y, \phi/2)$ for various aspect ratios alongside the planar result in Eq. (8.4) on pp. 97.[8.16] Some such curves are plotted in Fig. 8.7. It is immediately clear that the centerline temperature profiles converge very quickly to the 1–D result and that for $\phi > 2$, wall effects will be negligible strictly at the center. Conversely, small values of ϕ result in asymmetries that shift the regions of maximum temperature toward the sliding

[8.12] We will handle the overall finite problem using the integral transform technique established in chapter 6 specifically, as developed by Wendl and Agarwal (2002). Therefore, we are not actually making the simplification for mathematical expedience, but rather simply because the lengths of mathematical expressions themselves borderline on being unwieldy.

[8.13] The basic Brinkman number is defined in Eq. (7.21) on pp. 91. The specific form for the planar Couette problem was given in footnote 8.7 on pp. 97.

[8.14] We note that the momentum problem has actually been solved already in the form of the equations shown in §E.3, though the coordinate basis is different. Also, that particular treatment is based on the physical (dimensional) rather than a dimensionless one. While the appropriate transform could be devised to use this result directly, it will arguably be more convenient to rework it according to the problem specification here.

[8.15] The reader is referred to those results. They are not repeated here because $T(y,z)$ is especially lengthy.

[8.16] Here, we must redefine boundary conditions on the 1 dimensional problem to match those of the 2–D problem, i.e. equal temperatures at the boundaries.

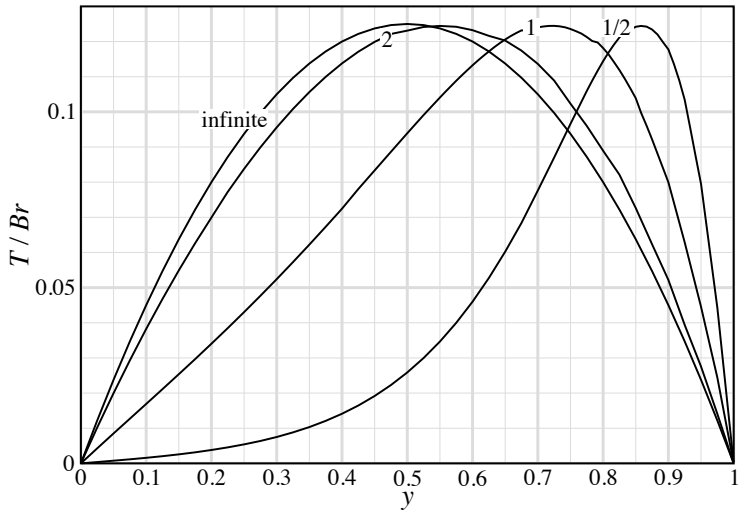


FIGURE 8.7. *Universal temperature profiles (temperature divided by Brinkman number) at the centerline for aspect ratios of 2, 1, 0.5 compared to planar profile, which implies an infinite aspect ratio.*

surface. The reason for this behavior is relatively easy to see if we look at the corresponding velocity profiles at the centerline (Fig. 8.8). Smaller aspect ratios, which imply increased wall effects from

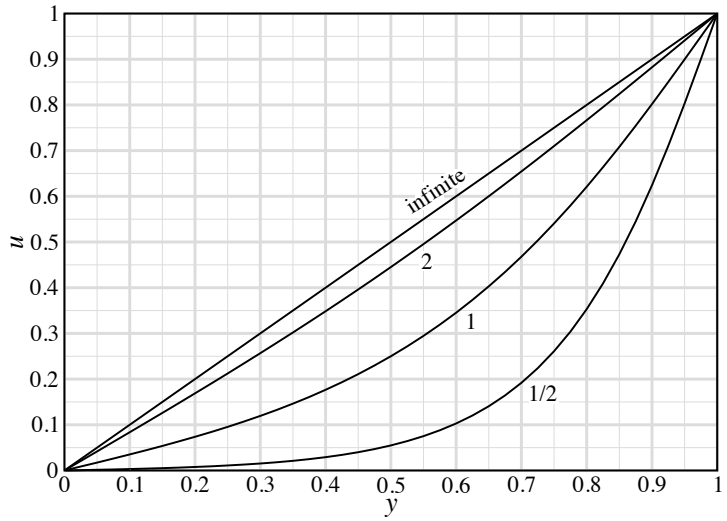


FIGURE 8.8. *Centerline velocity profiles corresponding to the respective temperature curves in Fig. 8.7.*

the closer proximities of the solid surfaces at $z = 0$ and $z = \phi$, impart more "drag" on the fluid motion and velocities in the middle of the channel, i.e. around $y = 1/2$, are dramatically reduced. More importantly, the *gradient*, $\partial u/\partial y$, is reduced, meaning that the dissipation source term in the conservation of energy statement, Eq. (8.8) on pp. 101, is smaller. Conversely, the velocity profile must recover for $y > 1/2$ such that it still satisfies the boundary condition at the sliding

surface, so the gradient is even higher in this region because it has to "make up the shortfall" from the higher–drag, lower–velocity center region. This is particularly obvious for the $\phi = 1/2$ velocity curve. Consequently, the dissipation effect is much higher near the sliding boundary for these cases, which is commensurately reflected by shifts in the temperature profile.

The other, broader point–of–view is to examine the differences over the whole flow domain. The planar 1–D solution in the depth dimension is simply an infinite continuation of the symmetric (parabolic) temperature curve in Fig. 8.7 and it has no sense of satisfying any sort of "wall boundary conditions" in that dimension. Conversely, the actual 2–D profile is significantly different because of these wall boundaries. The profile for the particular case of $\phi = 1$ is plotted in Fig. 8.9. The

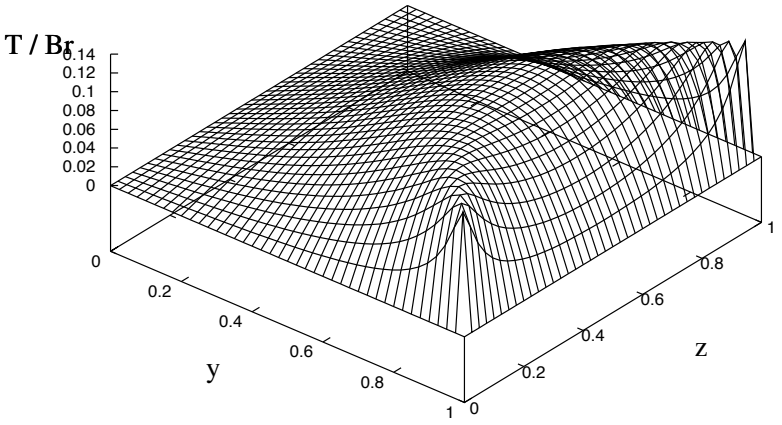


FIGURE 8.9. *$T(y, z)/Br$ plotted as a surface for 2–D Couette flow having homogeneous boundary conditions and an aspect ratio of $\phi = 1$.*

shifting of the maximum temperature at the centerline of $z = 1/2$ is apparent. However, it is also clear that $T(y, 1/2)$ taken alone leaves much to tell. For example, it is easy to "ridges" of high temperature that run roughly from the middle region of the domain along diagonals toward the 2 corners where the velocity discontinuities are: $(y, z) = (1, 0)$ and $(1, 1)$. Of course, this phenomenon makes complete sense in light of our discussion above regarding the gradient $\partial u/\partial y$. Specifically, the flow not only has to meet the sliding plate boundary condition as $y \to 1$, but it also has to meet the no–slip conditions of the stationary walls as $z \to 0, 1$. Consequently, not only will $\partial u/\partial y$ be high in these corner regions, but $\partial u/\partial z$ will be, as well! Thus, the source term in the energy equation is even commensurately higher. The very large gradients shown in Fig. 8.9 near these corner regions indicates that heat transfer in both the y and z directions will be highest here. In general, heat transfer will also be much higher to the rotor at $y = 1$ than to the bearing itself at $y = 0$, although even along the rotor the heat transfer is clearly varying, being again highest at the corners and lower in the middle.

8.4. Convection in a Circular Duct

Finally, let us now consider the very common configuration of steady, fully–developed, laminar flow in a pipe (Fig. 8.10). The equations are nominally the same as those discussed in §8.1 for

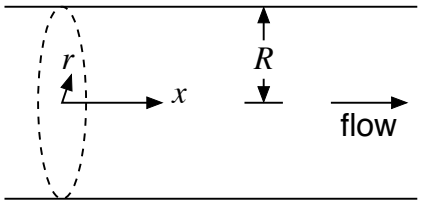


FIGURE 8.10. *Convection resulting from fully–developed laminar flow in a circular cross–section pipe.*

parallel flow, except of course where we write them in cylindrical coordinates. Referencing appendix I and invoking the same restrictions for steady, parallel flow discussed in §8.1, we find that the momentum equation simplifies to

$$\frac{1}{r}\,\frac{d}{dr}\left(r\,\frac{du}{dr}\right) \;=\; \frac{1}{\mu}\,\frac{\partial P}{\partial x}\,, \tag{8.11}$$

where it is important to note that we have changed the partial differential signs, ∂, to regular differential signs, d, to reflect the fact that the cylindrical coordinate system describes the circular pipe cross–section using only r, i.e. just 1 coordinate.[8.17] Consequently, this is an ODE whose solution has the form $u = u(r)$.

For the energy problem, we will invoke the common simplification of no heat generation (Özişik, 1985; Bergman et al., 2011), as well as the temperature profile being axi–symmetric, i.e. $\partial/\partial\theta = 0$, whereby the energy equation is

$$u\,\frac{\partial T}{\partial x} \;=\; \alpha\,\left[\frac{\partial^2 T}{\partial x^2} \;+\; \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)\right]\,. \tag{8.12}$$

Here, we have not assumed single–variable dependence upon r and are instead allowing for the possibility that the temperature could develop along the flow axis, x, as well, i.e. $T = T(x, r)$. This contingency is intuitive, given that we might specify, for instance, some particular heating rate along the axis.

Eq. (8.11) describes the basic flow for all laminar fully–developed convection scenarios in a circular pipe. It is readily solved by direct integration as[8.18]

$$u(r) \;=\; \frac{1}{\mu}\,\frac{\partial P}{\partial x}\,\frac{r^2}{4} \;+\; C_1\,\ln(r) \;+\; C_2\,,$$

[8.17] Contrast this, for example, to the rectangular cross–section for the Couette problem §8.3, which required 2 coordinates, y and z. This necessitated a partial rather than ordinary differential equation.

[8.18] That is, the pressure gradient and the viscosity are assumed to be known and constant. Consequently, Eq. (8.11) is simply integrated twice.

where C_1 and C_2 are constants of integration. Using the boundary conditions for evaluation, we then find the standard parabolic laminar flow profile[8.19]

$$u(r) \;=\; 2\,\overline{u}\left[1-\left(\frac{r}{R}\right)^2\right]\,, \tag{8.13}$$

where $\overline{u}$ is the *prescribed* average velocity (see footnote 8.19). Eq. (8.13) is then the velocity profile in Eq. (8.12), for which a variety of configurations can be modeled with appropriate boundary conditions.

8.5. The Case of Constant Prescribed Heat Flux

There are 2 common cases of laminar pipe flow convection that are widely discussed in texts, the first being the scenario where a constant heat flux is applied to the pipe wall along the flow direction. This is a very good physical model not only for actual configurations where uniform heating is applied, e.g. electric resistance heating or radiant heating, but also for counter–flow heat exchangers where the heat capacity rates are comparable.[8.20] Mathematically speaking, this problem is relatively straightforward, once we have resolved a subtle issue in its formulation. That issue involves identifying a suitable reference temperature is much more a matter of insightful deduction, as we now show.

Thus far, we have always had the benefit of an obvious reference temperature to serve as a basis for non–dimensionalization, use in Newton's Law of Cooling, definition of the Brinkman number, etc. Here, there is neither a far–field "freestream" (constant) temperature, as there routinely is

---

[8.19] One boundary condition is immediately given by the no–slip requirement, $u = 0$ at $r = R$, where R is the pipe radius, but the second boundary condition may not be completely obvious. It can be derived from either of the following physical observations:

- the mathematical function is "well–behaved", so that the natural log term must remain finite at $r = 0$
- because of symmetry of the overall problem, the velocity profile must also be symmetric about $r = 0$

Both these observations yield the same result that $C_1 = 0$. For example, $u|_{r=0} \;=\; 0 + C_1 \ln 0 + C_2 \;=\; \text{finite}$, so that $C_1 = 0$. Applying no–slip, we find

$$u|_{r=R} \;=\; \frac{1}{\mu}\cdot\frac{dP}{dx}\cdot\frac{R^2}{4} \;+\; C_2 \;=\; 0 \qquad \text{so that} \qquad C_2 \;=\; -\,\frac{1}{\mu}\cdot\frac{dP}{dx}\cdot\frac{R^2}{4}\;\cdot$$

Substituting C_1 and C_2 and simplifying, we find the exact solution is

$$u(r) \;=\; -\,\frac{1}{4\,\mu}\cdot\frac{dP}{dx}\,R^2\left[1-\left(\frac{r}{R}\right)^2\right]\,,$$

however, it is even more convenient to represent this result in a more compact form using the average velocity, $\overline{u}$. Using the integral volume flow rate formula (Munson et al., 2006) with a differential "ring element", we find

$$\begin{aligned} Q \;&=\; \int_A \mathbf{V}\cdot\hat{n}\,dA \;=\; \int_0^R u\,2\pi r\,dr \;=\; -\int_0^R \frac{1}{4\,\mu}\cdot\frac{dP}{dx}\,R^2\left[1-\left(\frac{r}{R}\right)^2\right]\,2\,\pi\,r\,dr \\ &=\; -\,\frac{\pi\,R^2}{2\,\mu}\cdot\frac{dP}{dx}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]\,r\,dr \;=\; -\,\frac{\pi\,R^2}{2\,\mu}\cdot\frac{dP}{dx}\left[\frac{r^2}{2}-\frac{r^4}{4\,R^2}\right]\Bigg|_0^R \\ &=\; -\,\frac{\pi\,R^2}{2\,\mu}\cdot\frac{dP}{dx}\cdot\frac{R^2}{4} \;=\; -\,\frac{\pi\,R^4}{8\,\mu}\cdot\frac{dP}{dx} \qquad \text{then} \qquad \overline{u} \;=\; \frac{Q}{A} \;=\; -\,\frac{R^2}{8\,\mu}\cdot\frac{dP}{dx}\,\cdot \end{aligned}$$

Eq. (8.13) follows directly.

[8.20] In such instances, the bulk temperature difference between the hot–side and cold–side flows will be approximately constant, as can be discerned from a "thumbnail diagram" (Özişik, 1985), whereby heat flux between the streams is likewise approximately constant.

for external flows, nor a defined (also usually constant) boundary temperature on the pipe wall. Instead, we define what is commonly called a "bulk temperature" (Özişik, 1985; Bergman et al., 2011) as our reference based on the rate at which thermal energy passes through an arbitrary cross–section. Here, the dimensional argument observes that c_p has physical units of $J/(kg\,K)$, so c_pT is the thermal energy per unit mass, J/kg, of the flow and, furthermore, this quantity multiplied by the mass flow rate through a cross–section is the rate at which thermal energy passes through that cross–section. Setting the integrated quantity equal to the definition of bulk temperature according to the mass flow rate, we have

$$\iint_A \underbrace{c_p\,T}_{J/kg}\,\underbrace{\rho\,u\,dA}_{kg/sec} \;=\; \dot{m}\,c_p\,T_m\;, \tag{8.14}$$

where T_m is the bulk (mean) temperature of the thermal energy passing through the section. For a circular cross–section, this general principle takes the specific form[8.21]

$$T_m \;=\; \frac{2}{\overline{u}\,R^2}\,\int_0^R u\,T\,r\,dr\;, \tag{8.15}$$

where $\overline{u}$ is again the average velocity derived in footnote 8.19.

Note that T_m as defined by Eq. (8.15) is generally not constant, but will vary with axial distance. Because we will define a constant flux for the boundary, we should also presume that the *unknown* wall boundary temperature of the pipe, T_R, *also* varies axially:

$$T_m \;=\; T_m(x) \qquad \text{and} \qquad T_R \;=\; T_R(x)\;.$$

If the fluid is being heated ($T_R > T_m$), then T_m will increase along the flow direction, and *vice versa*, and, if that is the case, then clearly $dT_m/dx \neq 0$. Invoking the "star" notation (*) again for the moment, let us define the dimensionless form of temperature for this problem as[8.22]

$$T^* \;=\; \frac{T(x,r)\;-\;T_R(x)}{T_m(x)-T_R(x)}\;. \tag{8.16}$$

Here, it is clear that all 3 temperatures change along the axial direction, x.

An important experimental observation in this class of flows is that the dimensionless temperature in Eq. (8.16) is approximately fully–developed, i.e. is not a function of x (Kays and Crawford, 1980), which has several important implications. First, if $\partial T^*/\partial x = 0$, then T^* cannot be a function

[8.21] Here, c_p is assumed constant. For a circular cross–section $A = \pi R^2$ and for the corresponding differential "ring element", $dA = 2\,\pi\,r\,dr$, so that

$$T_m \;=\; \frac{1}{\dot{m}}\iint_A T\,\rho\,u\,dA \;=\; \frac{1}{\rho\,\overline{u}\,\pi\,R^2}\int_0^R T\,\rho\,u\,2\,\pi\,r\,dr$$

implies Eq. (8.15). Note that some texts also call T_m the "mixing cup temperature" (Kays and Crawford, 1980; Burmeister, 1983).

[8.22] Note that this definition of the dimensionless temperature switches the order of the boundary conditions somewhat from what we are used to seeing, i.e. if $r^* = r/R$, then

$$T^*\Big|_{r^*=0} \;=\; 1 \qquad \text{and} \qquad T^*\Big|_{r^*=1} \;=\; 0\;.$$

of x, which implies that $\partial T^*/\partial r$ likewise cannot be a function of x. We can evaluate

$$\left.\frac{\partial T^*}{\partial r}\right|_{r=R} = \left.\frac{\partial}{\partial r}\left[\frac{T(r,x)-T_R(x)}{T_m(x)-T_R(x)}\right]\right|_{r=R} = \frac{1}{T_m(x)-T_R(x)}\left.\frac{\partial T}{\partial r}\right|_{r=R}$$

and by the same argument conclude that this result is also not a function of x. In other words, $\partial T^*/\partial r$ at the boundary remains fixed as the flow proceeds in the x direction. Let us define Newton's Law of Cooling the pipe surface as

$$q''_R = h\,(T_R - T_m)\,, \tag{8.17}$$

so that applying heat *to* the pipe from the outside is positive transfer of thermal energy *to* the flow, even though this is in the $-r$ direction. With this sign convection, Fourier's Law of Conduction is[8.23]

$$q''_R = k\left.\frac{\partial T}{\partial r}\right|_{r=R}, \tag{8.18}$$

so that substituting these expressions, we find

$$\frac{1}{T_m(x)-T_R(x)}\left.\frac{\partial T}{\partial r}\right|_{r=R} = -\frac{h}{q''_R}\cdot\frac{q''_R}{k} = \left|\frac{h}{k}\right|,$$

from which we must conclude that the local convection coefficient h is independent of x for fully developed flow.[8.24] Note from Fourier's Law that if h is now constant and q''_R is certainly constant because it is our chosen scenario, then the difference between T_R and T_m must also be constant, which implies that they must be changing at the same rate, i.e.

$$\frac{dT_m}{dx} = \frac{dT_R}{dx}\,.$$

Let us now go even one step further. We evaluate $\partial T^*/\partial x$ using T^* as defined above. Our initial proposition of fully developed conditions requires $\partial T^*/\partial x = 0$. Evaluating, we find

$$\begin{aligned}\frac{\partial T^*}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{T(x,r)\;-\;T_R(x)}{T_m(x)\;-\;T_R(x)}\right]\\ &= \frac{1}{T_m(x)\;-\;T_R(x)}\cdot\frac{\partial\left[T(x,r)\;-\;T_R(x)\right]}{\partial x}\;-\;\frac{T(x,r)\;-\;T_R(x)}{\left[T_m(x)\;-\;T_R(x)\right]^2}\cdot\frac{d\left[T_m(x)\;-\;T_R(x)\right]}{dx}\\ &= 0\,.\end{aligned}$$

[8.23] Note here the conspicuous absence of the negative sign as compared to the conventional definition, e.g. Eq. (2.1) on pp. 8. The absence is because, if $\partial T/\partial r$ is indeed positive, then temperatures closer to the pipe wall are *larger* than those further near the center, and heat transfer is "positive into the fluid", consistent with what we have just defined as the sign convection. The issue is basically that the coordinate r is in the opposite orientation with respect to the conventional flow/boundary arrangement that we have been using thus far. Footnote 8.26 on pp. 111 formalizes this notion.

[8.24] Since the thermal conductivity k is constant, h must not be a function of x if the expression as a whole cannot be a function of x.

Noting that we can multiply through by the non–zero quantity $T_m - T_R$, we can use this expression to *solve* for $\partial T/\partial x$ as

$$\frac{\partial T}{\partial x} \; - \; \frac{dT_R}{dx} \; - \; \frac{T \; - \; T_R}{T_m \; - \; T_R} \left(\frac{dT_m}{dx} \; - \; \frac{dT_R}{dx} \right) \;\; = \;\; 0$$

$$\frac{\partial T}{\partial x} \;\; = \;\; \frac{dT_R}{dx} \; + \; T^* \left(\frac{dT_m}{dx} \; - \; \frac{dT_R}{dx} \right) \tag{8.19}$$

where we have again invoked the definition of T^* from Eq. (8.16). The second term vanishes since it was just shown that $dT_m/dx = dT_R/dx$, indicating $\partial T/\partial x = dT_R/dx$, as well. That is, we ultimately have

$$\frac{\partial T(x,r)}{\partial x} \; = \; \frac{dT_m}{dx} \; = \; \frac{dT_R}{dx} \, . \tag{8.20}$$

In other words, the streamwise (axial) temperature gradient anywhere in the cross–section is equal to the streamwise gradient of the mean temperature and of the boundary wall temperature, if the applied heat flux at the boundary is constant. This result will be used in solving for the temperature distribution $T(x,r)$ in the pipe and, ultimately, the Nusselt number for laminar fully developed flow.

Stepping back, we see that there is still insufficient information to flesh–out the terms in Eq. (8.12) on pp. 105. Specifically, while we have connected $\partial T/\partial x$ to the gradients of T_m and T_R, we still do not know what these terms are, nor do we yet know how to specify $\partial^2 T/\partial x^2$. However, let us consider a simple energy balance that relates the applied heat flux to the change in mean temperature (Fig. 8.11). The rate at which heat is added to the element from the boundary is dq.

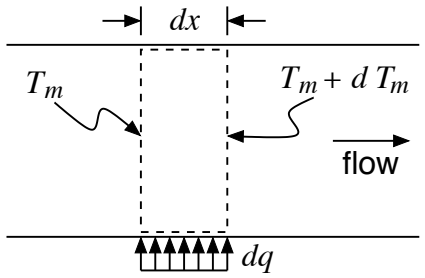


FIGURE 8.11. *Energy balance for a differential heated element laminar flow in a circular cross–section pipe.*

Recalling from Eq. (8.14) that the rate at which thermal energy is convected through a cross–section is $\dot{m} \cdot c_p$ multiplied by the local mean temperature, we can write the balance

$$dq \; + \; \dot{m} \, c_p \, T_m \; = \; \dot{m} \, c_p \left(T_m \; + \; \frac{dT_m}{dx} \, dx \right) \, ,$$

which says simply that the rate of energy entering the left from the flow plus the rate of energy entering from the applied heat at the boundary must equal the rate at which thermal energy leaves via the flow out of the right–side cross–section. Given that the boundary heating rate is simply the

flux times the differential area, i.e. $dq = q''_R\, dA = q''_R\, P\, dx$ (where P is the perimeter, or actually the circumference for the special case of a circular cross–section), a little algebra shows

$$\frac{dT_m}{dx} \;=\; \frac{q''_R\, P}{\dot{m}\, c_p} \;=\; \text{a constant.}$$

This observation now unlocks the basic framing of the problem, because, by virtue of Eq. (8.20), we can prescribe $\partial T/\partial x$ in Eq. (8.12) and, because that quantity is now seen to be a constant, $\partial^2 T/\partial x^2$ is clearly 0. Moreover, we know u from the solution of conservation of momentum given by Eq. (8.13) on pp. 106. Assembling all of these components into the conservation of energy statement in Eq. (8.12) on pp. 105, we see that the raw "constant–flux" version of convective pipe flow, i.e. as it looks before any simplification, is described by

$$2\,\overline{u}\left[1-\left(\frac{r}{R}\right)^2\right]\frac{q''_R\, P}{\dot{m}\, c_p} \;=\; \alpha\,\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right),$$

and, after a little work can be put into the form

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right) \;=\; \frac{4\,q''_R}{k\,R}\,r\left[1-\left(\frac{r}{R}\right)^2\right],$$

where we have simplified using $A = \pi R^2$, $\dot{m} = \rho\overline{u}A$, $P = 2\pi R$, etc. This expression is something of an anomaly because formally $T = T(x, r)$ suggesting that this is indeed a PDE, as shown on the left–hand side, however we have also already exploited several observations that show that the right–hand side is only a function of r and not of x. Let us reconcile this issue by re–framing in terms of the dimensionless temperature, T^* in Eq. (8.16) on pp. 107, which is fully–developed and therefore not a function of x[8.25]

$$\frac{d}{dr}\left(r\,\frac{dT^*}{dr}\right) \;=\; \varphi\, r\left[1-\left(\frac{r}{R}\right)^2\right] \qquad \text{where} \qquad \varphi \;=\; \frac{4\,q''_R}{k\,R\,(T_m \;-\; T_R)} \quad \text{is constant} \tag{8.21}$$

and where the equation is now clearly an ODE.

As with the flow profile problem, this equation can be solved by direct integration, although there are a few artful operations, including re–using the mean temperature definition to eliminate parameters. These operations, shown in appendix J, specifically §J.1 on pp. 184, ultimately yield a "universal profile" of

$$T^* \;=\; \frac{96}{11}\left(\frac{r^{*\,4}}{16} \;-\; \frac{r^{*\,2}}{4} \;+\; \frac{3}{16}\right), \tag{8.22}$$

where $r^* = r/R$, as originally defined in footnote 8.22 on pp. 107. A similar "universal profile" is easily written directly from Eq. (8.13) on pp. 106 by defining $u^* = u/\overline{u}$, so that

$$u^* \;=\; 2\left(1-r^{*\,2}\right). \tag{8.23}$$

These universal profiles are plotted in Fig. 8.12.

As a final step, let us calculate the Nusselt number for the constant heat flux configuration. Owing to the interpretation of Eq.(7.23) on pp. 93 of Nu being simply the dimensionless temperature gradient at the boundary, we could, in principle determine the Nusselt number directly from

[8.25] It is a subtle point worth repeating. Even though Eq. (8.16) nominally indicates that T^* is a function of x, we have shown that T, T_R, and T_m all vary with x at the same rate, which was the finding of Eq. (8.20). This means their *differences* in Eq. (8.16) are *constant* with respect to x.

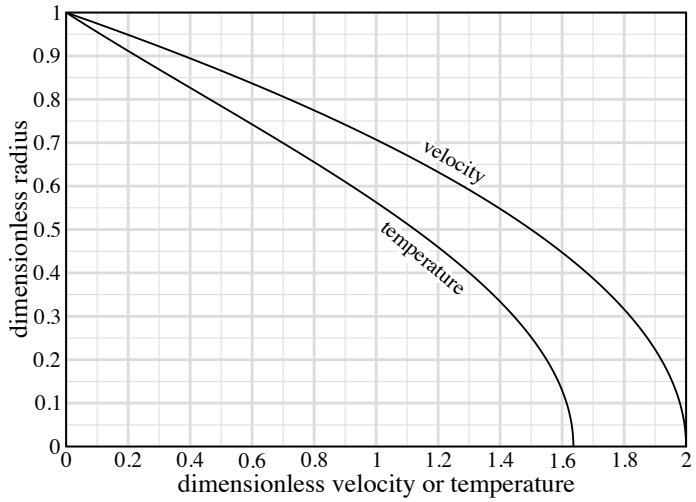


FIGURE 8.12. *Universal profiles for steady, fully–developed flow in a circular pipe, as specified by Eq.* (8.23)*, and the corresponding temperature profile for constant flux boundary condition, as given by Eq.* (8.22)*. Profiles are axi–symmetric, so they are actually "shapes of revolution" around* $r^* = 0$*.*

Eq. (8.22). However, there are 2 subtleties for which we would have to account. First, the conventional length scale for pipe flow is the pipe diameter, not the radius upon which r^* is predicated. Second, the Nu was defined in the context of a coordinate whose origin was the wall and pointed into the flow domain. Here, r originates at the center (axis) is the flow and points toward the wall, so an extra negative sign would be needed to resolve this difference.[8.26] Conversely, we could also pursue the more conventional argument introduced in §7.4 of equating convection and conduction at the boundary, in this case Eqs. (8.17) and (8.18), respectively, to find

$$h\,(T_R - T_m) \;=\; k\,\frac{\partial T}{\partial r}\,\bigg|_{r=R}\,, \tag{8.24}$$

from which we ultimately find[8.27]

$$Nu \;=\; \frac{48}{11} \;\approx\; 4.364\,.$$

[8.26] It is very easy to demonstrate this formally, as well. Let $y = R - r$ be a new coordinate with an origin at the pipe wall and pointing toward the center. Then, by Chain Rule

$$\frac{d}{dr} \;=\; \frac{d}{dy}\,\frac{dy}{dr} \;=\; -\,\frac{d}{dy}\,\cdot$$

This formalizes the notion discussed previously in Footnote 8.23 on pp. 108.

[8.27] Since Eq. (8.22) for T^* can be trivially re–written in terms of dimensional quantities, i.e.

$$\frac{T \,-\, T_R}{T_m \,-\, T_R} \;=\; \frac{96}{11}\left[\frac{1}{16}\left(\frac{r}{R}\right)^4 \,-\, \frac{1}{4}\left(\frac{r}{R}\right)^2 \,+\, \frac{3}{16}\right]$$

it is easy to see that

$$\frac{\partial T}{\partial r}\bigg|_{r=R} \;=\; (T_m \,-\, T_R)\,\frac{96}{11}\left[\frac{1}{4}\,\frac{r^3}{R^4} \,-\, \frac{1}{2}\,\frac{r}{R^2}\right]\bigg|_{r=R} \;=\; \frac{T_m \,-\, T_R}{R}\cdot\frac{96}{11}\left(-\,\frac{1}{4}\right) \;=\; -\,\frac{24\,(T_m \,-\, T_R)}{11\,R}\,\cdot$$

Equating the conduction and convection heat transfers at the wall, as shown in Eq. (8.24), a little algebra readily shows the result $Nu = hD/k = 48/11$ if substituting $R = D/2$.

8.6. Remarks on the Case of Constant Prescribed Temperature

The constant heat flux configuration in the previous section appears in most introductory books (e.g. Özişik, 1985; Bergman et al., 2011) because the energy equation reduces to an ODE and is therefore not terribly difficult to solve.[8.28] As a general rule, these same texts will report the solution for another important case, the constant temperature boundary, but invariably do not show its derivation. Indeed, the same is true for advanced texts (Kays and Crawford, 1980; Burmeister, 1983). The result that is uniformly reported is

$$Nu \;\approx\; 3.66\,.$$

The constant–temperature boundary configuration is an important model for a broad range of heat–exchangers having nearly constant wall temperatures, for example where there is a phase change (evaporator or condenser), or where the heat capacity of one of the streams is much larger than the other.

Here, we will only make a few remarks on this configuration to point out that its proper consideration is more complex than most previous treatments generally acknowledge.[8.29] The usual approach is again based on the premise that T^*, as defined in Eq. (8.16) on pp. 107, does not change in the flow direction, i.e. is fully–developed. That observation, taken with the prescribed constant–temperature boundary condition, leads to the conclusion from Eq. (8.19) on pp. 109 that

$$\frac{\partial T}{\partial x} \;=\; T^*\,\frac{dT_m}{dx}\,, \tag{8.25}$$

so that the conservation of energy statement that properly governs this case is derived from Eq. (8.12) on pp. 105

$$u\,T^*\,\frac{1}{\alpha}\cdot\frac{dT_m}{dx} \;=\; \underbrace{\frac{\partial^2 T}{\partial x^2}}_{\substack{\text{axial}\\ \text{conduction}}} + \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial T}{\partial r}\right)\,.$$

For the constant flux boundary condition in the previous section, $\partial T/\partial x = dT_m/dx$ was likewise a constant, which subsequently meant that the phenomenon of axial conduction, as quantified by the second–derivative term, vanished *identically*. Strictly speaking, there is no such simplification here. Indeed, it can be shown that[8.30]

$$T_m(x) \;=\; \big[T_m(0) \;-\; T_R\big]\,e^{-\frac{h\,P}{\dot m\,c_p}\,x} \;+\; T_R\,,$$

i.e. that $T_m(x)$ asymptotically approaches the constant boundary temperature, T_R, which leads to

$$\frac{\partial^2 T}{\partial x^2} \;=\; \big[T_m(0) \;-\; T_R\big]\,\left(\frac{h\,P}{\dot m\,c_p}\right)^2 e^{-\frac{h\,P}{\dot m\,c_p}\,x}\,.$$

The contribution of this term may or may not be negligible, depending upon the combination of parameters, although, as mentioned above, most texts do in fact examine this configuration in

[8.28] The solution is *tedious*, but does not require advanced mathematical techniques of the kind we have called on for some other problems.

[8.29] We are not making the claim that $Nu \approx 3.66$ is an unreasonable model for certain *specific* cases, only that the problem formulation, as posed in almost all texts that one might care to examine, omits some important details relevant to the inherently more general nature of this case versus that of constant heat flux.

[8.30] See Eqs. (J.1) and (J.3) on pp. 186, respectively, for T_m and the second–derivative of $\partial^2 T/\partial x^2$.

the context of dropping it altogether (Kays and Crawford, 1980; Burmeister, 1983; Özişik, 1985; Bergman et al., 2011). One might argue that $\partial^2 T/\partial x^2$ becomes negligible if x is large enough or if $T_m(0) - T_R$ is small enough, but under these conditions, we also see that $T_m(x) \to T_R$, meaning the flow would have essentially attained the (uninteresting) state of thermal equilibrium. Rather, the admissibility of neglecting axial conduction depends on $(hP)/(\dot{m}c_p)$, which, if its value is sufficiently small, will result in a shallow curve for $T_m(x)$, whose slope changes very slowly, and small $\partial^2 T/\partial x^2$.

It is not entirely trivial to assess the size of $(hP)/(\dot{m}c_p)$ for a typical application, but dimensionless analysis again "comes to the rescue" if we observe that[8.31]

$$\frac{h\,P}{\dot{m}\,c_p} \;=\; \frac{4\,Nu}{D\cdot Re\,Pr} \;\approx\; \frac{10}{D\cdot Re\cdot Pr}\,, \qquad \text{where} \qquad Re \le 2100\,. \tag{8.26}$$

The figure of 2100 for the Reynolds number is the rough limit under which the laminar flow model can be assumed to be valid (Munson et al., 2006).

It is true that Nu is precisely what we do *not* know for this configuration. However, if we assume for estimation purposes that it does not differ significantly, i.e. by orders of magnitude, from the figure of $Nu \approx 3.6$ quoted above, we arrive at Eq. (8.26) as a plausible approximation. The picture is now much more clear: any flow for which the combination of parameters in the denominator is large admits the assumption of negligible axial conduction in the flow, while small combinations negate it. The latter includes low Reynolds number flows, especially when pipe diameter itself is small, as well as for any flow where Pr is small, e.g. liquid metals. For example, mercury has a Prandtl number on the order of 0.01 (Özişik, 1985). This aspect is critical in determining to what degree the classical solution actually applies.

If the vanishing axial conduction assumption is not valid, the problem becomes *significantly* more complicated, not so much because the conservation law for energy, Eq. (8.12) on pp. 105, is itself especially difficult to solve, but rather because of uncertainty in its boundary conditions. That is, the equation cannot be reduced to an ODE, remaining a full PDE, and therefore requiring boundary conditions for T at suitable upstream and downstream locations. For the latter, one might specify a "long pipe", i.e. $T(x, r) \to T_R$ for $x \to \infty$, implying the flow eventually reaches thermal equilibrium.

The more problematic boundary is the upstream one. An obvious boundary condition would be a uniform "inlet" temperature profile, but in such cases, the velocity flow profile will itself generally *not* be fully developed. In other words, the developing flow "entrance" problem will have to be solved, as well, meaning we are now dealing with non–linear terms in the momentum equations. The exception, of course, is for high–Pr fluids, whose hydrodynamic entry length is very short compared to their thermal entry length (Özişik, 1985) and for such problems we would simply

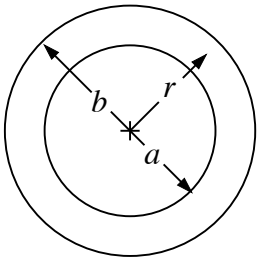
[8.31] For the circular pipe, the perimeter is $P = 2\pi R$ and $\dot{m} = \rho\overline{u}\pi R^2$. Observing that the diffusivity is $\alpha = k/(\rho c_p)$ and furthermore, $Pr = \nu/\alpha$, as defined in Eq. (7.20) on pp. 91. Moreover, given that D and $\overline{u}$ are the length and velocity scales, respectively, for pipe flow, we have $Nu = hD/k$ from Eq. (7.23) on pp. 93 and $Re = \overline{u}D/\nu = 2\overline{u}R/\nu$ from Eq. (7.18) on pp. 90.

$$\frac{h\,P}{\dot{m}\,c_p} \;=\; \frac{h\,2\,\pi\,R}{\rho\,\overline{u}\,\pi\,R^2\;c_p} \;=\; \frac{2\,h}{\rho\,\overline{u}\,R\,c_p} \;=\; \underbrace{\frac{2\cdot k\cdot Nu}{D\,(\rho\,c_p)}}_{\frac{2\,h}{\rho\,c_p}} \times \underbrace{\frac{2}{\nu\cdot Re}}_{\frac{1}{\overline{u}\,R}} \;=\; \frac{4\,\alpha\cdot Nu}{\nu\,D\cdot Re} \;=\; \frac{4\,Nu}{D\cdot Re\,Pr}$$

invoke the original assumption of vanishing axial conduction, since the combination in Eq. (8.26) is likely to be acceptably small.

8.7. Exercises

8.1 The most basic model of Couette flow is the so–called "thin gap" configuration for infinite parallel plates. However, if we want to account for *curvature* of the kind that would approximate a rotating cylinder within a journal with a wide gap, the governing equations for momentum and energy are



$$\frac{d}{dr}\left[\frac{1}{r}\,\frac{d}{dr}\Big(r\,u\Big)\right] \;=\; 0$$

$$k\,\frac{1}{r}\,\frac{d}{dr}\left[r\,\frac{dT}{dr}\right] \;=\; -\,\mu\left(\frac{du}{dr}\;-\;\frac{u}{r}\right)^2\,,$$

where u is the velocity in the rotation direction, T is temperature, and μ and k are fluid viscosity and thermal conductivity, respectively. The inner and outer cylinder radii are a and b, respectively.

Determine the velocity profile, $u(r)$, if the inner cylinder rotates counter–clockwise at a rate ω, while the outer is stationary.

8.2 Referring to exercise 8.1: Determine the temperature profile, $T(r)$, if both the inner and outer cylinder surfaces are maintained at a constant boundary temperature of T_0.

8.3 Referring to exercise 8.2: Show that the solution can be non–dimensionalized *post hoc*, being expressible as

$$T^*(r^*) \;=\; \frac{Br}{1\;-\;R^{-2}}\left[1\;-\;\frac{1}{r^{*2}}\;-\;\left(1\;-\;\frac{1}{R^2}\right)\frac{\ln(r^*)}{\ln(R)}\right]\;+\;1\,,$$

where $R = b/a$ is the radius ratio, $1 \le r^* \le R$ is the dimensionless radius (independent coordinate), $T^*(r^*) = T(r)/T_0$ is the dimensionless temperature, and

$$Br \;=\; \frac{\mu\,(\omega\,a)^2}{k\,T_0}$$

is the Brinkman number.

8.4 Referring to exercise 8.3: From our study of the idealized problem, we already understand the physical effect of the Brinkman number on the temperature distribution and consequent heat transfer. Use the solution in the previous section to make some brief observations of how the geometry affects the temperature distribution by plotting the solution for an arbitrary Brinkman number, say $Br = 4$, for several different radius ratios.

CHAPTER 9

External Convection

In chapter 8 we discussed several basic cases of convection in confined domains, i.e those domains being fully bounded in directions lateral to the primary flow direction. Although the underlying flows were all viscous, we did not place any real emphasis on discussing the phenomenon of the *boundary layer* because they were all fully–developed, i.e. the partial derivative of a quantity in the flow direction was 0. What this meant in essence was that viscous effects were important throughout the entire flow domain, not just a small part of it.

External flows are a fundamentally different animal in the sense that they are only partially bounded, so there is no basic requirement that the flow ever become fully–developed. Consider the canonical example of 2–D flow over a flat plate, (Fig. 9.1), where a uniform approach flow of

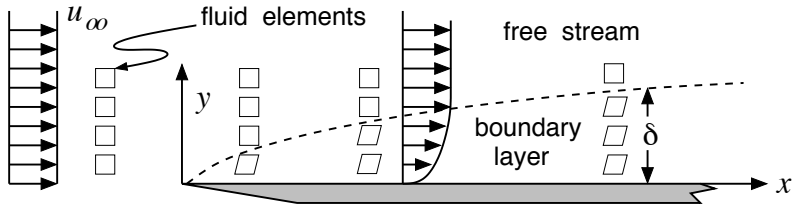


Figure 9.1. *Anatomy of a simple boundary layer for flow over a sharp–leading–edge flat plate. The dashed curve represents the boundary layer edge, under which viscous effects are important and outside of which they can be neglected. The phenomenon is illustrated by the small squares, which represent differential fluid elements.*

constant velocity u_∞ encounters a flat plate having a sharp leading edge,[9.1] and where (u, v) are the velocity components in the (x, y) directions, respectively. The no–slip boundary condition applies at the wall surface, $y = 0$, from which viscous effects immediately begin to manifest themselves. Fluid elements start to distort and rotate, with a velocity profile developing such that u is higher at the top of each element than on its bottom. Macroscopically, the velocity profile decays within the boundary layer, but remains uniform outside of it, as depicted in the diagram. The dashed curve denotes the edge of the region in which viscous effects are acting. This region is the boundary layer and its vertical size, as quantified by the boundary layer thickness, $\delta = \delta(x)$, grows with x. However, the domain is not bounded in the vertical direction, so this development process is not

[9.1] The sharp leading edge is a pedagogical device routinely invoked to justify ignoring any complicated fluid motion that would otherwise occur in the immediate neighborhood of $(x, y) = (0, 0)$ as the flow impinges on the plate edge. In actuality, the so–called boundary layer equations we will derive in §9.1 are not even strictly valid in the immediate small neighborhood of the this edge because the distance x down the plate is comparable to the upstream distance $x < 0$, which streamwise diffusion effects act.

constrained. It either continues indefinitely, if the plate is infinite in the x direction, or it evolves into a some sort of trailing flow motion behind the plate if the plate is finite.

The defining feature of the boundary layer is that it is very *thin*, so that gradients of the type shown in Fig. 9.1 for velocity are extremely pronounced.[9.2] This property has important implications for simplifying the governing equations for momentum and energy, as we will show in the following section. This class of flows will place us in the "boundary layer" section of the phylogeny in Fig. 8.1 on pp. 94.

9.1. Prandtl's Order–of–Magnitude Derivation of the Boundary Layer Equations

Ludwig Prandtl originally conceived of the boundary layer in 1904 as a thin region near the boundary within which viscous effects are confined and outside of which the flow remains basically inviscid (Blasius, 1908). This idea was the key to unifying the somewhat paradoxical observations of the day that drag measurements (for which viscous effects are important for streamline shapes, e.g. ship hulls) were not consistent with the seemingly inviscid nature of a flow.[9.3] The physical implications are actually quite simple, as depicted in Fig. 9.1. The primary motion is in the x direction, so that $u \gg v$, while the primary gradient is in the y direction, so that $\partial/\partial y \gg \partial/\partial x$. The mathematical implications of these 2 observations are significant, as we will demonstrate using Prandtl's "order–of–magnitude" argument to simplify the convection (Navier–Stokes) equations given by Eqs. (7.16), (7.17), and (7.19) starting on pp. 90. Let us consider the "standard" version of the problem, which is steady, two–dimensional, and neglects body forces, dissipation, and other types of heat generation (Schlichting, 1979; Kays and Crawford, 1980; Panton, 1984). Under these conditions, the governing equations are

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\
u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{1}{\rho}\cdot\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \\
u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= -\frac{1}{\rho}\cdot\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \\
u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) .
\end{aligned}$$

For the purposes of this section only, let us define an "order–of–magnitude" notation using square braces, e.g. [1], [ε], and [ε^2], which represent orders–of–magnitude of unity, ε, and ε^2, respectively, where ε is a small parameter. We will denote the orders–of–magnitude of the individual terms by displaying this notation directly below each one.

Following the basic presentation by Schlichting (1979), let $u \sim [1]$ and $\partial/\partial x \sim [1]$, so that $\partial u/\partial x \sim [1]\cdot[1]$. Given $u \gg v$ and $\partial/\partial y \gg \partial/\partial x$ from above, we assign $v \sim [\varepsilon]$ and $\partial/\partial y \sim 1/\varepsilon$,

[9.2] Fig. 9.1 is not drawn to scale, but rather is stretched vertically so as to make the velocity profile visible.

[9.3] It is not overstating matters to say that the idea of the boundary layer opened the modern era of the science of fluid mechanics, as we recognize it today. Eckert (2006, chapter 2) relates the interesting history of Prandtl's developmental work at the University of Göttingen in the important period from 1904 up to the First World War.

so that conservation of mass is

$$\underbrace{\frac{\partial u}{\partial x}}_{[1]\cdot[1]} + \underbrace{\frac{\partial v}{\partial y}}_{[\varepsilon^{-1}]\cdot[\varepsilon]} = 0\,,$$

meaning both terms are of the same order of magnitude. Consequently, neither of these terms drops.

Given what we now know about u, v, and the derivatives, we can deduce the orders–of–magnitude of the other terms in this system, dropping those terms that are much smaller than the overall order of the system, which is order [1] according to conservation of mass. For example, the x momentum equation is

$$\underbrace{u\,\frac{\partial u}{\partial x}}_{[1]\cdot[1]\cdot[1]} + \underbrace{v\,\frac{\partial u}{\partial y}}_{[\varepsilon]\cdot[\varepsilon^{-1}]\cdot[1]} = -\frac{\partial(P/\rho)}{\partial x} + \underbrace{\nu}_{[\varepsilon^2]}\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{[1]\cdot[1]} + \underbrace{\frac{\partial^2 u}{\partial y^2}}_{[\varepsilon^{-2}]\cdot[1]}\right).$$

The size of all the terms involving velocity components are immediately determined by what we have already discerned above. Notice the important assignment of the kinematic viscosity as $\nu \sim [\varepsilon^2]$ to force the *overall order* of the viscous contribution to be order [1].[9.4] We invoke this condition because the viscous contribution should be neither significantly more nor significantly less than the convective contribution within the boundary layer. Both are important. Note that the overall order of the equation is [1], but that $\partial^2 u/\partial x^2$ is of a significantly lesser order–of–magnitude, $[\varepsilon^2]$. This term will be dropped in the final boundary layer equation set. Note also that it is not yet clear how to assign the order–of–magnitude of the pressure equation

Given that $\nu \sim [\varepsilon^2]$ is now fixed for the system, the y momentum equation is

$$\underbrace{u\,\frac{\partial v}{\partial x}}_{[1]\cdot[1]\cdot[\varepsilon]} + \underbrace{v\,\frac{\partial v}{\partial y}}_{[\varepsilon]\cdot[\varepsilon^{-1}]\cdot[\varepsilon]} = -\frac{\partial(P/\rho)}{\partial y} + \underbrace{\nu}_{[\varepsilon^2]}\left(\underbrace{\frac{\partial^2 v}{\partial x^2}}_{[1]\cdot[\varepsilon]} + \underbrace{\frac{\partial^2 v}{\partial y^2}}_{[\varepsilon^{-2}]\cdot[\varepsilon]}\right).$$

The important observation here is that the overall order of the entire equation is $[\varepsilon]$, which has 2 implications. First, it suggests that the term containing the pressure gradient is of order $[\varepsilon]$, as well because the pressure gradient is not obviously more or less important than the other terms.[9.5] Furthermore, because $\partial(P/\rho)/\partial y$ is very small, the pressure is essentially constant across the boundary layer. In essence, the pressure reduces to $P = P(x)$, implying that the streamwise pressure gradient can be derived simply by considering the inviscid flow outside the boundary layer. This observation then ultimately implies we can drop the pressure term in the x momentum equation, as well, because we are considering the horizontal flat plate with constant freestream velocity, u_∞.[9.6] The

[9.4] That is, the product of the viscosity and the $\partial^2 u/\partial y^2$ term is order [1].

[9.5] Note that if we *were* to have made the apparently reasonable assumption of $\partial(P/\rho)/\partial x \sim [1]$ in the x–momentum equation, then we would have had $(P/\rho) \sim [1]$ and, consequently, $\partial(P/\rho)/\partial y \sim [\varepsilon^{-1}]$ since $\partial/\partial y \sim [\varepsilon^{-1}]$. This would mean that this single pressure gradient term would be significantly larger than every other term in the entire set of equations!

[9.6] Since $P = P(x)$, the streamwise pressure gradient is the same in both the boundary layer and the inviscid freestream domain. In the latter, the Bernoulli equation applied in the streamwise direction requires a constant pressure, if both the kinetic energy and potential energy terms are fixed, the former being true because the freestream velocity is constant and the latter being true because the plate is horizontal. The same conclusion would be reached if arguing on the basis of the order–of–magnitude of P/ρ itself based on the fact that indeed $\partial(P/\rho)/\partial y \sim [\varepsilon]$.

second major implication is that the entire y momentum equation, being only of order $[\varepsilon]$, is small compared to the x momentum equation, being of order [1]. The entire y momentum equation can therefore be dropped.

For the energy equation, we assume $T \sim [1]$ and write

$$\underbrace{u\,\frac{\partial T}{\partial x}}_{[1]\cdot[1]\cdot[1]} + \underbrace{v\,\frac{\partial T}{\partial y}}_{[\varepsilon]\cdot[\varepsilon^{-1}]\cdot[1]} = \underbrace{\alpha}_{[\varepsilon^2]}\left(\underbrace{\frac{\partial^2 T}{\partial x^2}}_{[1]\cdot[1]} + \underbrace{\frac{\partial^2 T}{\partial y^2}}_{[\varepsilon^{-2}]\cdot[1]} \right),$$

where we have assigned $\alpha \sim [\varepsilon^2]$ for reasons similar to ν above. Again, the overall order of the equation is [1], but $\partial^2 T/\partial x^2$ is of a significantly lesser order–of–magnitude, $[\varepsilon^2]$. This term will likewise be dropped in the final boundary layer equation set.

Given all the above considerations, we can write the final set of Prandtl's boundary layer equations for the flat plate as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9.1}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = \nu\,\frac{\partial^2 u}{\partial y^2} \tag{9.2}$$

$$u\,\frac{\partial T}{\partial x} + v\,\frac{\partial T}{\partial y} = \alpha\,\frac{\partial^2 T}{\partial y^2}\,, \tag{9.3}$$

where $u = u(x,y)$, $v = v(x,y)$, and $T = T(x,y)$. Recall that the momentum equation in the y direction has vanished. While the conservation of momentum statement in Eq. (9.2) still retains non–linear terms, this system is obviously still enormously more amenable to solution than the full Navier–Stokes equations.

Boundary conditions simplify, too. There is only 1 remaining derivative in v, so that variable needs only 1 boundary condition, the obvious one being no–slip at the plate surface, $y = 0$. The velocity component u still has a second–derivative in y, signaling 2 boundary conditions are still required. Here, we will take the obvious no–slip at $y = 0$ and freestream conditions at the edge of the boundary layer, $y = \delta$. The first–derivative in x for u also requires 1 boundary condition, which we will take as some prescribed upstream flow condition. We summarize these as

$$u(x,0) = v(x,0) = 0 \qquad \text{no–slip} \tag{9.4}$$

$$u(x,\delta) = u_\infty \qquad \text{freestream} \tag{9.5}$$

$$u(0,y) = u_\infty \qquad \text{upstream (approach flow)} \tag{9.6}$$

The similarity of the energy equation to the momentum equation suggests the boundary conditions for T might be similar to those for u. We will indeed examine the case where T itself is specified at the 3 required boundary locations, but remind the reader that other boundary conditions are possible, e.g. for specified boundary flux at $y = 0$.

9.2. Kármán–Pohlhausen Approximate Integral Method

Although the boundary layer equations are significantly simplified, as compared to the full Navier–Stokes equations, the statement of conservation of momentum still retains non–linear terms

in the likes of $u \cdot \partial u/\partial x$ and $v \cdot \partial u/\partial y$, so the overall system remains somewhat formidable. We mentioned already on several occasions, e.g. in §7.2, that there is not yet a general mathematical theory for non–linear differential equations (Hildebrand, 1976). Perhaps somewhat surprisingly, non–linear problems can sometimes be solved to a very good degree of approximation using operations that are no more complicated than those in introductory calculus. In this section, we discuss one such method due to the apparently independent work of von Kármán (1921) and Pohlhausen (1921).[9.7]

One of the main hurdles or advantages, depending upon one's perspective, for the semi–infinite plate is that there is no obvious length scale. The key aspect of the Kármán–Pohlhausen approach is the development of an integral equation that relates streamwise developments to conditions at the plate surface, $y = 0$. This is shown in detail in §K.1 in appendix K on pp. 187. The result, given by Eq. (K.3) on pp. 188, can be re–arranged slightly to give[9.8]

$$\frac{d}{dx} \int_0^\delta \rho \left(u_\infty \, - \, u \right) u \, dy \; = \; \mu \, \frac{\partial u}{\partial y} \, \bigg|_{y=0} \, .$$

In this form, it can be seen that the Kármán–Pohlhausen equation says that the rate of change of the so–called *momentum boundary layer thickness* (Munson et al., 2006) along the plate (left side) is equal to the shear stress at the plate surface, $y = 0$, (right side). This equation permits one to derive the boundary layer growth law, $\delta = \delta(x)$, for any estimated "trial" velocity profile. Here, δ is indeed the sufficient length scale to furnish the needed closure.

The Kármán–Pohlhausen equation is obtained by integrating the boundary layer conservation laws, Eqs. (9.1) through Eqs. (9.3), over the boundary layer thickness, thus transforming an exact differential system into an approximate integral system. The latter only requires satisfaction of conditions at the boundaries and has no say in what happens "inside the control volume", so to speak. This observation is easily demonstrated by applying the Kármán–Pohlhausen equation in practice. We may assume any boundary layer profile that can be made to satisfy certain conditions at $y \in \{0, \delta\}$. For example, it is very common to model u as a 3–rd order polynomial, from which we ultimately find[9.9]

$$u \; = \; \frac{3 \, u_\infty}{2 \, \delta} \, y \; - \; \frac{u_\infty}{2 \, \delta^3} \, y^3 \qquad \text{and} \qquad \delta \; = \; \frac{4.641 \, x}{\sqrt{Re_x}} \, . \tag{9.7}$$

It is worth noting again that this solution requires nothing more than basic calculus operations. Moreover, the boundary layer growth law compares favorably to the exact solution, which specifies a constant of 5.00, as opposed to 4.64 in the polynomial approximation. Table 9.1 gives the corresponding constants for several such approximations.

A similar procedure is adopted for the temperature distribution in the thermal boundary layer. Indeed, a 3–rd order polynomial approximation for temperature, coupled with slightly more involved mathematical considerations as compared to the momentum problem, yield T and ultimately the convective solution in the form of the Nusselt number

$$Nu_x \; \approx \; 0.3313 \, Re_x^{1/2} \, Pr^{1/3} \, . \tag{9.8}$$

[9.7] Much of Theodore von Kármán's work emphasized the ability to solve complex flow physics problems using only fairly straightforward mathematical methods (Eckert, 2006, chapter 5).

[9.8] There is a similar statement for the thermal boundary layer given by Eq. (K.4) on pp. 189.

[9.9] These results are derived in detail in §K.3 starting on pp. 189. The profile described by Eq. (9.7) is plotted in Fig. 9.2 on pp. 122 as part of a comparison of a number of solution methods for the boundary layer equations.

TABLE 9.1. Constants in $\delta = C\,x/\sqrt{Re_x}$ for several u approximations

model	C	model	C
exact	5.00	cubic	4.64
linear	3.46	sine	4.79
parabolic	5.48		

9.3. Similarity Transform for Laminar Flow Over a Semi–Infinite Flat Plate

As we said, the flat plate is one of the simplest representations of external flow, but the fact that there is no obvious finite length scale makes for some peculiarity. The Kármán–Pohlhausen method can be thought of as a tool to *deduce* a length scale in the form of the boundary layer thickness. Another method exploits a similarity approach to transform the mass–momentum PDE system to an ODE, for which the solution procedure will be yet further simplified.[9.10] In particular, it is shown in appendix L, specifically Eq. (L.1) on pp. 198, that if we take the similarity variable

$$\varphi \;=\; y\,\sqrt{\frac{u_\infty}{\nu\,x}}\;,$$

the boundary layer PDEs can be reduced to a single ODE for a function $f(\varphi)$

$$f''' \,+\, \frac{f\,f''}{2} \;=\; 0 \qquad\qquad f(0) \;=\; f'(0) \;=\; 0 \qquad f'(\infty) \;=\; 1\,, \tag{9.9}$$

where the velocity profile is given by $u = u_\infty\,f'$.

While solving an ODE is ordinarily much preferable to a PDE, this equation is still non–linear, and, as we pointed out previously, there is no general mathematical theory for non–linear differential equations (Hildebrand, 1976). This aspect, coupled with the problem's basic importance in thermofluids, means that Eq. (9.9) has actually been probed with a variety of rather different and interesting methods. These methods include the following.

Blasius (1908) first solved this problem using an expansion in small values of φ patched to an asymptotic expansion for large values of φ. Such methods are fairly sophisticated (van Dyke, 1964; Bender and Orszag, 1999; Nayfeh, 2004), rather involved, and beyond the scope of our discussion. The more direct approach is numerical integration. Eq. (9.9) can be solved with high–order accuracy methods, e.g. fourth–order Runge–Kutta (Conte, 1965; Bejan, 1984), for which solutions can be computed extremely accurately. The solution by Howarth (1938) is perhaps the *de facto* standard (Schlichting, 1979; Burmeister, 1983). Yet a third approach is due to Piercy and Preston (1936), who *split* Eq. (9.9) into different levels of approximation in such a way as to recast the problem into one of iterating on $f(\varphi)$ directly. This method seems to have receded into obscurity to some degree,[9.11] but we feel it is worth detailed discussion for several reasons. First, it is conceptually straightforward, being derivable with standard calculus techniques. Second, it offers the possibility of an exact solution, in the sense that accuracy can be as good as one likes, without the extreme tedium and mathematical caveats of asymptotic expansions (van Dyke, 1964). Third, convergence

[9.10] Recall that lack of a length scale also prompted a similarity approach for the Rayleigh conduction problem discussed in §D.1 on pp. 146.

[9.11] While mentioned briefly in Kays and Crawford (1980), it seems to be absent from many of the leading texts that discuss the boundary layer.

seems to be very fast, for example exceeding the Kármán–Pohlhausen method on the first round if using a good initialization of $f(\varphi)$. We will examine this method in the next section.

9.4. Momentum Solution via the Piercy–Preston Successive Integration Method

As mentioned, this technique cleverly splits Eq. (9.9) into 2 iterative levels

$$f_i''' \,+\, \frac{f_{i-1}\cdot f_i''}{2} \;=\; 0\,, \tag{9.10}$$

where $i-1$ denotes the previous solution and i is the improved solution. §L.3 shows the derivation of the so–called Piercy–Preston integrals, Eqs. (L.9) and (L.10) on pp. 202, which specify the updating process, essentially providing $f_i(f_{i-1})$, i.e.

$$f_i(\varphi) \quad = \; \frac{\int_0^\varphi \left[\int_0^\varphi \exp\left(-\,\frac{1}{2}\int_0^\varphi f_{i-1}(\varphi)\,d\varphi\right) d\varphi\right] d\varphi}{\int_0^\infty \exp\left(-\,\frac{1}{2}\int_0^\varphi f_{i-1}(\varphi)\,d\varphi\right) d\varphi}\,.$$

One initializes $f_1(\varphi)$, then obtains successively better solutions, $f_2, f_3, \ldots$ using this integral. In their original paper, Piercy and Preston (1936) simply took f_1 representing the freestream flow, $u = u_\infty$, from which a single iteration shows[9.12]

$$\frac{u}{u_\infty} \;\approx\; f_2'(\varphi) \;=\; \mathrm{erf}\Big(\frac{\varphi}{2}\Big)\;, \tag{9.11}$$

whereas in a later paper Watson and Preston (1951) obtained improved results taking f_1 describing a uniform flow $u = \kappa\, u_\infty$, where κ is an undetermined constant, a degree–of–freedom essentially, that can be deduced from the Kármán–Pohlhausen integral, i.e. Eq. (K.4) on pp. 189. A single iteration shows

$$\frac{u}{u_\infty} \;\approx\; f_2'(\varphi) \;=\; \mathrm{erf}\left(\frac{\sqrt{\sqrt{2}\;-\;1}\;\varphi}{2}\right)\,. \tag{9.12}$$

In both Eqs. (9.11) and (9.12), erf is the Gauss error function. Watson and Preston (1951) also discuss more complicated initializations of $f_1(\varphi)$, as well as some standard generalizations of the boundary layer problem.

It is a natural question to wonder how accurate Eqs. (9.11) and (9.12) are, which will be suggestive of just how fast the Piercy–Preston method converges, as well as how these solutions compare to the Kármán–Pohlhausen method, whose solution is fixed once the general model for a profile is prescribed. Before a comparison can be made, however, we must reconcile the 2 different bases for measuring vertical distance from the plate: the similarity variable, φ, in the Piercy–Preston results, which *is* the vertical coordinate vs. the boundary layer thickness, δ, in the Kármán–Pohlhausen method, which serves as a length scale against which the distance y is measured. The former depends upon similarity arguments and is fixed, while the latter varies, depending upon one's choice for the general model of the prescribed profile. For convenience, we restate the definitions

$$\delta \;=\; \sqrt{\frac{280}{13}\;\frac{\nu\,x}{u_\infty}} \qquad\qquad \varphi \;=\; y\,\sqrt{\frac{u_\infty}{\nu\,x}}\;,$$

[9.12] The results in Eqs. (9.11) and (9.12) are derived in §L.4 starting on pp. 202.

the former, being the exact result in Eq. (K.6) on pp. 190 from prescribing a 3–rd order polynomial profile for the Kármán–Pohlhausen method and the latter being the general similarity parameter for flat plate boundary layer flow derived in Eq. (L.1) on pp. 198.[9.13] Given that the Kármán–Pohlhausen method measures with y/δ, we find the relationship[9.14]

$$\frac{y}{\delta} \,=\, y\,\sqrt{\frac{13}{280}}\,\sqrt{\frac{u_\infty}{\nu\,x}} \,=\, \sqrt{\frac{13}{280}}\;\varphi\,,$$

so that the 3–rd order polynomial Kármán–Pohlhausen result from Eq. (9.7) on pp. 119 can be re–written in the compatible form

$$\frac{u}{u_\infty} \,=\, \left[\frac{3}{2}\,\sqrt{\frac{13}{280}}\;\varphi \,-\, \frac{1}{2}\,\left(\sqrt{\frac{13}{280}}\;\varphi\right)^3\right] \qquad\qquad \varphi \,\leq\, \sqrt{\frac{280}{13}}\;. \tag{9.13}$$

The accompanying limit is a consequence of a subtle technicality in that *all* Kármán–Pohlhausen profiles are defined only within $0 \leq y \leq \delta$, with conditions outside of this domain being taken identically as freestream. Therefore, the boundary layer solution is restricted to $y/\delta \leq 1$, which implies the limit.

Fig. 9.2 compares these 3 solutions to the numerical result by Howarth (1938), which is consid-

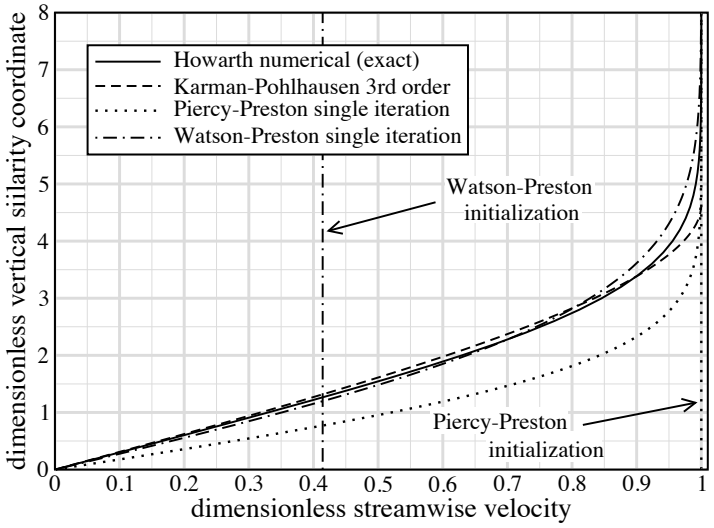


FIGURE 9.2. *Comparison of the boundary layer profile, as obtained by several different methods: Howarth (1938) numerical integration (solid curve), considered "exact", Kármán–Pohlhausen solution (dashed curve) in Eq.*(9.13)*, Piercy–Preston solution (dotted curve) from Eq.* (9.11)*, and Watson–Preston solution (dash–dot curve) from Eq.* (9.12)*. Plot also shows initializations of the Piercy–Preston and Watson–Preston solutions.*

ered the *de facto* standard (Schlichting, 1979; Burmeister, 1983). The first observation is that the Kármán–Pohlhausen (KP) and Watson–Preston (WP) solutions recapitulate the Howarth profile

[9.13]The expression for δ in Eq. (9.7) on pp. 119 is an approximation, but the velocity profile in that equation in which δ is used is exact.

[9.14]Note that both φ and y/δ are inherently dimensionless.

quite well, especially near the plate boundary. The slopes of these 3 curves seem to be visually indistinguishable at $\varphi = 0$, suggesting that both KP and WP should give very good estimates of shear stress at the wall. At the other end, KP and WP seem to a little short and long, respectively, with respect to the edge of the boundary layer. Conversely, the original Piercy–Preston (PP) solution is clearly inferior to the other 2 results in that it plainly over–predicts the gradient at the wall and underpredicts the boundary layer thickness.

Fig. 9.2 also shows the initializations of the PP and WP methods, vertical lines (uniform profiles) of $u/u_\infty = 1$ and $u/u_\infty \approx 0.414$, respectively. Given that we performed only a single iteration for each method, the accuracy that we *did* realize actually seems quite impressive. It suggests that the basic iterative method does indeed converge relatively fast and that improved initializations and/or additional iterations will yet further improve the results. In practice, further refinement will not be without increased mathematical challenge.

9.5. Solving for Convection in the Context of the Similarity Model

As we discussed in §9.3, the similarity framework leads to an ODE for $f(\varphi)$, which can be solved by a variety of methods, one of which we discussed in §9.4. The remaining task is then to use the momentum solution within the energy equation to solve for the temperature distribution, and finally the Nusselt number. Appendix L, specifically §L.5 starting on pp. 205, shows that under the similarity framework, the energy PDE also collapses to an ODE of the form

$$\frac{d^2T}{d\varphi^2} \,+\, \frac{f \cdot Pr}{2}\,\frac{dT}{d\varphi} \;=\; 0\,,$$

which can be integrated in very much the same manner as to how the Piercy and Preston (1936) method processes the momentum equation.[9.15] The general solution for T is

$$T(\varphi) \;=\; \frac{\int_0^\varphi \exp\left(-\,\frac{Pr}{2}\int_0^\varphi f(\varphi)\,d\varphi\right) d\varphi}{\int_0^\infty \exp\left(-\,\frac{Pr}{2}\int_0^\varphi f(\varphi)\,d\varphi\right) d\varphi}\,,$$

which, again presuming $f(\varphi)$ is known, is a formality to integrate.[9.16] §L.5 further shows that, under the similarity framework, the Nusselt number is given by[9.17]

$$Nu_x \;=\; \sqrt{Re_x}\cdot T'(0) \;=\; \frac{\sqrt{Re_x}}{\int_0^\infty \exp\left(-\,\frac{Pr}{2}\int_0^\varphi f(\varphi)\,d\varphi\right) d\varphi}\,\cdot \tag{9.14}$$

For example, the Howarth (1938) numerical approach leads to the well–known "exact" result

$$Nu_x \;\approx\; 0.332\, Re_x^{1/2}\, Pr^{1/3}\,, \tag{9.15}$$

[9.15] Once the momentum problem is solved, f is presumed known. Thus, $f \cdot Pr/2$, which is the coefficient of the 1–st derivative term, is entirely known and the equation is seen to be *separable*, c.f. the discussion in §L.5. While the momentum ODE in Eq. (9.9) may look different, it has the same structure in that its 2 terms differ by exactly 1 level of their derivative: 2–nd and 1–st derivatives in the energy equation, and 3–rd and 2–nd derivatives in the momentum equation. The cleverness of the Piercy and Preston (1936) method lies in the way they split the momentum Eq. (9.10), such that the coefficient of the lesser–order term becomes known, whereby that equation also becomes separable.

[9.16] Though a formality, this equation will generally be very difficult to complete in actual practice without resorting to numerical integration at some point, c.f. §L.6 starting on pp. 208.

[9.17] This is Eq. (L.16) on pp. 208.

which is sometimes called the "Blasius curve".

Generally speaking, modern texts like Kays and Crawford (1980), Burmeister (1983), and Bejan (1984), give very little detail as to how such results as Eq. (9.15) are ultimately derived. An important point in this regard is that it is *not* like how the corresponding result for the Kármán–Pohlhausen method in Eq. (9.8) is determined. Recall that in that case, the "construction" of the Nusselt number is essentially an *algebraic* process, once the boundary layer growth laws are determined.[9.18] Conversely, in the case of the similarity framework, and irrespective of the actual mathematical or numerical technique used to find $f(\varphi)$, the integral in Eq. (9.14) implies that Nu_x is actually some sort of (invariably) complicated function of Pr.[9.19] The much–simplified form in which results are usually reported, e.g. the "Blasius curve" in Eq. (9.15), actually comes from *regressing* these curves. Specifically, the complicated dependence upon the Prandtl number, Pr, is assumed to be expressible in a much simpler regressed form of $a \cdot Pr^b$, where a and b are undetermined constants. The entire process is shown in detail, including regression, for the Piercy–Preston examples we have examined here in §L.6 starting on pp. 208. In particular, we find that the original Piercy and Preston (1936) velocity profile in Eq. (9.11) on pp. 121 leads to

$$Nu_x \approx 0.383\, Re_x^{1/2}\, Pr^{0.348}\,, \tag{9.16}$$

while the improved Watson and Preston (1951) profile in Eq. (9.12) on pp. 121 leads to

$$Nu_x \approx 0.339\, Re_x^{1/2}\, Pr^{0.342}\,. \tag{9.17}$$

These curves, *sans* Reynolds number, are plotted in Fig. L.2 on pp. 212, along with the "Blasius curve" in Eq. (9.15) for comparison. The results are surprisingly good, given that only 1 iteration was executed.

9.6. Additional Remarks

We have only examined arguably *the* most basic external flow. There are numerous other relevant flows that exhibit boundary–layer–like dynamics. A very interesting generalization was reported by Falkner and Skan (1931), who allowed the approach flow from a non–zero angle of attack (Fig. 9.3). It can be shown that a similarity parameter of

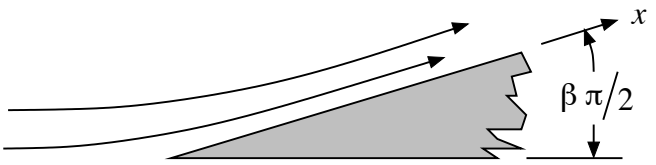


FIGURE 9.3. *Boundary layer configuration for a wedge, which is equivalent to the approach flow at non–zero angle of attack.*

$$\varphi \,=\, y\,\sqrt{\frac{(m+1)\,u_\infty}{2\,\nu}}\;x^{(m-1)/2} \qquad \text{where} \qquad m \,=\, \frac{\beta}{2\,-\,\beta}$$

[9.18] These operations are given in detail in §K.3 starting on pp. 189.

[9.19] For example, Eq. (L.18) on pp. 209 represents the general form for the single–iteration Piercy and Preston (1936) method and appears sufficiently complex so as not even to be obviously integrable without resorting to numerical techniques.

leads to reduction to an ODE of

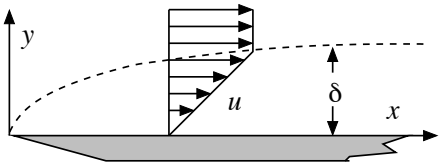
$$f''' \,+\, f\, f'' \,+\, \beta\left(1 \,-\, f'^2\right) \,=\, 0$$

with the same boundary conditions as for the flat plate ODE in Eq. (9.9) on pp. 120. In fact, the flat plate configuration is clearly a special case of this so–called Falkner–Skan ODE for $\beta = 0$.[9.20] Many texts examine this configuration in detail (e.g. White, 1974; Schlichting, 1979; Panton, 1984).

There are broader classes of boundary layer flows where the similarity approach will fail outright and others where obtaining any analytical results requires combining similarity variables with other approaches. White (1974, §4-4) recounts quite a number of examples. In general, these cases are fraught with mathematical difficulty and require additional techniques beyond what we have discussed here.

9.7. Exercises

9.1 The Kármán–Pohlhausen integral method allows for various kinds of approximations (trial profiles) for the velocity and temperature distributions of Prandtl's laminar boundary layer flow. Consider the flat plate configuration, where we use *linear* approximations for the profiles, as shown for example for $u(x, y)$ in the diagram.



Denote the freestream velocity for the flow as u_∞ and the momentum boundary layer thickness as $\delta = \delta(x)$. If we approximate the velocity distribution by the linear relationship

$$\frac{u}{u_\infty} \,=\, a_1 \,+\, a_2 \frac{y}{\delta}\,,$$

where a_1 and a_2 are constants to be determined, show that δ grows as

$$\delta \,\approx\, \frac{3.464\, x}{\sqrt{Re_x}}$$

using the Kármán–Pohlhausen momentum integral equation, where Re_x is the appropriately defined Reynolds number. Pay attention to selecting the 2 most appropriate boundary conditions to evaluate the 2 constants.

[9.20] Actually, most discussions of the Falkner–Skan problem use the similarity variable for this problem quoted above that makes the factor of 1/2 in Eq. (9.9) disappear. We again make the point that the successful similarity variable is not necessarily unique, c.f. footnote D.1 on pp. 147.

9.2 Referring to exercise 9.1: Denote the freestream temperature of the flow as T_∞, the constant plate temperature as T_S, and the thermal boundary layer thickness as $\delta_t = \delta_t(x)$. If we approximate the temperature distribution by the linear relationship

$$\frac{T - T_S}{T_\infty - T_S} = b_1 + b_2 \frac{y}{\delta_t} ,$$

where b_1 and b_2 are constants to be determined, show that δ_t grows as

$$\delta_t = \delta \, Pr^{-1/3} \approx \frac{3.464 \, x}{Re_x^{1/2} \, Pr^{1/3}}$$

using the Kármán–Pohlhausen energy integral equation, where Pr is the fluid Prandtl number. Pay attention to selecting the 2 most appropriate boundary conditions to evaluate the 2 constants.

9.3 Referring to exercises 9.1 and 9.2: Use the results obtained thus far to show that the Nusselt number for the Kármán–Pohlhausen method using linear approximations for u and T in the boundary layer is

$$Nu_x \approx 0.289 \, Pr^{1/3} \, Re_x^{1/2} .$$

CHAPTER 10

Epilogue

As we mentioned in the preface, our scope was limited to a number of fundamental conduction and convection problems, for which we were able, for the most part, to both pose and solve in analytical terms. It might be said that these problems were examined from "first principles", with the commensurate benefits being not only exposure to the physical configurations themselves, but also to the underlying mathematical techniques that form a sound foundation for more difficult problems. Here, the word "difficult" can be taken as synonymous with "realistic", or perhaps "real world".

There is no shortage of complications that will occur in real problems, for example non–linearities due to property variation and complicated fluid motion, unsteady effects (including at the boundaries), complex and perhaps varying geometry, turbulence, phase change, chemical reaction, etc. In fact, in many instances one can expect that the actual heat transfer dynamics will involve both conduction and convection, for example heat exchangers, and maybe even radiation, as well. There is obviously great worth in having a command of the techniques for treating such problems, not only in mathematical terms, but also in experimental and numerical ones, too.[10.1] However, in the real world, the practitioner should also pay close attention to what might be called the "parsimony principle of problem solving". That is to say, one should arrive at a solution having the appropriate level of sophistication and precision dictated by relevant considerations of economy, time, and the problem's importance within any larger context.

To take heat exchangers again as an example, a problem may be nothing more than to perform "sizing analysis" to replace a retired unit and there may only be a few possible answers based on availability from a manufacturer. Here, "back of the envelope" calculations more than suffice. Conversely, one may be trying to arrive at an optimal design for a new type of heat exchanger whose precise operation is integral to a larger instrument or process and in this case the analysis and testing procedures would be much more substantial. In short, the latter solution would be significantly more expensive and could take much longer to obtain, even though these factors would be entirely justified.

Looking back in history, it seems many of the greatest scientists and engineers were very good at this aspect of problem solving. John von Neumann certainly was, likewise for Stanislaw Ulam, Richard Feynman, the duo of Simon Ramo and Dean Wooldridge, Vannevar Bush, Claude Shannon, Richard Hamming, etc. A very long list of such individuals could easily be recited. Even Einstein himself had a pragmatic view of the role of parsimony in technical work. Needless to say, these people were all highly proficient across the methodological spectrum, which perhaps gave them the

[10.1] Recall the discussion of the "methodological triumvirate" in §1.5 on pp. 5. Heat transfer, like most areas of mechanics, stands squarely at the intersections of pure mathematical theory, empirical science, and engineering application and design optimization. While one may study the topic from one particular perspective, as we have done here, it is wise to remember its significantly broader nature.

luxury of picking the most fitting approach for a particular problem. I cannot say there is an exact recipe of steps to follow in order to become good at this aspect, but ability will certainly tend to develop with experience.

The whole matter has been eloquently discussed many times, but I think it is especially well summed–up by Fontolliet (1986) who, in his book on the analysis and design of telecommunication systems, wrote:

> "The ability to leave things out is as important a requirement in engineering as the ability to calculate. Knowing how to omit things is a difficult and subtle art which requires in–depth knowledge of nature and techniques, and at the same time sure judgment to evaluate the degree of approximation necessary, which can still be compatible with the prescribed goals. The engineer must know that simple models are false, but also that those which are not simple are unusable. Between the perfectionism which is fatal because of its excessive cost and an empiricism which is dangerous because of its unpredictable nature, an appropriate compromise must be found which satisfies quality requirements and economic imperatives ... Reality is always more complex than the theory which attempts to describe it. In effect, to understand this reality, we are obliged to simplify and schematize it."

APPENDIX A

Bessel's Modified Differential Equation for Annular Fins

EQ. (3.21) ON PP. 24 is the specific form of the fin equation that is obtained when specifying the geometric attributes of the annular fin:

$$\frac{d}{dr}\left(2\,\pi\,r\,L\,\frac{d\theta}{dr}\right)\;-\;\frac{4\,\pi\,h}{k}\,r\,\theta\;=\;0\,,$$

where $\theta = \theta(r)$ is the temperature distribution in the radial direction. Gathering constants and taking the derivative of the product in the first term, we can write this in the canonical form of a Bessel equation, specifically a modified, or hyperbolic Bessel equation:

$$\frac{d^2\theta}{dr^2}\;+\;\frac{1}{r}\,\frac{d\theta}{dr}\;-\;m^2\,\theta\;=\;0\qquad\text{where}\qquad m^2\;=\;\frac{2\,h}{k\,L} \tag{A.1}$$

Note that the coefficient of the first–derivative term, r^{-1}, is *not* constant, so this equation is rather more difficult than the special case of constant cross section fins in Eq. (3.13) on pp. 21.

A.1. Application of Frobenius' Method

Here, we can use the extended power series, a.k.a. Frobenius' method (Ross, 1965), which casts the solution in terms of the power series

$$\theta\;=\;r^{\varphi}\sum_{i=0}^{\infty}a_i\,r^i\;=\;\sum_{i=0}^{\infty}a_i\,r^{i+\varphi}\;=\;r^{\varphi}\Big(a_0\;+\;a_1\,r\;+\;a_2\,r^2\;+\;\cdots\Big)\,, \tag{A.2}$$

where $a_0 \neq 0$ and φ and the sequence of a_i are to be determined. This expression can be differentiated term–by–term to obtain first and second derivatives, as

$$\begin{aligned}
\frac{d\theta}{dr} &= \sum_{i=0}^{\infty}(i+\varphi)\,a_i\,r^{i+\varphi-1}\;=\;r^{\varphi-1}\Big(\varphi\,a_0\;+\;(\varphi+1)\,a_1\,r\;+\;(\varphi+2)\,a_2\,r^2\;+\;\cdots\Big)\\
\frac{d^2\theta}{dr^2} &= \sum_{i=0}^{\infty}(i+\varphi)(i+\varphi-1)\,a_i\,r^{i+\varphi-2}\\
&= r^{\varphi-2}\Big(\varphi\,(\varphi-1)\,a_0\;+\;(\varphi+1)\,\varphi\,a_1\,r\;+\;(\varphi+2)\,(\varphi+1)\,a_2\,r^2\;+\;\cdots\Big)\,.
\end{aligned}$$

Direct substitution into the differential equation gives

$$\begin{aligned}
r^{\varphi-2}\Big[\varphi\,(\varphi-1)\,a_0\;+\;(\varphi+1)\,\varphi\,a_1\,r\;+\;(\varphi+2)\,(\varphi+1)\,a_2\,r^2\;+\;\cdots\Big]\\
+\,r^{\varphi-2}\Big[\varphi\,a_0\;+\;(\varphi+1)\,a_1\,r\;+\;(\varphi+2)\,a_2\,r^2\;+\;\cdots\Big]\\
-\,m^2 r^{\varphi}\Big[a_0\;+\;a_1\,r\;+\;a_2\,r^2\;+\;\cdots\Big]\;=\;0\,,
\end{aligned}$$

and, gathering like terms in r, we find

$$\begin{aligned} & r^{\varphi-2}\Big[\varphi\,(\varphi-1)\,a_0 \;+\; \varphi\,a_0\Big] \\ & + r^{\varphi-1}\Big[(\varphi+1)\,\varphi\,a_1 \;+\; (\varphi+1)\,a_1\Big] \\ & + r^{\varphi}\Big[(\varphi+1)\,(\varphi+2)\,a_2 \;+\; (\varphi+2)\,a_2 \;-\; m^2\,a_0\Big] \\ & + r^{\varphi+1}\Big[(\varphi+2)\,(\varphi+3)\,a_3 \;+\; (\varphi+3)\,a_3 \;-\; m^2\,a_1\Big] \\ & + \cdots \;=\; 0\,. \end{aligned} \tag{A.3}$$

In general $r \neq 0$, so each term within square brackets must vanish in order for the equation to be satisfied. The term in the lowest power of r, here being $r^{\varphi-2}$, is the so–called *indicial equation* and it determines the further nature of the solution. Here, we find

$$\varphi\,(\varphi-1)\,a_0 \;+\; \varphi\,a_0 \;=\; \varphi^2\,a_0 \;=\; 0\,.$$

Since $a_0 \neq 0$, it must be the case that $\varphi = 0$.

This is a special case where there are two linearly independent solutions: one which is identically Eq. (A.2), where φ is now taken as 0, and another having the form $\theta \ln r + \sum b_i\, r^i$, where the b_n must also be determined. Let us first develop the solution θ in Eq. (A.2).

We have now processed the first term in square brackets in Eq. (A.3), from which $\varphi = 0$, but which likewise does not give any value for a_0. *All succeeding terms are now evaluated using* $\varphi = 0$. For instance, substitution shows the second term leads to $a_1 = 0$. The third and fourth terms are

$$1 \cdot 2 \cdot a_2 \;+\; 2 \cdot a_2 \;-\; m^2 a_0 \;=\; 0$$

$$2 \cdot 3 \cdot a_3 \;+\; 3 \cdot a_3 \;-\; m^2 a_1 \;=\; 0$$

and, in fact, it is clear that these and all succeeding terms follow the pattern

$$(j-1)\,j\,a_j \;+\; j\,a_j \;-\; m^2 a_{j-2} \;=\; 0\,,$$

from which a little algebra indicates a *recurrence equation* for the entire set of coefficients:

$$a_j \;=\; \left(\frac{m}{j}\right)^2 a_{j-2}\,.$$

This expression indicates $a_3 = a_5 = a_7 = \cdots = 0$ because $a_1 = 0$, i.e. only the even–indexed coefficients are meaningful. Without loss of generality, we can set $j = 2 \cdot i$, so that the series in Eq. (A.2) only includes these even–indexed coefficients, i.e.

$$a_{2i} \;=\; \left(\frac{m}{2\,i}\right)^2 a_{2i-2} \qquad \text{for} \quad i = 1, 2, 3, 4, \ldots$$

Let us write a few of these out in order to discern an even more basic pattern:

$$
\begin{aligned}
i=1: \quad a_2 &= \left(\frac{m}{2\cdot 1}\right)^2 a_0 \\
i=2: \quad a_4 &= \left(\frac{m}{2\cdot 2}\right)^2 a_2 \;=\; \left(\frac{m}{2\cdot 2}\right)^2 \left(\frac{m}{2\cdot 1}\right)^2 a_0 \\
i=3: \quad a_6 &= \left(\frac{m}{2\cdot 3}\right)^2 a_4 \;=\; \left(\frac{m}{2\cdot 3}\right)^2 \left(\frac{m}{2\cdot 2}\right)^2 \left(\frac{m}{2\cdot 1}\right)^2 a_0 \\
i=4: \quad a_8 &= \left(\frac{m}{2\cdot 4}\right)^2 a_6 \;=\; \left(\frac{m}{2\cdot 4}\right)^2 \left(\frac{m}{2\cdot 3}\right)^2 \left(\frac{m}{2\cdot 2}\right)^2 \left(\frac{m}{2\cdot 1}\right)^2 a_0
\end{aligned}
$$

so that, in general, we find

$$
a_{2i} \;=\; \left[\left(\frac{m}{2}\right)^2\right]^i \frac{a_0}{i!\,i!} \;=\; \left(\frac{m}{2}\right)^{2i} \frac{a_0}{(i!)^2}\,.
$$

Recalling our change in the index to retain only even terms, Eq. (A.2) becomes

$$
\theta \;=\; \sum_{i=0}^{\infty} a_{2i}\, r^{2i} \;=\; \sum_{i=0}^{\infty} \left(\frac{m}{2}\right)^{2i} \frac{a_0}{(i!)^2}\, r^{2i} \;=\; a_0 \sum_{i=0}^{\infty} \left(\frac{m\,r}{2}\right)^{2i} \frac{1}{(i!)^2} \;=\; a_0\, I_0(m\,r)\,, \tag{A.4}
$$

where

$$
I_0(x) \;=\; \sum_{i=0}^{\infty} \left(\frac{x}{2}\right)^{2i} \frac{1}{(i!)^2} \tag{A.5}
$$

is the so–called *modified Bessel function of the first kind* (Gray et al., 1952), in this case of order zero (since $\varphi = 0$ for this case). In other words, Eq. (A.4) gives one of the two linearly independent solutions up to an as of yet undetermined constant a_0.

As mentioned above, there is a second solution, now seen to be of the form

$$
a_0\, I_0(m\,r)\, \ln r \;+\; \sum_{b=0}^{\infty} b_i\, r^i\,.
$$

We could go through a similar, though rather more involved process to find this solution to be $b_0\, K_0(m\,r)$, where b_0 is an undetermined constant, K_0 is the *modified Bessel function of the second kind* (Gray et al., 1952)

$$
K_0(x) \;=\; -\,I_0(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] \;+\; \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^{2i} \frac{1}{(i!)^2} \sum_{j=1}^{i} \frac{1}{j}\,.
$$

Here, $\gamma \approx 0.5772157$ is Euler's constant (Beyer, 1984). Consequently, the general solution for the annular fin temperature distribution is

$$
\theta(r) \;=\; a_0\, I_0(m\,r) \;+\; b_0\, K_0(m\,r)\,, \tag{A.6}
$$

where a_0 and b_0 must still be determined and m is a parameter comprised of the flow, geometric, and material properties of the fin, as specified in Eq. (A.1).

A.2. Boundary Conditions

Eq. (A.6) is the general solution for conduction in the common annular fin. Constants a_0 and b_0 will take on appropriate values depending upon the boundary conditions. Recall that we conventionally assume that the temperature at the base of the fin is known (§3.4). Consequently,

$$\theta(r_i) \;=\; T(r_i) \;-\; T_\infty \;=\; \theta_b \,,$$

where θ_b is the known temperature at the base $r = r_i$ (c.f. Fig. 3.6). There are various possibilities for the tip boundary condition, but we will consider the particular case of an adiabatic boundary

$$\left.\frac{d\theta}{dr}\right|_{r=r_o} \;=\; 0 \,.$$

The latter requires us to take derivatives of I_0 and K_0 in Eq. (A.6). While derivatives can be written in a very general and elegant recursive fashion for Bessel functions of arbitrary order (Gray et al., 1952), it may be more instructive to derive the necessary expressions here in longhand. We can expand the definition of I_0 in Eq. (A.5) to

$$I_0(x) \;=\; \frac{1}{0!\,0!}\left(\frac{x}{2}\right)^0 \;+\; \frac{1}{1!\,1!}\left(\frac{x}{2}\right)^2 \;+\; \frac{1}{2!\,2!}\left(\frac{x}{2}\right)^4 \;+\; \frac{1}{3!\,3!}\left(\frac{x}{2}\right)^6 \;+\; \frac{1}{4!\,4!}\left(\frac{x}{2}\right)^8 \;+\; \cdots$$

and differentiate term–by–term. Denoting dI_0/dx as I_0', we find

$$\begin{aligned}
I_0'(x) \;&=\; \frac{2}{1!\,1!}\left(\frac{x}{2}\right)^1\cdot\frac{1}{2} \;+\; \frac{4}{2!\,2!}\left(\frac{x}{2}\right)^3\cdot\frac{1}{2} \;+\; \frac{6}{3!\,3!}\left(\frac{x}{2}\right)^5\cdot\frac{1}{2} \;+\; \frac{8}{4!\,4!}\left(\frac{x}{2}\right)^7\cdot\frac{1}{2} \;+\; \cdots \\
&=\; \frac{1}{1!\,1!}\left(\frac{x}{2}\right)^1 \;+\; \frac{2}{2!\,2!}\left(\frac{x}{2}\right)^3 \;+\; \frac{3}{3!\,3!}\left(\frac{x}{2}\right)^5 \;+\; \frac{4}{4!\,4!}\left(\frac{x}{2}\right)^7 \;+\; \cdots \\
&=\; \frac{1}{0!\,1!}\left(\frac{x}{2}\right)^1 \;+\; \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^3 \;+\; \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 \;+\; \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^7 \;+\; \cdots \\
&=\; \frac{x}{2}\left[\frac{1}{0!\,1!}\left(\frac{x}{2}\right)^0 \;+\; \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 \;+\; \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 \;+\; \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 \;+\; \cdots\right] \\
&=\; \frac{x}{2}\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^{2i}\frac{1}{i!\,(i+1)!} \\
&=\; I_1(x) \,.
\end{aligned}$$

Of course, this is consistent with the general definition of $I_n(x)$, given by

$$I_n(x) \;=\; \left(\frac{x}{2}\right)^n\sum_{i=0}^{\infty}\left(\frac{x}{2}\right)^{2i}\frac{1}{i!\,(i+n)!} \,,$$

for integer values of n (Gray et al., 1952). Similar examination shows $K_0' = -K_1$, so that the derivative is $d\theta/dr = a_0 I_0' + b_0 K_0' = a_0 I_1 - b_0 K_1$.

It is now a relatively straightforward matter to use the two boundary conditions to solve for the unknown constants a_0 and b_0. At r_i and r_o, we have, respectively

$$a_0\, I_0(m\, r_i) \;+\; b_0\, K_0(m\, r_i) \;=\; \theta_b \qquad \text{and} \qquad a_0\, I_1(m\, r_o) \;-\; b_0\, K_1(m\, r_o) \;=\; 0 \,,$$

so that a little algebra shows

$$a_0 \;=\; \frac{K_1(m\,r_o)\,\theta_b}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}$$

$$b_0 \;=\; \frac{I_1(m\,r_o)\,\theta_b}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,.$$

Finally, we can write the exact solution by substituting these expression into the general solution, obtaining

$$\theta(r) \;=\; \frac{I_0(m\,r)\,K_1(m\,r_o) \;+\; K_0(m\,r)\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\theta_b\,. \tag{A.7}$$

A.3. Heat Transfer

The heat transfer rate of the fin can be evaluated in the usual way of applying Fourier's Law at the base of the fin, c.f. §3.6 (pp. 24), which in this case we write

$$q \;=\; -\,k\,\left(A_c\,\frac{d\theta}{dr}\right)\bigg|_{r=r_i}\,,$$

where we are careful to evaluate A_c at the base, as well. (This matters because A_c is not constant in r, whereas it would not matter for constant cross–section fins.) At the base, we have $A_c = 2\,\pi\,r_i\,L$, which is the circumference of the fin at its base times the fin thickness. The derivative involves the Bessel functions, but these no longer pose any real difficulty since we have already established their derivative relationships.[A.1] We find

$$\begin{aligned}
\frac{d\theta}{dr} \;&=\; \frac{I_0'(m\,r)\,K_1(m\,r_o)\,m \;+\; K_0'(m\,r)\,m\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\theta_b \\
&=\; \frac{I_1(m\,r)\,K_1(m\,r_o) \;-\; K_1(m\,r)\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\theta_b\,m \\
\frac{d\theta}{dr}\bigg|_{r=r_i} \;&=\; \frac{I_1(m\,r_i)\,K_1(m\,r_o) \;-\; K_1(m\,r_i)\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\theta_b\,m\,,
\end{aligned}$$

so that

$$\begin{aligned}
q \;&=\; -\,k\,\Big(2\,\pi\,r_i\,t\Big)\,\theta_b\,m\,\frac{I_1(m\,r_i)\,K_1(m\,r_o) \;-\; K_1(m\,r_i)\,I_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)} \\
&=\; 2\,\pi\,r_i\,t\,k\,m\,\theta_b\,\frac{K_1(m\,r_i)\,I_1(m\,r_o) \;-\; I_1(m\,r_i)\,K_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\cdot \\
&=\; 2\,\pi\,r_i\,\theta_b\,\sqrt{2\,t\,k\,h}\,\frac{K_1(m\,r_i)\,I_1(m\,r_o) \;-\; I_1(m\,r_i)\,K_1(m\,r_o)}{I_0(m\,r_i)\,K_1(m\,r_o) \;+\; I_1(m\,r_o)\,K_0(m\,r_i)}\,\cdot \qquad \text{(A.8)}
\end{aligned}$$

where the expression for m in Eq. (3.22) on pp. 24 has been substituted into the leading product.

[A.1] Note however, that we are taking d/dr of, for example, $I_0(m\,r)$. One can readily check the expansion in the previous section to see that each term will be multiplied by an additional m, according to Chain Rule for the product $m\,r$. Consequently, $dI_0(m\,r)/dr = m\cdot I_0'(m\,r)$.

APPENDIX B

Separation of Variables for the 1–D Unsteady Dirichlet Problem

The Dirichlet problem is governed by the one–dimensional transient conduction law, Eq. (4.3) on pp. 30, repeated here for convenience

$$\frac{\partial T}{\partial t} \;=\; \alpha\, \frac{\partial^2 T}{\partial x^2}\;,$$

within the Cartesian domain $0 \le x \le L$ for time $t \ge 0$. Take the boundary conditions to be homogeneous of the first kind, i.e. $T(0,t) = T(L,t) = 0$. Finally, allow for an arbitrary initial distribution of T, i.e. the initial conditions are given by an unspecified function $T(x,0) = F(x)$.

B.1. Separation of Variables Method

The method begins with a conjecture that the solution can be written in the form

$$T(x,t) \;=\; \Psi(x)\;\Gamma(t)\;. \tag{B.1}$$

That is, we assume that the physical problem is such that the contribution related to temporal response can be separated from the contribution related to spatial variation (Carrier and Pearson, 1976). For example, consider the hypothetical case $T = (x^2-1)t$. This equation clearly represents a "separable" problem, where $\Psi = (x^2-1)$ and $\Gamma = t$. Conversely, the case $T = \sqrt{xt} + \tanh(x^2\sqrt{t} + \sqrt{x/t}\;)$ is not readily separable. In general, separation of variables works optimally when the governing equation is homogeneous, as in the case here.

Under the conjecture of Eq. (B.1), partial derivatives have certain forms. Using the Chain Rule of Calculus, we see

$$\frac{\partial T}{\partial t} \;=\; \Gamma(t)\,\frac{\partial \Psi(x)}{\partial t} \;+\; \Psi(x)\,\frac{\partial \Gamma(t)}{\partial t} \;=\; \Psi(x)\,\Gamma\,'(t)\;, \tag{B.2}$$

where the prime symbol denotes the derivative of a univariate function. Notice that the derivative of $\Psi(x)$ with respect to t vanishes because Ψ is only a function of x, not of t. Similarly, we can apply Chain Rule twice to find

$$\frac{\partial^2 T}{\partial x^2} \;=\; \Psi\,''(x)\,\Gamma(t)\;. \tag{B.3}$$

Proceeding, we now substitute Eqs. (B.2) and (B.3) into the conduction equation to obtain

$$\frac{1}{\alpha}\;\frac{\Gamma\,'(t)}{\Gamma(t)} \;=\; \frac{\Psi\,''(x)}{\Psi(x)}\;. \tag{B.4}$$

We have now cast the problem in a *separated* form where the left hand side is only a function of time t and the right hand side is only a function of the spatial coordinate x. However, according to principle, Eq. (B.4) must be valid for all x and t in the problem domain. It follows that each side

must be equal to a constant. Otherwise, either of x or t could be held fixed while the other could be varied such that Eq. (B.4) would be contradicted. Therefore,

$$\frac{1}{\alpha}\,\frac{\Gamma\,'(t)}{\Gamma(t)} \;=\; C_0 \qquad \text{and} \qquad \frac{\Psi\,''(x)}{\Psi(x)} \;=\; C_0\,,$$

where C_0 is a constant. Positive values for C_0 lead to exponentially increasing behavior and imaginary values (involving $i = \sqrt{-1}$) lead to periodic behavior (Carrier and Pearson, 1976). These responses can be verified by substitution. We are instead interested in the case where C_0 is negative. This leads to behavior that decays exponentially in time, a phenomenon compatible with the type of boundary and initial conditions we have specified. Therefore, we define

$$C_0 \;=\; -\zeta^2\,,$$

where $\zeta > 0$.

We have now developed two individual ordinary differential equations from the single partial differential equation

$$\Gamma\,'(t) \;+\; \alpha\zeta^2\Gamma(t) \;=\; 0 \qquad t \geq 0 \tag{B.5}$$

and

$$\Psi\,''(x) \;+\; \zeta^2\Psi(x) \;=\; 0 \qquad 0 \leq x \leq L\,. \tag{B.6}$$

These taken separately are each easier to solve than the original partial differential equation.

We must apply the same separation process to the boundary conditions. Using Eq. (B.1), the two boundary conditions can be written as

$$\Psi(0)\;\Gamma(t) \;=\; \Psi(L)\;\Gamma(t) \;=\; 0\,.$$

Now, $\Gamma(t)$ cannot vanish for arbitrary values of t, otherwise the whole solution would be trivial. Therefore we see that $\Psi(0)$ and $\Psi(L)$ must vanish instead. The boundary conditions are therefore

$$\Psi(0) \;=\; 0 \tag{B.7}$$

and

$$\Psi(L) \;=\; 0\,. \tag{B.8}$$

We do not perform a similar reduction on the initial condition. This is handled differently as discussed below. The problem in (x,t) has now been completely separated into two simpler problems: a spatial problem in x and a temporal problem in t.

B.2. Solution Procedure

We are now faced with solving the individual problems in x and t. As mentioned above, we are focusing specifically on homogeneous boundary conditions of the first kind. Cases involving other combinations of boundary conditions are more complex and often require a more generalized treatment (Özişik, 1980).

Because its coefficients are constant, Eq. (B.5) is readily solvable using the method of the auxiliary equation (Ross, 1965), which in this case is $\varphi + \alpha\zeta^2 = 0$, so that $\varphi = -\alpha\zeta^2$ is the only

root.[B.1] Consequently, Eq. (B.5) has a solution of the form

$$\Gamma(t) \;=\; C_t\, e^{-\alpha\zeta^2 t}\,, \tag{B.9}$$

where we leave the constant C_t undetermined for the moment. Eq. (B.6) can be solved in the same way, the auxiliary equation being $\varphi^2 + \zeta^2 = 0$, so that $\varphi = \pm i\zeta$, where $i = \sqrt{-1}$, as usual. It can be shown[B.2] that the solution can be expressed by the elementary trigonometric functions

$$\Psi(x) \;=\; C_1 \sin(\zeta x) \;+\; C_2 \cos(\zeta x)\,, \tag{B.10}$$

where C_1 and C_2 are constants of integration. If we apply the boundary condition in Eq. (B.7), we get

$$\Psi(0) \;=\; C_1 \cancelto{0}{\sin 0} \;+\; C_2 \cos 0 \;=\; 0\,,$$

which implies that $C_2 = 0$. Now apply the other condition in Eq. (B.8) to obtain

$$\Psi(L) \;=\; C_1 \sin(\zeta L) \;=\; 0\,.$$

It is clear that $C_1 \neq 0$, otherwise the entire solution would once again be trivial. Therefore, it must instead be the case that $\sin(\zeta L) = 0$, so that

$$\zeta L \;=\; n\pi\,,$$

where $n = 1, 2, 3, \ldots$ That is, the problem is only satisfied for certain values of ζ, called *eigen–values*, i.e. $\zeta = n\pi/L$. This is a so–called eigen–value problem.[B.3]

Note, we will now change the notation to reflect the fact that there are many admissible eigen–related values, thus $\zeta \to \zeta_n$, $C_1 \to C_n$, and $\zeta_n = n\pi/L$. Also note that in this case "$\sin(\zeta_n x)$" is called the eigen–function. Each mode n yields an elementary solution to the problem. A general solution is therefore obtained by the linear superposition of all modes $n = 1 \to \infty$. This can be written as

$$T(x,t) \;=\; \sum_{n=1}^{\infty} C_n \,\sin\left(\zeta_n x\right)\, e^{-\alpha\zeta_n^2 t}\,. \tag{B.11}$$

Eq. (B.11) now satisfies both the governing equation and the boundary conditions. Note that the single constant C_t from Eq. (B.9) has been subsumed into the set of mode coefficients, C_n. The C_n must now be determined such that the initial conditions are satisfied.

[B.1] We will find this same solution procedure is needed if solving this problem using different methods. See footnote 6.9 on pp. 67.

[B.2] The roots $\varphi = \pm i\zeta$ indicate the solution

$$\Psi(x) \;=\; \overline{C_1} e^{i\zeta x} \;+\; \overline{C_2} e^{-i\zeta x} \;=\; \overline{C_1}\Big(\cos(\zeta x) + i\,\sin(\zeta x)\Big) \;+\; \overline{C_2}\Big(\cos(\zeta x) - i\,\sin(\zeta x)\Big)\,.$$

The latter expression, obtained by Euler's Formula, is easily shown to be equivalent to Eq. (B.10), where $C_2 = \overline{C_1} + \overline{C_2}$ and $C_1 = i(\,\overline{C_1} - \overline{C_2}\,)$.

[B.3] Özişik (1980) discusses eigen–value problems in detail. This problem is part of a broader mathematical theory of boundary–value problems for ordinary differential equations called *Sturm–Liouville Theory* (e.g. Ross, 1965, chapter 12).

B.3. Determining Mode Coefficients

Directly plugging in initial conditions at $t = 0$ yields[B.4]

$$T(x,0) \;=\; F(x) \;=\; \sum_{n=1}^{\infty} C_n \,\sin(\zeta_n x)\,, \tag{B.12}$$

however, this appears to be a dead end at first glance since there are an infinite number of unknown C_n values and only one equation. In actuality, we see that this problem boils down to one of expanding an arbitrary function $F(x)$ into a series. In this case, the series will be a Fourier "sin" series (Boyce and DiPrima, 1977). Let us review how such a series is constructed.

First, we make use of the fact that the eigen–functions are *orthogonal*, that is, they obey[B.5]

$$\int_0^L \sin(\zeta_n x)\,\sin(\zeta_m x)\,dx \;=\; 0$$

for $m \neq n$ and

$$\int_0^L \sin(\zeta_n x)\,\sin(\zeta_m x)\,dx \;=\; N(\zeta_n)$$

for $m = n$, where $N(\zeta_n)$ is the *normalization integral*, commonly called the "norm", and is defined for this particular case as

$$N(\zeta_n) \;=\; \int_0^L \sin^2(\zeta_n x)\,dx\,.$$

Özişik (1980) and Ross (1965) have extensive discussions of orthogonality.

Next, we operate on Eq. (B.12) with $\int_0^L \sin(\zeta_n x)\,dx$ to obtain

$$\int_0^L F(x)\,\sin(\zeta_n x)\,dx \;=\; \int_0^L \sum_{m=1}^{\infty} C_m\,\sin(\zeta_m x)\,\sin(\zeta_n x)\,dx\,. \tag{B.13}$$

[B.4] Eq. (B.12) is an expansion of $F(x)$ in a sine series and is often called a "half–range" expansion in newer texts (Kreyszig, 1993).

[B.5] The cancellation property implied by orthogonality can be demonstrated directly in this case by evaluating the general form of the integral. Here, we refer directly to an integral table (e.g. Beyer, 1984). Noting that $\zeta_n = n\pi/L$ and $\zeta_m = m\pi/L$, where n and m are both positive integers, we find

$$\begin{aligned}
\int_0^L \sin(\zeta_n x)\,\sin(\zeta_m x)\,dx \;&=\; \left(\frac{\sin[(\zeta_n-\zeta_m)\,x]}{2(\zeta_n-\zeta_m)} \;-\; \frac{\sin[(\zeta_n+\zeta_m)\,x]}{2(\zeta_n+\zeta_m)}\right)\Bigg|_0^L \\
&=\; \left(\frac{\sin[(n-m)\pi\,x/L]}{2(n-m)\pi/L} \;-\; \frac{\sin[(n+m)\pi\,x/L]}{2(n+m)\pi/L}\right)\Bigg|_0^L \\
&=\; \left(\frac{\cancelto{0}{\sin[(n-m)\pi]}}{2(n-m)\pi/L} \;-\; \frac{\cancelto{0}{\sin[(n+m)\pi]}}{2(n+m)\pi/L}\right) \;-\; \left(\frac{\cancelto{0}{\sin 0}}{2(n-m)\pi/L} \;-\; \frac{\cancelto{0}{\sin 0}}{2(n+m)\pi/L}\right) \\
&=\; 0\,.
\end{aligned}$$

The two sine functions evaluated at $x = L$ vanish because n and m are both integers, so that $(n-m)\pi$ and $(n+m)\pi$ are both always some integer multiple of π, for which the sine is identically zero. It is clear by inspection that these same functions vanish when evaluated at $x = 0$. Consequently, the entire integral vanishes whenever $m \neq n$. Footnote C.2 on pp. 142 shows a similar derivation for orthogonality of the cosine function. A general proof of the orthogonality property is given in §6.6 starting on pp. 78 and is also discussed in various other texts, see e.g. Cotta (1993, appendix A) and Özişik (1980, note 1 on pp. 587). See also Ross (1965).

Note, the mode number symbol is arbitrary and we changed $n \to m$ in the summation to avoid ambiguity. We recall that the mode coefficients are constants rather than functions of x, so Eq. (B.13) can be simplified to

$$\int_0^L F(x)\, \sin(\zeta_n x)\, dx \;=\; \sum_{m=1}^{\infty} C_m \int_0^L \sin(\zeta_m x)\, \sin(\zeta_n x)\, dx\,. \tag{B.14}$$

The summation sign and coefficients have been taken outside of the integral. Now we make the observation that the right hand side vanishes except in the case where $m = n$ due to the orthogonality property. This can be better visualized if we write out the terms explicitly. We obtain

$$\begin{aligned}
\int_0^L F(x)\, \sin(\zeta_n x)\, dx \;=\;& C_1 \int_0^L \sin(\zeta_1 x)\, \sin(\zeta_n x)\, dx \;\to\; 0 \\
&+\; C_2 \int_0^L \sin(\zeta_2 x)\, \sin(\zeta_n x)\, dx \;\to\; 0 \\
&+\; C_3 \int_0^L \sin(\zeta_3 x)\, \sin(\zeta_n x)\, dx \;\to\; 0 \quad + \quad \cdots \\
&+\; C_n \int_0^L \sin(\zeta_n x)\, \sin(\zeta_n x)\, dx \qquad \boxed{\text{only term that does not vanish}} \\
&+\; C_{n+1} \int_0^L \sin(\zeta_{n+1} x)\, \sin(\zeta_n x)\, dx \;\to\; 0 \\
&+\; C_{n+2} \int_0^L \sin(\zeta_{n+2} x)\, \sin(\zeta_n x)\, dx \;\to\; 0 \quad + \quad \cdots
\end{aligned}$$

as the explicit representation of the series. Clearly, only mode C_n is non–zero, as implied by orthogonality. The rest of the modes are trivial and the summation itself vanishes. We can then simplify Eq. (B.14) to

$$\int_0^L F(x)\, \sin(\zeta_n x)\, dx \;=\; C_n \int_0^L \sin(\zeta_n x)\, \sin(\zeta_n x)\, dx \;=\; C_n\, N(\zeta_n)\,,$$

from which we find that the mode coefficients are given by

$$C_n \;=\; \frac{1}{N(\zeta_n)} \int_0^L F(x)\, \sin(\zeta_n x)\, dx \;=\; \frac{2}{L} \int_0^L F(x)\, \sin(\zeta_n x)\, dx\,, \tag{B.15}$$

where $N(\zeta_n) = L/2$ for this case.[B.6]

Eq. (B.11) satisfies the governing equation and the boundary conditions while Eq. (B.15) determines coefficients such that the initial condition $F(x)$ is also (simultaneously) satisfied. These

[B.6] The norm in this particular case is readily evaluated using standard tables (e.g. Beyer, 1984) as

$$N(\zeta_n) \;=\; \int_0^L \sin^2\left(\frac{n\,\pi\,x}{L}\right) dx \;=\; \left[\frac{x}{2} \;-\; \frac{L}{4\,n\,\pi}\, \sin\left(\frac{2\,n\,\pi\,x}{L}\right)\right]\Bigg|_0^L \;=\; \frac{L}{2}\,. \tag{B.16}$$

two equations give the final solution to the problem specified above, although if $F(x)$ is very complicated, evaluation of Eq. (B.15) may not be straightforward. Physically, the solution means that we know the temperature distribution $T(x,t)$ for the entire problem domain $0 \le x \le L$ and $t \ge 0$. We can therefore compute quantities of engineering interest such as heat flux at any position and any time.

B.4. Example: The Unit Initial Condition

Let us consider the unit initial condition $F(x) = 1$ as an actual example, i.e. the temperature at any x at time $t = 0$ is unity. We evaluate the mode coefficients using Eq. (B.15) as

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L 1\,\sin(\zeta_n x)\,dx \;=\; \frac{2}{L}\int_0^L \sin(\zeta_n x)\,dx \\ &= -\frac{2}{L\,\zeta_n}\cos(\zeta_n x)\Big|_0^L \;=\; -\frac{2\,L}{L\,n\,\pi}\cos\Big(\frac{n\,\pi\,x}{L}\Big)\Big|_0^L \\ &= -\frac{2}{n\,\pi}\left[\cos\left(\frac{n\,\pi\,L}{L}\right) \;-\; \cos 0\right] \\ &= \frac{2}{n\pi}\,[1 \;-\; \cos(n\,\pi)] \\ C_n &= \frac{2}{n\pi}\,[1 \;-\; (-1)^n] \end{aligned}$$

The last result is obtained by noting that n varies as $1, 2, 3, \ldots$, for which the cosine of $n\pi$ must correspondingly alternate between negative and positive 1. In fact, we can construct a table of how the term in brackets varies with n (Table B.1). Clearly, all even–numbered modes drop out

TABLE B.1. Values of the Alternating Term: $[1 - (-1)^n]$

n	1	2	3	4	5	6	$7\cdots$
value	2	0	2	0	2	0	$2\cdots$

of the problem because their resulting mode coefficients are all 0. However, odd–numbered modes remain. We can simplify C_n further by writing it only for the non–trivial odd modes as

$$C_n \;=\; 2\,\frac{2}{n\,\pi} \;=\; \frac{4}{n\,\pi} \qquad \text{where} \qquad n \;=\; 1, 3, 5, 7, \ldots$$

Finally, we can substitute C_n into the general solution in Eq. (B.11) to obtain

$$T(x,t) \;=\; \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\,\pi}\,\sin(\zeta_n x)\;e^{-\alpha\zeta_n^2 t}\,. \tag{B.17}$$

This expression can be used to evaluate temperature at any arbitrary (x,t) by writing a simple computer loop to add terms until a suitable convergence threshold has been reached.

B.5. Example: Transient Decay of Steady 1–D Conduction

In elementary one–dimensional steady conduction, the general solution for the temperature distribution is $T(x) = C_1\, x + C_2$ (c.f. footnote 3.2 on pp. 18). An interesting extension of this configuration is to examine its decay transient. For example, consider the specific steady–state conduction $T(x) = x/L$, which obviously implies $T(0) = 0$ and $T(L) = 1$. Now let the boundary temperature at L be changed to $T(L) = 0$, so that $T(x)$ decays to a final state of thermal equilibrium. We again evaluate the mode coefficients using Eq. (B.15) as

$$\begin{aligned}
C_n &= \frac{2}{L}\int_0^L \frac{x}{L}\,\sin(\zeta_n x)\,dx \;=\; \frac{2}{L^2}\int_0^L x\,\sin(\zeta_n x)\,dx \\
&= \frac{2}{L^2}\left[\frac{L^2}{n^2\,\pi^2}\,\sin\Big(\frac{n\,\pi\,x}{L}\Big) \;-\; \frac{x\,L}{n\,\pi}\,\cos\Big(\frac{n\,\pi\,x}{L}\Big)\right]\Bigg|_0^L \\
&= \frac{2}{L^2}\left[\left(\frac{L^2}{n^2\,\pi^2}\,\sin\,(n\,\pi) \;-\; \frac{L^2}{n\,\pi}\,\cos\,(n\,\pi)\right) \;\; - \right. \\
&\qquad\qquad \left.\left(\frac{L^2}{n^2\,\pi^2}\,\sin 0 \;-\; \frac{0\cdot L}{n\,\pi}\,\cos 0\right)\right] \\
&= -\,\frac{2}{n\,\pi}\,\cos\,(n\,\pi) \;=\; -\,\frac{2}{n\,\pi}\,(-1)^n \\
C_n &= \frac{2\,(-1)^{n+1}}{n\,\pi}\,.
\end{aligned}$$

Notice that, unlike the unit initial condition configuration, this problem retains all the eigen–modes, though they alternate sign. Substituting C_n into the general solution in Eq. (B.11), we then obtain the final solution for the decay transient as

$$T(x,t) \;=\; \sum_{n=1}^{\infty} \frac{2\,(-1)^{n+1}}{n\,\pi}\,\sin\,(\zeta_n x)\;e^{-\alpha\zeta_n^2 t}\,. \tag{B.18}$$

APPENDIX C

Separation of Variables for the 1–D Unsteady Robbins Problem

Similar to the procedures shown in appendix B, we apply the separation of variables technique to solve the Robbins problem, here cast in dimensionless form. Because much of the process is the same as for the Dirichlet problem, we will only show those parts that differ materially. Also, for this derivation only, *we will temporarily drop the "star" notation* (*) for convenience of presentation. That is, all quantities here are dimensionless, despite the absence of the "starred" variables.

Eq. (4.9) on pp. 34 is the one–dimensional transient conduction law in dimensionless form

$$\frac{\partial \theta}{\partial t} \;=\; \frac{\partial^2 \theta}{\partial x^2}$$

where θ is the dimensionless temperature, as defined in Eq. (4.6) on pp. 34. (Note the absence of the thermal diffusivity from the equation.) We consider the behavior of θ within the non–dimensionalized Cartesian domain $0 \le x \le 1$ for dimensionless time (Fourier number) $t \ge 0$. According to Eqs. (4.10) and (4.11), the dimensionless initial and (Robbins) boundary conditions are

$$\theta(x,0) \;=\; 1 \qquad \text{and} \qquad \left.\frac{\partial \theta(x,t)}{\partial x}\right|_{x=0} \;=\; 0 \qquad \text{and} \qquad -Bi \cdot \theta(1,t) \;=\; \left.\frac{\partial \theta(x,t)}{\partial x}\right|_{x=1} .$$

C.1. Solution Method

The separation of variables method used here follows the basic approach of Eq. (B.1) in appendix B, i.e. $\theta(x,t) = \Psi(x)\,\Gamma(t)$. The separation process leads to

$$\frac{\Gamma\,'(t)}{\Gamma(t)} \;=\; \frac{\Psi\,''(x)}{\Psi(x)} \;=\; -\,\zeta^2 \,,$$

where again ζ is a positive value. The separated, ordinary equations are

$$\Gamma\,'(t) \;+\; \zeta^2 \Gamma(t) \;=\; 0 \qquad \text{and} \qquad \Psi\,''(x) \;+\; \zeta^2 \Psi(x) \;=\; 0$$

and their respective general solutions are[C.1]

$$\Gamma(t) \;=\; C_t\, e^{-\zeta^2 t} \qquad \text{and} \qquad \Psi(x) \;=\; C_1 \sin(\zeta x) \;+\; C_2 \cos(\zeta x) \,,$$

where C_t, C_1, and C_2 are all undetermined constants. The reduced boundary conditions for $\Psi(x)$ are obtained by substituting the presumed form of θ into the physical boundary conditions, from which we obtain

$$\Psi\,'(0)\;\Gamma(t) \;=\; 0 \qquad \text{and} \qquad -\,Bi\;\Psi\,(1)\;\Gamma(t) \;=\; \Psi\,'(1)\;\;\Gamma(t)\,.$$

[C.1] See §B.2 on pp. 135 for the derivation of these solutions. Note that the problem in Γ here appears slightly different because of the dimensionless nature in which the current problem was cast.

As with the Dirichlet problem, we observe that $\Gamma(t)$ cannot vanish for arbitrary values of t, otherwise the whole solution would be trivial. Therefore, the boundary conditions imply $\Psi\,'(0) = 0$ and $-Bi = \Psi\,'(1)/\Psi\,(1)$.

The former means

$$\Psi\,'(0) \;=\; \zeta\, C_1\, \cos 0 \;-\; \zeta\, C_2 \cancelto{0}{\sin 0} \;=\; 0\,,$$

which implies that $C_1 = 0$. (Therefore, $C_2 \neq 0$ and the eigen–function is "cosine".) Using this observation, the latter requirement can now be written in the form

$$-Bi \;=\; \frac{-\,C_2\,\zeta\,\sin\zeta}{C_2\,\cos\zeta}\,,$$

which simplifies to

$$\zeta_n\,\tan\zeta_n \;=\; Bi\,. \tag{C.1}$$

Eq. (C.1) is the expression that governs the eigen–values. The "tangent" is a cyclical function and there are an infinite number of ζ_n that will satisfy this equation for a given value of Bi. However, it cannot be solved by elementary algebraic manipulation, except for a few cases, and will instead normally require a numerical solution (see below). Unlike in the Dirichlet problem where we had to explicitly discern the eigen phenomenon, the constant C_2 vanishes identically here. The eigen phenomenon is more obvious. Combining the solution for the separate x and t and using superposition, the general solution can be expressed as

$$\theta(x,t) \;=\; \sum_{n=1}^{\infty} C_n\,\cos\left(\zeta_n\,x\right)\,e^{-\zeta_n^2 t}\,,$$

where we still must determine the mode coefficients, C_n.

The mode coefficients are derived in the exact same fashion as in §B.3, i.e. by exploiting the orthogonality property of the associated eigen–function, here $\cos(\zeta_n x)$.[C.2] Specifically, the normalization integral can be evaluated using standard tables (e.g. Beyer, 1984) as

$$N(\zeta_n) \;=\; \int_0^1 \cos^2(\zeta_n\,x)\,dx \;=\; \left[\frac{x}{2} \;+\; \frac{1}{4\,\zeta_n}\,\sin(2\,\zeta_n\,x)\right]\Bigg|_0^1 \;=\; \frac{1}{2} \;+\; \frac{\sin(2\,\zeta_n)}{4\,\zeta_n}$$

and the C_n are then

$$C_n \;=\; \frac{1}{N(\zeta_n)}\int_0^1 \theta(x,0)\,\cos(\zeta_n\,x)\,dx \;=\; \frac{1}{N(\zeta_n)}\int_0^1 \cos(\zeta_n\,x)\,dx \;=\; \frac{1}{N(\zeta_n)}\,\frac{\sin(\zeta_n\,x)}{\zeta_n}\,\Bigg|_0^1\,,$$

since the initial condition is $\theta(x,0) = 1$. It can be shown without much trouble that evaluating the limits in this expression, multiplying by the ratio of $4\,\zeta_n$ with itself, and substituting the expression

[C.2] The cancellation property implied by orthogonality can be demonstrated directly in this case by evaluating the general form of the integral. Here, we refer directly to an integral table (e.g. Beyer, 1984). Noting that $\zeta_n = n\pi/L$ and $\zeta_m = m\pi/L$, where n and m are both positive integers, we find

$$\int_0^L \cos(\zeta_n x)\,\cos(\zeta_m x)\,dx \;=\; \left(\frac{\sin[(\zeta_n-\zeta_m)\,x]}{2(\zeta_n-\zeta_m)} \;+\; \frac{\sin[(\zeta_n+\zeta_m)\,x]}{2(\zeta_n+\zeta_m)}\right)\Bigg|_0^L\,,$$

which, with the exception of the sign between the two terms in parenthesis, is identical to the orthogonality integral for the sine function and which therefore vanishes according to the same reasoning given in footnote B.5 on pp. 137. See §6.6 starting on pp. 78 for a general discussion.

for the norm yields the mode coefficients, which are obvious in the final form of the solution

$$\theta(x,t) \;=\; \sum_{n=1}^{\infty} \frac{4\,\sin\zeta_n}{2\,\zeta_n + \sin(2\,\zeta_n)}\;\cos\left(\zeta_n\,x\right)\;e^{-\zeta_n^2 t}\;, \tag{C.2}$$

with the eigen–values specified by Eq. (C.1).

C.2. Determining the Eigen–Values

Unlike the Dirichlet problem in appendix B, most eigen–values here are not algebraic, but must rather be obtained numerically. The exceptions are

$$Bi = 0: \quad \tan\zeta_n = 0 \qquad \zeta_n = 0, \pi, 2\pi, \ldots, (n-1)\pi \qquad \text{and}$$

$$Bi \to \infty: \quad \tan\zeta_n = \infty \qquad \zeta_n = \frac{\pi}{2}, \frac{3\,\pi}{2}, \frac{5\,\pi}{2}, \cdots, \frac{(2n-1)\,\pi}{2}\;.$$

The clever numerical method for finding other eigen–values observes that the "tangent" function is non–negative and increases monotonically from $0 \to \infty$ only within each sub–domain between the two above limits

$$(n-1)\,\pi \;\le\; \zeta_n \;\le\; \frac{(2n-1)\,\pi}{2}$$

for $n = 1, 2, 3, \ldots$, as shown in Fig. C.1. Therefore, we would expect to find exactly one root in

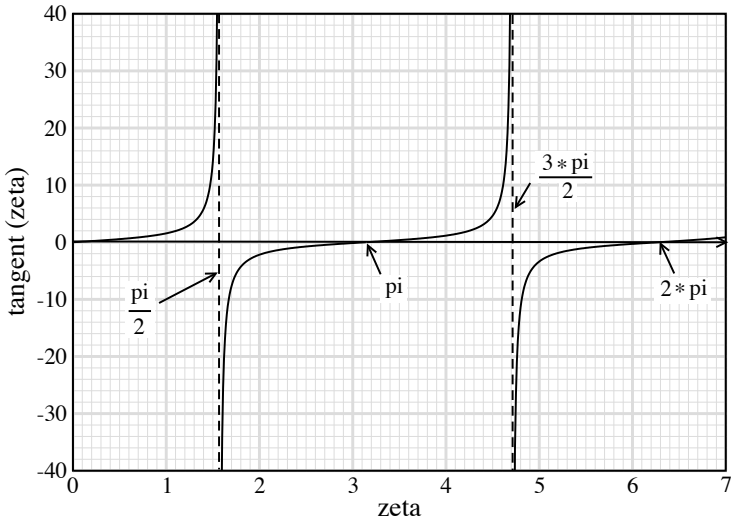


FIGURE C.1. *Cyclical nature of the "tangent" function.*

each of these sub–domains where Eq. (C.1) is satisfied for some particular value of the Biot number. Once this fact is observed, it is fairly straightforward to apply any standard numerical root–finding algorithm, e.g. bisection (Hamming, 1962), to find as many eigen–values as might be necessary for the convergence of Eq. (C.2). Algorithm C.1 can be implemented in any suitable language for this task. The first 8 eigen–values for a variety of Biot number values are furnished in Table C.1 using this method.

Algorithm C.1 Solving for Eigen–values from Eq.(C.1) Using the Bisection Method

input value of Biot number, Bi
set value of the mode number, n
set convergence threshold for the iteration, ϵ, suitably near 0
set lower bound of the initial bisection iteration as $\zeta_n^L = (n-1)\,\pi$
set upper bound of the initial bisection iteration as $\zeta_n^H = (2n-1)\,\pi/2$
while true **do**
  $\zeta_n = \zeta_n^L + (\zeta_n^H - \zeta_n^L)/2$
  $f = \zeta_n \tan\zeta_n - Bi$
  if $f < 0$ then $\zeta_n^L = \zeta_n$ else $\zeta_n^H = \zeta_n$
  if $|f| \leq \epsilon$ then exit loop
end while
print ζ_n

TABLE C.1. The First 8 Eigen–values for the Unsteady 1–D Robbins Problem

Bi	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6	ζ_7	ζ_8
0	0	π	$2\,\pi$	$3\,\pi$	$4\,\pi$	$5\,\pi$	$6\,\pi$	$7\,\pi$
0.001	0.0316	3.1419	6.2833	9.4249	12.5665	15.7080	18.8496	21.9912
0.002	0.0447	3.1422	6.2835	9.4250	12.5665	15.7081	18.8497	21.9912
0.004	0.0632	3.1429	6.2838	9.4252	12.5667	15.7082	18.8498	21.9913
0.006	0.0774	3.1435	6.2841	9.4254	12.5668	15.7083	18.8499	21.9914
0.008	0.0893	3.1441	6.2845	9.4256	12.5670	15.7085	18.8500	21.9915
0.01	0.0998	3.1448	6.2848	9.4258	12.5672	15.7086	18.8501	21.9916
0.02	0.1410	3.1479	6.2864	9.4269	12.5680	15.7092	18.8506	21.9921
0.04	0.1987	3.1543	6.2895	9.4290	12.5696	15.7105	18.8517	21.9930
0.06	0.2425	3.1606	6.2927	9.4311	12.5711	15.7118	18.8527	21.9939
0.08	0.2791	3.1668	6.2959	9.4333	12.5727	15.7131	18.8538	21.9948
0.1	0.3111	3.1731	6.2991	9.4354	12.5743	15.7143	18.8549	21.9957
0.2	0.4328	3.2039	6.3148	9.4459	12.5823	15.7207	18.8602	22.0002
0.3	0.5218	3.2341	6.3305	9.4565	12.5902	15.7270	18.8655	22.0048
0.4	0.5932	3.2636	6.3461	9.4670	12.5981	15.7334	18.8707	22.0093
0.6	0.7051	3.3204	6.3770	9.4879	12.6139	15.7460	18.8813	22.0184
0.8	0.7910	3.3744	6.4074	9.5087	12.6296	15.7587	18.8919	22.0275
1	0.8603	3.4256	6.4373	9.5293	12.6453	15.7713	18.9024	22.0365
2	1.0769	3.6436	6.5783	9.6296	12.7223	15.8336	18.9547	22.0815
3	1.1925	3.8088	6.7040	9.7240	12.7966	15.8945	19.0061	22.1259
4	1.2646	3.9352	6.8140	9.8119	12.8678	15.9536	19.0565	22.1697
5	1.3138	4.0336	6.9096	9.8928	12.9352	16.0107	19.1055	22.2126
6	1.3496	4.1116	6.9924	9.9667	12.9988	16.0654	19.1531	22.2545
8	1.3978	4.2264	7.1263	10.0949	13.1141	16.1675	19.2435	22.3351
10	1.4289	4.3058	7.2281	10.2003	13.2142	16.2594	19.3270	22.4108
12	1.4505	4.3636	7.3070	10.2869	13.3004	16.3414	19.4034	22.4814
14	1.4664	4.4074	7.3694	10.3586	13.3746	16.4142	19.4729	22.5468
16	1.4786	4.4416	7.4198	10.4184	13.4386	16.4786	19.5358	22.6071
18	1.4883	4.4690	7.4610	10.4688	13.4939	16.5357	19.5926	22.6624
20	1.4961	4.4915	7.4954	10.5117	13.5420	16.5864	19.6439	22.7131
25	1.5105	4.5330	7.5603	10.5947	13.6378	16.6901	19.7517	22.8221
30	1.5202	4.5615	7.6057	10.6543	13.7085	16.7691	19.8361	22.9098
35	1.5272	4.5822	7.6391	10.6989	13.7625	16.8305	19.9033	22.9809
40	1.5325	4.5979	7.6647	10.7334	13.8048	16.8794	19.9576	23.0394
50	1.5400	4.6202	7.7012	10.7832	13.8666	16.9519	20.0392	23.1287
60	1.5451	4.6353	7.7259	10.8172	13.9094	17.0026	20.0971	23.1931
80	1.5514	4.6543	7.7573	10.8606	13.9644	17.0686	20.1733	23.2788
100	1.5552	4.6658	7.7764	10.8871	13.9981	17.1093	20.2208	23.3327
1000	1.5692	4.7077	7.8461	10.9846	14.1230	17.2615	20.4000	23.5384
10000	1.5706	4.7119	7.8532	10.9945	14.1358	17.2770	20.4183	23.5596
∞	$\pi/2$	$3\,\pi/2$	$5\,\pi/2$	$7\,\pi/2$	$9\,\pi/2$	$11\,\pi/2$	$13\,\pi/2$	$15\,\pi/2$

APPENDIX D

The Similarity Technique Applied to the Rayleigh Problem

Like the Dirichlet problem, the Rayleigh problem is governed by the one–dimensional transient conduction law in the form of Eq. (4.3) on pp. 30

$$\frac{\partial T}{\partial t} \;=\; \alpha\,\frac{\partial^2 T}{\partial x^2}$$

that we consider within the Cartesian domain $0 \le x < \infty$ for time $t \ge 0$, where $T = T(x,t)$. We take the initial distribution of the temperature to be constant throughout,

$$T(x,0) = T_i$$

and the boundary conditions to be

$$T(0,t) \;=\; T_0 \qquad \text{and} \qquad T(x\to\infty,t) \;=\; T_i\,.$$

This problem has a direct analog in fluid mechanics called the Rayleigh Problem and we shall refer to it by that same name here (c.f. footnote 4.9 on pp. 37).

As it is stated, the Rayleigh problem has no obvious finite length scale and this characteristic is often a hint that a similarity transform might exist. In this transformation process, the partial differential equation, which depends upon two independent variables, x and t, is reduced to an ordinary differential equation, which depends upon only one independent variable: a single, unified similarity parameter. The routinely non–trivial first step in such a process is the mere identification of the similarity variable itself. The successful transform law must process both the equation and the boundary and initial conditions properly. Once the ordinary differential equation system is obtained, it must still be solved by conventional methods.

D.1. Derivation of the Similarity Parameter

There is no single, general theory for similarity analysis and a number of procedures are available. Here, we will opt for using the "stretching variables" procedure, a special case of group theoretic transformations, which govern the similarity parameter based on the concept of invariance (Hansen, 1967). Define

$$\overline{T} \;=\; \xi^a\,T\,, \qquad \overline{x} \;=\; \xi^b\,x\,, \qquad \overline{t} \;=\; \xi^c\,t\,,$$

where ξ is the parameter for the group. To obtain the equation in the "bar" (coordinate stretched) system, we apply Chain Rule in much the same way as we use it for non–dimensionalizing an equation (c.f. Eqs. (4.7) and (4.8) on pp. 34). We find

$$\xi^{c-a}\,\frac{\partial \overline{T}}{\partial \overline{t}} \;=\; \xi^{2b-a}\,\alpha\,\frac{\partial^2 \overline{T}}{\partial \overline{x}^2}\,.$$

To maintain invariance between the original and the stretched system, the ξ must be able to be canceled, implying $c-a = 2b-a$, or $c = 2b$. We immediately see that $a = 0$, otherwise the condition at the surface boundary in the stretched system, $\xi^{-a}\,\overline{T} = T_0$, would not be invariant.

Next, we must find a parameter, φ, that is invariant under transformations of the independent variables. We have two independent variables here, x and t, so we choose a power expression of their product of the form

$$\varphi \;=\; x\,t^d\,,$$

noting that there is no need to assume a power of x other than unity because d would be determined commensurately (Hansen, 1967). The invariance condition just mentioned for φ requires

$$x\,t^d \;=\; \overline{x}\,\overline{t}^d \;=\; \Big(\xi^b\,x\Big)\,\Big(\xi^c\,t\Big)^d \;=\; \xi^{b+cd}\,x\,t^d\,,$$

where $b + c\,d = 0$ for invariance to hold. Given $c = 2\,b$ from above, a little algebra shows

$$d \;=\; -\,\frac{b}{c} \;=\; -\,\frac{b}{2\,b} \;=\; -\,\frac{1}{2}\,,$$

so that the final similarity parameter is[D.1]

$$\varphi \;=\; \frac{x}{\sqrt{t}}\,. \tag{D.1}$$

Eq. (D.1) shows the two physical independent variables, x and t, incorporated into one unified similarity variable, φ, and this combination will allow reduction of the partial differential equation system into an ordinary one.

D.2. Reduction of the Partial Differential Equation System to an Ordinary One

A successful similarity parameter must do two things. First, it must reduce the governing equation from a PDE to an ODE. Second, it must also consistently transform the boundary and initial conditions in a manner such that the total of 3 conditions in x and t collapse to 2 conditions in φ.

The Chain Rule can be used to determine what the derivatives are with respect to the new (transformed) variable φ, specifically

$$\frac{\partial}{\partial t} \;=\; \frac{d}{d\varphi}\,\frac{\partial\varphi}{\partial t} \;=\; -\,\frac{x}{2\,t^{3/2}}\,\frac{d}{d\varphi}$$

and

$$\frac{\partial}{\partial x} \;=\; \frac{\partial\varphi}{\partial x}\,\frac{d}{d\varphi} \qquad \text{which implies} \qquad \frac{\partial^2}{\partial x^2} \;=\; \left(\frac{\partial\varphi}{\partial x}\right)^2\frac{d^2}{d\varphi^2} \;=\; \left(\frac{1}{t^{1/2}}\right)^2\frac{d^2}{d\varphi^2} \;=\; \frac{1}{t}\,\frac{d^2}{d\varphi^2}\,.$$

Substituting these into the governing equation we find

$$-\,\frac{x}{2\,t^{3/2}}\,\frac{dT}{d\varphi} \;=\; \alpha\,\frac{1}{t}\,\frac{d^2T}{d\varphi^2}\,,$$

[D.1] The successful similarity parameter is not necessarily unique to a specific problem and Eq. (D.1) is only one example of a parameter that works for the Rayleigh problem. Other commonly used parameters are $x/\sqrt{4\,\alpha\,t}$ (Schlichting, 1979; Panton, 1984) and $x/\sqrt{\alpha\,t}$ (Batchelor, 1967), which clearly differ from Eq. (D.1) by only a constant. Incidentally, the forms specified here are also dimensionless, a property that is not actually necessary for the solution process. Note that that the relationship between the independent variables, x and t, remains the same for all 3 forms and this is the salient property.

which can be simplified using substitution of Eq. (D.1) and a little algebra to

$$\frac{d^2T}{d\varphi^2} \;+\; \frac{1}{2\,\alpha}\,\varphi\,\frac{dT}{d\varphi} \;=\; 0\,. \tag{D.2}$$

Note that Eq. (D.2) is now an ordinary differential equation (second–order), having the single independent variable φ.

Being second–order, this equation requires precisely 2 boundary conditions. However, we have 3 conditions in the original problem, one on time and two on space. We must apply the similarity transform to the original conditions, too, which is simply a matter of substituting the given values of x and t (as combined in Eq. (D.1)) into the assigned boundary values of $T(x,t)$, as follows:

$$T(x,0) \;=\; T_i \quad \text{transforms to} \quad T(\varphi\to\infty) \;=\; T_i \qquad \text{since } \varphi \;\to\; \frac{x}{\sqrt{0}}$$

$$T(0,t) \;=\; T_0 \quad \text{transforms to} \quad T(\varphi=0) \;=\; T_0 \qquad \text{since } \varphi \;=\; \frac{0}{\sqrt{t}}$$

$$T(x\to\infty,t) \;=\; T_i \quad \text{transforms to} \quad T(\varphi\to\infty) \;=\; T_i \qquad \text{since } \varphi \;\to\; \frac{\infty}{\sqrt{t}}\,.$$

Here, we note the significant result that the initial condition and one of the boundary conditions collapse into a single, redundant statement. We now have the necessary two boundary conditions

$$T(\varphi=0) \;=\; T_0 \qquad \text{and} \qquad T(\varphi\to\infty) \;=\; T_i \tag{D.3}$$

and, taken with the ODE above in Eq. (D.2), we have fully transformed the problem of $T(x,t)$ to one of $T(\varphi)$.

D.3. Solution Procedure and the Gaussian Error Function

The most straightforward method to solve the ODE in Eq. (D.2) is to make the substitution

$$\xi = \frac{dT}{d\varphi}\,,$$

so that this expression becomes

$$\frac{d\xi}{d\varphi} \;+\; \frac{1}{2\,\alpha}\,\varphi\,\xi \;=\; 0\,.$$

A moment's inspection reveals that this differential equation is separable, and can therefore be solved by direct integration. In particular, further development shows

$$\frac{d\xi}{\xi} \;=\; -\,\frac{1}{2\,\alpha}\,\varphi\,d\varphi\,,$$

which is integrated once (as an indefinite integral) to obtain

$$\ln\xi \;=\; -\,\frac{1}{4\,\alpha}\,\varphi^2 \;+\; C_1\,,$$

where C_1 is the as–of–yet undetermined integration constant. According to the substitution above, ξ still involves an additional differential of the temperature T, for which we are trying to solve. The second integration is staged by first exponentiating the current expression

$$\xi \;=\; e^{-(\varphi/\sqrt{4\,\alpha}\,)^2+C_1} \qquad \text{or} \qquad \frac{dT}{d\varphi} \;=\; \overline{C_1}\,e^{-(\varphi/\sqrt{4\,\alpha}\,)^2}\,, \tag{D.4}$$

which we immediately see is once again separable. However, the apparently straightforward right–hand–side is misleading: integrands[D.2] having the form $e^{-\eta^2}\,d\eta$ cannot be expressed in terms of elementary functions, but rather involve the so–called Gaussian error function "erf" (Andrews, 1985). Before proceeding to integrate Eq. (D.4), let us report, without derivation, a few of the basic properties of erf.[D.3]

$$\text{(D.5)} \qquad \text{definition} \qquad \operatorname{erf}(\eta) \;=\; \frac{2}{\sqrt{\pi}} \int_0^{\eta} e^{-\chi^2}\,d\chi$$

$$\text{(D.6)} \qquad \text{"erf" is an odd function} \qquad \operatorname{erf}(-\eta) \;=\; -\operatorname{erf}(\eta)$$

$$\text{(D.7)} \qquad \text{zero value} \qquad \operatorname{erf}(0) \;=\; 0$$

$$\text{(D.8)} \qquad \text{asymptotic behavior} \qquad \operatorname{erf}(+\infty) \;=\; 1$$

Returning to the topic of Eq. (D.4), we see that our problem is related, but not yet in the precise form of the definition of the error function in Eq. (D.5). We can readily make the following change of variables:

$$\eta \;=\; \frac{\varphi}{\sqrt{4\,\alpha}} \qquad\qquad d\eta \;=\; \frac{1}{\sqrt{4\,\alpha}}\,d\varphi\,,$$

so that we can formally integrate Eq. (D.4) as

$$T \;=\; \overline{C_1}\,\frac{\sqrt{\pi}}{2}\cdot\frac{2}{\sqrt{\pi}}\cdot\int e^{-\eta^2}\,\sqrt{4\,\alpha}\,d\eta \;=\; \overline{\overline{C_1}}\left(\frac{2}{\sqrt{\pi}}\int e^{-\eta^2}\,d\eta\right) \;=\; \overline{\overline{C_1}}\,\operatorname{erf}(\eta) \;+\; C_2\,,$$

so that going back from η to φ

$$T(\varphi) \;=\; \overline{\overline{C_1}}\,\operatorname{erf}\left(\frac{\varphi}{\sqrt{4\,\alpha}}\right) \;+\; C_2\,.$$

Note that, in this particular case, the various involved constants are superfluous because of the indefinite nature of the integration and we simply absorb them into a new constant $\overline{\overline{C_1}}$. The remainder of the problem is now simply to use the transformed boundary conditions in Eq. (D.3) to determine the two constants, $\overline{\overline{C_1}}$ and C_2. We find

$$T(0) \;=\; T_0 \;=\; \overline{\overline{C_1}}\,\operatorname{erf}(0) \;+\; C_2 \qquad\rightarrow\qquad C_2 \;=\; T_0 \quad \text{c.f. Eq. (D.7)}$$

$$T(\infty) \;=\; T_i \;=\; \overline{\overline{C_1}}\,\operatorname{erf}(\infty) \;+\; T_0 \qquad\rightarrow\qquad \overline{\overline{C_1}} \;=\; T_i \;-\; T_0 \quad \text{c.f. Eq. (D.8)}\,.$$

Using these results and substituting back the original x and t from the similarity parameter in Eq. (D.1), we see that the final solution to the Rayleigh problem can be expressed as

$$\frac{T(x,t) \;-\; T_0}{T_i \;-\; T_0} \;=\; \operatorname{erf}\left(\frac{x}{\sqrt{4\,\alpha\,t}}\right)\,.$$

We note that "erf" is asymptotic in both the positive and negative x directions (Fig. D.1). In this particular case, the independent variables do not take on negative values, i.e. $x \geq 0$ and $t \geq 0$,

[D.2] Incidentally, $e^{-\eta^2}$ is also the Gaussian or normal distribution from probability theory and is often colloquially referred to as the "bell curve" (Feller, 1968). This is one of the truly numerous examples of how seemingly disparate areas of mathematics are fundamentally related to one another. We will encounter the error function again in §L.4 starting on pp. 202 in examining the boundary layer profile.

[D.3] The error function is discussed in detail in many sources, e.g. Abramowitz and Stegun (1972), Özişik (1980), and Andrews (1985).

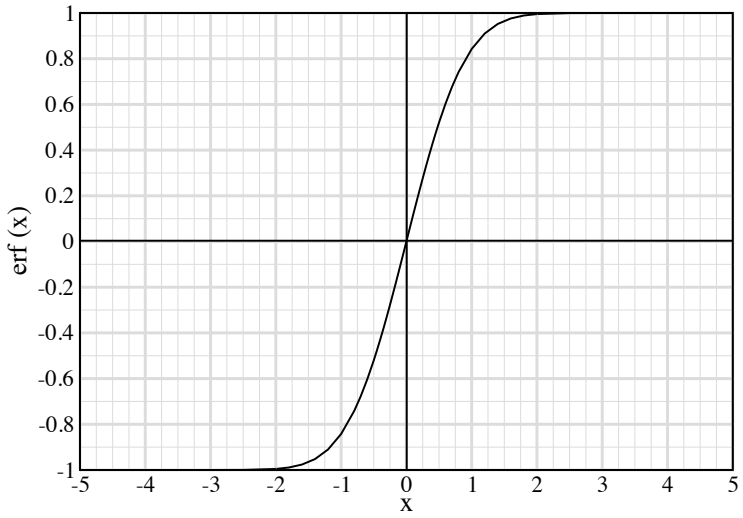


FIGURE D.1. *The Gaussian error function*

so "erf" will vary between 0 and 1. The implications that can be discerned from the solution in light of Fig. D.1 are that the T remains roughly T_i where $x > 2\sqrt{4\,\alpha\,t}$ and has changed to T_0 where $x \ll \sqrt{4\,\alpha\,t}$.

D.4. Heat Transfer

As with other configurations, the heat transfer can now be derived directly from Fourier's Law in Eq. (2.2) on pp. 8

$$q'' \;=\; -k\,\frac{dT}{dx}\,,$$

for which it is clear we must be able to take the derivative in x of the above solution. The form of this derivative can be deduced from the definition of the error function itself in Eq. (D.5) as

$$\begin{aligned} q'' &= -k\,\frac{d}{dx}\left[(T_i \,-\, T_0)\,\mathrm{erf}\left(\frac{x}{\sqrt{4\,\alpha\,t}}\right) \,+\, T_0\right] \\ &= -k\,(T_i \,-\, T_0)\,\frac{2}{\sqrt{\pi}}\,e^{-x^2/(4\alpha t)}\,\frac{1}{\sqrt{4\,\alpha\,t}} \\ &= \frac{k\,(T_0 - T_i)}{\sqrt{\pi\,\alpha\,t}}\,e^{-x^2/(4\alpha t)}\,, \end{aligned}$$

so that at the boundary we see

$$q\,\Big|_{x=0} \;=\; \frac{k\,(T_0 - T_i)}{\sqrt{\pi\,\alpha\,t}}\,,$$

i.e. the heat transfer at $x = 0$ responds as $t^{-1/2}$.

APPENDIX E

Solving the 2–D Steady Dirichlet Problem

THE TWO–DIMENSIONAL STEADY PROBLEM is governed by the Laplace equation, written here in rectangular coordinates

$$\frac{\partial^2 T}{\partial x^2} \,+\, \frac{\partial^2 T}{\partial y^2} \,=\, 0\,,$$

which we will solve for the case of Dirichlet boundary conditions for both a finite field and a semi–infinite field, as discussed in §5.1 on pp. 40.

E.1. The Finite Problem

The finite problem is described by the boundary conditions

$$T(0,y) \,=\, T(L,y) \,=\, T(x,H) \,=\, 0 \qquad \text{and} \qquad T(x,0) \,=\, F(x)\,,$$

repeated here for convenience from Eq. (5.2). The solution procedure for this problem is very similar to that for the unsteady 1–D Dirichlet problem (c.f. §B.1 on pp. 134). That is, assume a separation law of the form $T(x,t) = \Psi(x)\,\Gamma(y)$, so that

$$\frac{\Psi\,''(x)}{\Psi(x)} \,=\, -\,\frac{\Gamma\,''(y)}{\Gamma(y)} \,=\, -\,\zeta^2\,.$$

In fact, the problem in Ψ is identical to the one shown in §B.1: the corresponding governing equation being Eq. (B.6) on pp. 135, and boundary conditions being Eqs. (B.7) and (B.8). There, we found the solution to be

$$\Psi_n \,=\, C_n\,\sin(\zeta_n x) \qquad\qquad \zeta_n \,=\, \frac{n\,\pi}{L}\,,$$

where ζ_n are the eigen–values and C_n are the yet–to–be–determined coefficients. The problem in Γ is, however, different:

$$\Gamma_n'' \,-\, \zeta_n^2\,\Gamma_n \,=\, 0\,,$$

where our notation now reflects the solution for the n–th eigen–value, Γ_n. The method of the auxiliary equation (Ross, 1965) gives $\varphi^2 - \zeta^2 = (\varphi+\zeta)(\varphi-\zeta) = 0$, so that the general solution is

$$\Gamma_n(y) \,=\, A_n\,e^{\zeta_n\,y} \,+\, B_n\,e^{-\zeta_n\,y}\,. \tag{E.1}$$

The homogeneous boundary condition quoted above separates as $T(x,H) = \Psi(x)\Gamma(H) = 0$, so that the solution evaluated at $y = H$ is

$$\Gamma_n(H) \,=\, A_n\,e^{\zeta_n\,H} \,+\, B_n\,e^{-\zeta_n\,H} \,=\, 0\,.$$

Because the domain is finite in y, neither of the terms vanishes and we must instead solve for one of the constants in terms of the other, e.g. as $B_n = -A_n e^{2\,\zeta_n\,H}$. Substituting this expression back

into the general solution, we may observe the following clever evaluation:

$$\begin{aligned}
\Gamma_n(y) &= A_n\, e^{\zeta_n\, y} \;-\; A_n\, e^{2\,\zeta_n\, H}\, e^{-\zeta_n\, y} \\
&= A_n \left(e^{\zeta_n\, y} \;-\; e^{2\,\zeta_n\, H}\, e^{-\zeta_n\, y} \right) \\
&= \frac{A_n}{e^{-\zeta_n\, H}} \left(e^{\zeta_n\, y}\, e^{-\zeta_n\, H} \;-\; e^{2\,\zeta_n\, H}\, e^{-\zeta_n\, H}\, e^{-\zeta_n\, y} \right) \\
&= \frac{A_n}{e^{-\zeta_n\, H}} \left(e^{-\zeta_n\,(H-y)} \;-\; e^{\zeta_n\,(H-y)} \right) \\
&= -\,\frac{2\,A_n}{e^{-\zeta_n\, H}} \left(\frac{e^{\zeta_n\,(H-y)} \;-\; e^{-\zeta_n\,(H-y)}}{2} \right) \\
\Gamma_n(y) &= \overline{A_n}\, \sinh\Big[\zeta_n(H-y)\Big]
\end{aligned}$$

where we have noted the fact that the left term is still a constant, which we relabel as $\overline{A_n}$ and the term in parenthesis simplifies according to the identity in Eq. (3.14) on pp. 21.

Once again, the solution to the main problem is the linear superposition of all modes for Ψ and Γ, which can be written as

$$T(x,y) \;=\; \sum_{n=1}^{\infty} \overline{C_n}\, \sin(\zeta_n x)\, \sinh\Big[\zeta_n(H-y)\Big]\,,$$

where we have subsumed $\overline{A_n}$ into the original constant C_n, writing their product as $\overline{C_n}$.

These unknown mode coefficients, $\overline{C_n}$, are derived once again in the exact same fashion as in §B.3, i.e. by exploiting the orthogonality property of the associated eigen–function. While the eigen–function is exactly the same as in that problem, $\sin(\zeta_n x)$, the context is slightly different because this is a steady 2–D steady problem as opposed to a 1–D unsteady one. The procedure is the same, although we do not explicitly show the cancellation of terms using the orthogonality property, as was furnished in detail in §B.3. Starting with evaluation at the remaining (last) boundary,[E.1]

$$T(x,0) \;=\; F(x) \;=\; \sum_{n=1}^{\infty} \Big[\, \overline{C_n}\, \sinh(\zeta_n H) \Big]\, \sin(\zeta_n x)\,,$$

[E.1] Though not as obvious as Eq. (B.12) on pp. 137, this expression is also a "half–range" expansion of $F(x)$ in a sine series (Kreyszig, 1993). The $\sinh(\zeta_n H)$ term may initially seem not to belong, but it must be remembered that for every n, this term has a *value*. That is, it is not a function of the independent variable, x. In this sense, it can properly be thought of as part of the coefficient of each term in the expansion, which is why we have grouped it with $\overline{C_n}$ in square parenthesis. Eq. (E.2) below shows the final integral form for the $\overline{C_n}$ and it is clear that the $\sinh(\zeta_n H)$ term is divided out. In other words, the remainder of that term is precisely the conventional form for a half–range expansion in terms of sine (Kreyszig, 1993). This term does *not* appear in the solution for the problem of the semi–infinite domain. Why? A similar phenomenon to the above is found in half–range cosine expansions, see e.g. footnote F.1.

we integrate in terms of the eigen–function

$$\begin{aligned}\int_0^L F(x)\,\sin(\zeta_n x)\,dx &= \int_0^L \sum_{m=1}^{\infty} \overline{C_m}\,\sin(\zeta_m x)\,\sinh(\zeta_m H)\,\sin(\zeta_n x)\,dx \\ &= \sum_{m=1}^{\infty} \overline{C_m}\,\sinh(\zeta_m H)\int_0^L \sin(\zeta_m x)\,\sin(\zeta_n x)\,dx \\ &= \overline{C_n}\,\sinh(\zeta_n H)\int_0^L \sin^2(\zeta_n x)\,dx \\ &= \overline{C_n}\,\sinh(\zeta_n H)\,\frac{L}{2}\,.\end{aligned}$$

Note again that the simplification in the second–to–last line comes about because the integral is zero, i.e. vanishes, for every term where $m \neq n$. The only single term for which it survives is when $m = n$, meaning the summation vanishes, as well. In the last line, we observe that the normalization integral is identical to that in Eq. (B.16) on pp. 138, i.e. $L/2$. Solving for $\overline{C_n}$, we see

$$\overline{C_n} \;=\; \frac{2}{L\,\sinh(\zeta_n H)}\int_0^L F(x)\,\sin(\zeta_n x)\,dx\,, \tag{E.2}$$

so that the complete solution can be written

$$T(x,y) \;=\; \frac{2}{L}\sum_{n=1}^{\infty}\left(\frac{1}{\sinh(\zeta_n H)}\int_0^L F(x)\,\sin(\zeta_n x)\,dx\right)\sin(\zeta_n x)\,\sinh\Big[\zeta_n(H-y)\Big]\,, \tag{E.3}$$

where the eigen–values are explicit: $\zeta_n = n\pi/L$.

E.2. The Semi–Infinite Problem

The semi–infinite problem is obtained in the physical realm by merely moving the top boundary "very far away", whereby the associated boundary condition becomes

$$T(x, y\to\infty) \;=\; 0\,.$$

The separation assumption and the problem in Ψ are identical to those in the previous section. While the equation in Γ and its general solution in Eq. (E.1) are also the same, the boundary condition is obviously different. Here, we find

$$\Gamma_n(\infty) \;=\; A_n\,e^{\infty} \;+\; B_n\,e^{-\infty} \;=\; 0\,,$$

which immediately implies $A_n = 0$. Superposition then further implies a general solution of the form

$$T(x,y) \;=\; \sum_{n=1}^{\infty}\overline{C_n}\,\sin(\zeta_n x)\,e^{-\zeta_n y}\,,$$

where we have again combined constants, as now seems to be conventional. Again, we evaluate the constants by exploiting the orthogonality property when the remaining boundary condition

$T(x,0) = F(x)$ is imposed:

$$\begin{aligned}
\int_0^L F(x)\ \sin(\zeta_n x)\,dx &= \int_0^L \sum_{m=1}^{\infty} \overline{C_m}\ \sin(\zeta_m x)\cdot e^0 \cdot \sin(\zeta_n x)\,dx \\
&= \sum_{m=1}^{\infty} \overline{C_m}\cdot 1 \cdot \int_0^L \sin(\zeta_m x)\ \sin(\zeta_n x)\,dx \\
&= \overline{C_n}\int_0^L \sin^2(\zeta_n x)\,dx \\
&= \overline{C_n}\,\frac{L}{2}\ ,
\end{aligned}$$

so that

$$\overline{C_n} \;=\; \frac{2}{L}\int_0^L F(x)\ \sin(\zeta_n x)\,dx\ ,$$

which is identical to the coefficients for the 1–D unsteady Dirichlet problem (see Eq. (B.15) on pp. 138). It is fairly straightforward to see why this should be the case. Both solutions have the same eigen–function *and* a component having the form $e^{-\varphi}$ (where φ is t or y, as appropriate) whose specification at $\varphi = 0$ is the non–homogeneous boundary condition of the problem (Eqs. (B.11) and (B.12) on pp. 136 prove this statement for the unsteady problem). Consequently, that component is $e^0 = 1$ at this boundary when the mode coefficients are evaluated, leaving only the product of the identical eigen–function and $F(x)$. Similar to the finite problem, we can write the fully–incorporated general solution as

$$T(x,y) \;=\; \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L F(x)\ \sin(\zeta_n x)\,dx\right)\ \sin(\zeta_n x)\,e^{-\zeta_n y}\ , \tag{E.4}$$

where the eigen–values are once again explicit: $\zeta_n = n\pi/L$.

E.3. Example: Convergence of Geometries for a Constant Temperature Boundary

Suppose we take the example of $T(x,0) = F(x) = C$, where C is a constant. The boundary integral, identical in Eqs. (E.3) and (E.4), evaluates as

$$\begin{aligned}
\int_0^L F(x)\ \sin(\zeta_n x)\,dx &= C\int_0^L \sin(\zeta_n x)\,dx \\
&= -\frac{C}{\zeta_n}\cos(\zeta_n x)\Big|_0^L \\
&= -\frac{C}{\zeta_n}\ \left[\cos\left(\frac{n\,\pi\,L}{L}\right)\ -\ \cos 0\right] \\
&= \frac{C}{\zeta_n}\ [1\ -\ \cos(n\,\pi)] \\
&= \frac{C\,L}{n\,\pi}\ [1\ -\ (-1)^n]\ .
\end{aligned}$$

Like the result in §B.4 on pp. 139, the even modes vanish, leaving only the odd modes and an extra coefficient of 2. The solutions to the finite and semi–infinite cases can then be written respectively as

$$T(x,y) \;=\; \frac{4\,C}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n\,\sinh(\zeta_n H)}\,\sin(\zeta_n x)\,\sinh\Big[\zeta_n (H-y)\Big]\,,$$

$$T(x,y) \;=\; \frac{4\,C}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n}\,\sin(\zeta_n x)\,e^{-\zeta_n y}\,,$$

where again $\zeta_n = n\pi/L$ in both equations.

E.4. Uniqueness Demonstrated via Green's Theorem

Here, we demonstrate the standard proof that the Dirichlet problem for steady 2–D heat conduction, or any other mechanics problem governed by Laplace's equation, for that matter, has only one unique solution. We initially framed the proposition in §5.3 on pp. 49 in terms of a rectangular domain, assuming two existing solutions, T_1 and T_2, where $T_1 \neq T_2$. The observations established there so far are that the auxiliary solution we defined, $\theta = T_1 - T_2$, also satisfies Laplace's equation, $\nabla^2\theta = 0$ and θ is zero around the entire boundary. A moment of consideration should demonstrate that the very way we define θ as the difference of two solutions which *both* satisfy the boundary conditions will lead to θ vanishing around the boundary, regardless of its shape. We immediately observe then that the proposition we are about to demonstrate actually holds for the Dirichlet problem in general, independent of the geometry of the domain. Let us then assume a domain of arbitrary shape (Fig. E.1). The proposition can be extended to 3 dimensions using arguments

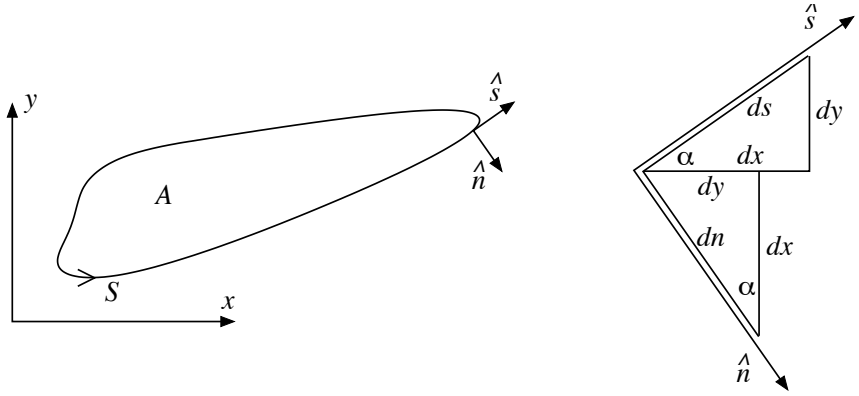


FIGURE E.1. *Two–dimensional domain A bounded by a closed boundary curve S with tangent and outward normal unit vectors, ŝ and n̂, respectively (left panel) and resolution of tangent and normal differentials in terms of Cartesian differentials (right panel).*

similar to what we will show here.

Start with a statement of Green's theorem in the plane.[E.2] If the domain A is smoothly bounded by closed curve S and $F_1(x,y)$ and $F_2(x,y)$ are continuous functions having continuous partial

[E.2]Green's theorem is fundamentally important in vector calculus and is derived and discussed extensively in standard texts, e.g. Courant (1937), Hildebrand (1976), and Kreyszig (1993). A short derivation of several additional forms of Green's identities is shown in appendix G on pp. 166.

derivatives with respect to y and x, respectively, then

$$\iint\limits_A \left(\frac{\partial F_2}{\partial x} \; - \; \frac{\partial F_1}{\partial y} \right) dx\, dy \; = \; \oint\limits_S \left(F_1 \, \frac{dx}{ds} \; + \; F_2 \, \frac{dy}{ds} \right) \, ds \, , \tag{E.5}$$

where the right hand side is a contour integral taken in the conventional counter–clockwise direction, as indicated in Fig. E.1.

The basic approach to proving uniqueness for the Dirichlet problem using this theorem will be to show that $\nabla\theta = 0$, in the domain so that θ must be a constant, then to further show that the constant must itself be 0 because θ is identically zero around the boundary. This process involves intelligently picking the functions F_1 and F_2 in Green's theorem:

$$F_1 \; = \; - \, \theta \, \frac{\partial \theta}{\partial y} \qquad \text{and} \qquad F_2 \; = \; \theta \, \frac{\partial \theta}{\partial x} \, .$$

Direct substitution shows

$$\iint\limits_A \left[\frac{\partial}{\partial x} \left(\theta \, \frac{\partial \theta}{\partial x} \right) \; - \; \frac{\partial}{\partial y} \left(- \, \theta \, \frac{\partial \theta}{\partial y} \right) \right] dx\, dy \; = \; \oint\limits_S \left(- \, \theta \, \frac{\partial \theta}{\partial y} \, \frac{dx}{ds} \; + \; \theta \, \frac{\partial \theta}{\partial x} \, \frac{dy}{ds} \right) \, ds \, ,$$

Expanding the integrand on the left and simplifying the one on the right

$$\iint\limits_A \left[\theta \, \frac{\partial^2 \theta}{\partial x^2} \; + \; \left(\frac{\partial \theta}{\partial x} \right)^2 + \; \theta \, \frac{\partial^2 \theta}{\partial y^2} \; + \; \left(\frac{\partial \theta}{\partial y} \right)^2 \right] dx\, dy \; = \; \oint\limits_S \theta \left(\frac{\partial \theta}{\partial x} \, \frac{dy}{ds} \; - \; \frac{\partial \theta}{\partial y} \, \frac{dx}{ds} \right) \, ds \, .$$

It will be more useful from this point further to convert to purely vector notation. First, let us observe that, given the convention of counter–clockwise contour integration, the tangent and outward unit normal vectors are defined as shown in (Fig. E.1). In particular, the outward unit normal is

$$\hat{n} \; = \; \sin\alpha \, \hat{i} \; - \; \cos\alpha \, \hat{j} \; = \; \frac{dy}{ds} \, \hat{i} \; - \; \frac{dx}{ds} \, \hat{j} \, .$$

The first assertion follows from a simple trigonometric observation on the triangle adjacent to $\hat{n}$, in particular that the vector length is indeed unity,[E.3] while the second is derived from the identical triangle adjacent to $\hat{s}$. The latter naturally leads to the incorporation of ds, which is appears in the theorem, rather than dn, which does not. Noting also that

$$\nabla\theta \; = \; \frac{\partial \theta}{\partial x} \, \hat{i} \; + \; \frac{\partial \theta}{\partial y} \, \hat{j} \, ,$$

we can regroup terms in the left–side integrand and utilize the vector quantities to write this instance of Green's theorem as

$$\iint\limits_A \Big(\theta \, \cancelto{0}{\nabla^2 \theta} + \; \nabla\theta \cdot \nabla\theta \Big) \, dx\, dy \; = \; \oint\limits_S \theta \, \nabla\theta \cdot \hat{n} \, ds \, ,$$

where we show the cancellation of the Laplacian term. Noting that $\nabla\theta \cdot \hat{n}$ is the directional derivative in the normal direction, let us finally write

$$\iint\limits_A |\nabla\theta|^2 \, dx\, dy \; = \; \oint\limits_S \theta \, \frac{\partial \theta}{\partial n} \, ds \, .$$

[E.3] The vector length is $|\hat{n}| = \sqrt{\hat{n} \cdot \hat{n}} = \sqrt{\sin^2\alpha + \cos^2\alpha} \; = 1$.

The proof of uniqueness now lies in properly interpreting this equation. With respect to the right hand side, we recall from above that the problem was cast such that θ itself is identically zero around the boundary, i.e. curve S, so that the contour integral actually vanishes. This implies the area integral on the left must also vanish, which will be the case if the integrand vanishes, the latter being the square of the magnitude of the vector $\nabla\theta$. The only means by which the vector $\nabla\theta$ vanishes is if all of its components are zero, i.e.

$$\frac{\partial\theta}{\partial x} \;=\; 0 \qquad \text{and} \qquad \frac{\partial\theta}{\partial y} \;=\; 0\,.$$

Integrating these two statements respectively would mean both $\theta = f_1(y)$ and $\theta = f_2(x)$. A moment of reflection will show the only way for both these statements to hold is if $f_2(x) = f_1(y) = C$, i.e. if they are constant. In other words, θ itself must be a constant.

The final deduction comes from the already–observed fact that the value of θ on the boundary is zero. That is, if θ within the domain is a constant C and θ around the boundary is zero, then it must be that $C = 0$. In other words, $\theta = 0$ identically. Finally, because we started with the proposition that $\theta = T_1 - T_2$, we finally see that

$$T_1 \;=\; T_2\,,$$

meaning that the two different solutions *must* be one and the same.

APPENDIX F

Solving the 2–D Steady Mixed Laplace Problem

THE MIXED BOUNDARY CONDITION PROBLEM in §5.4 on pp. 50 is again governed by the Laplace equation, which we again repeat for convenience

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \,.$$

Here, two of the boundary conditions are of the Dirichlet type and two are of the Neumann type

$$T(x,H) = F(x) \qquad \text{and} \qquad T(x,0) = 0 \qquad \text{and} \qquad \left.\frac{\partial T}{\partial x}\right|_{x=0} = \left.\frac{\partial T}{\partial x}\right|_{x=L} = 0 \,.$$

Note again that we have basically "inverted" the problem as compared to the one discussed in §5.1 on pp. 40 by swapping the assignments of $T(x,H)$ and $T(x,0)$. Even though the boundary conditions are now mixed, this system is still amenable to the separation of variables method.

F.1. Solution Method

The solution procedure for this problem is very similar to that for the steady 2–D Dirichlet problem (c.f. §E.1 on pp. 151) in that we assume a separation of variables form of $T(x,t) = \Psi(x)\,\Gamma(y)$, so that

$$\frac{\Psi''(x)}{\Psi(x)} = -\frac{\Gamma''(y)}{\Gamma(y)} = -\zeta^2 \,.$$

Although the ordinary equations for Ψ and Γ are the same, i.e.

$$\Psi''(x) + \zeta^2 \Psi(x) = 0 \qquad \text{and} \qquad \Gamma'' - \zeta^2 \Gamma = 0 \,, \tag{F.1}$$

the boundary conditions for each are different

$$\Psi'(0) = \Psi'(L) = \Gamma(0) = 0 \qquad \text{and} \qquad \Gamma(H) = F(x) \,. \tag{F.2}$$

(See e.g. §B.1 on pp. 134 for an example of separating the boundary conditions.) The general solution for Ψ is derived in §B.2, specifically it is given by Eq. (B.10) on pp. 136, i.e.

$$\Psi(x) = C_1 \sin(\zeta x) + C_2 \cos(\zeta x) \,. \tag{F.3}$$

Here, the constants depend upon the first derivative of Ψ, per the Neumann conditions, rather than on Ψ itself:

$$\Psi'(x) = C_1\,\zeta\,\cos(\zeta x) - C_2\,\zeta\,\sin(\zeta x) \,,$$

so that

$$\Psi'(0) = C_1\,\zeta\,\cos 0 - C_2\,\zeta\,\cancelto{0}{\sin 0} = 0 \qquad \text{implying} \quad C_1 = 0 \quad \text{and}$$

$$\Psi'(L) = \cancelto{0}{C_1\,\zeta\,\cos L} - C_2\,\zeta\,\sin(\zeta L) = 0 \,.$$

The eigen–function here is "cosine" (as opposed to "sine" for the purely Dirichlet problem, e.g. in §E.1 on pp. 151). In the boundary condition at $x = L$, we notice another interesting and very important difference from the purely Dirichlet problem. Namely, in addition to the usual countably infinite set of eigen–values,

$$\zeta_n \;=\; \frac{n\,\pi}{L} \qquad\qquad n \in \{1, 2, 3, \dots\}\,,$$

it appears that $\zeta = 0$ is also an eigen–value, i.e. another way to satisfy the boundary equation $\zeta\,\sin(\zeta L) = 0$. In effect, we have a "zero mode" eigen–value $\zeta_0 = 0$, which is consistent with the general expression for ζ_n if we add 0 to the set of mode numbers. This observation has the following implications. The usual eigen–modes apply for $n \in \{1, 2, 3, \dots\}$ when substituted into the general solution Eq. (F.3)

$$\Psi_n(x) \;=\; C_n \cos(\zeta_n x)\,,$$

however, the zero eigen–value actually implies an additional, subtle eigen–problem. When we substitute $\zeta = 0$ into the governing equation for Ψ, Eq. (F.1), we find another unexpected, but relevant ordinary differential equation

$$\Psi_0\,''(x) \;=\; 0\,,$$

which has its own *independent* solution

$$\Psi_0\,(x) \;=\; \overline{C_1}\,x \;+\; \overline{C_2}\,,$$

that is readily derived by integrating twice. Note that the same boundary conditions on Ψ in Eq. (F.2) still apply here. Interestingly, given $\Psi_0\,'(x) = \overline{C_1}$ implied by this general solution, both boundary conditions imply the same result

$$\Psi_0\,'(0) \;=\; \Psi_0\,'(L) \;=\; \overline{C_1} \;=\; 0\,,$$

leaving the zero–mode solution

$$\Psi_0(x) \;=\; C_0\,,$$

where we have appropriately relabeled the constant $\overline{C_2} \to C_0$. In other words, the solution to the zero–mode problem in Ψ is simply a constant.

The ordinary differential equation for Γ appears to be the same as for the Dirichlet problem, $\Gamma_n'' \;-\; \zeta_n^2\,\Gamma_n = 0$ and, therefore, has the same general solution as the one found for that problem, i.e. Eq. (E.1) on pp. 151

$$\Gamma_n(y) \;=\; A_n\,e^{\zeta_n\,y} \;+\; B_n\,e^{-\zeta_n\,y}\,, \tag{F.4}$$

but we must be careful to remember this applies strictly to the positive eigen–values $n \geq 1$. Like the Dirichlet problem, the domain here is finite and we solve for one of the constants in terms of the other using the boundary condition $\Gamma_n(0) = 0$, as follows

$$\Gamma_n(0) \;=\; A_n\,e^0 \;+\; B_n\,e^0 \;=\; 0\,,$$

so that $B_n = -\,A_n$, implying

$$\begin{aligned}
\Gamma_n(y) &= A_n\, e^{\zeta_n\, y} \;-\; A_n\, e^{-\zeta_n\, y} \\
&= A_n \left(e^{\zeta_n\, y} \;-\; e^{-\zeta_n\, y} \right) \\
&= 2\, A_n \left(\frac{e^{\zeta_n\, y} \;-\; e^{-\zeta_n\, y}}{2} \right) \\
\Gamma_n(y) &= \overline{A_n}\, \sinh(\zeta_n y)\,,
\end{aligned}$$

where we have relabeled the constant as $\overline{A_n}$. Here, we can now clearly see the implications of "inverting" the boundary conditions as we have done as compared to the purely Dirichlet problem in §5.1 on pp. 40 by swapping the assignments of $T(x,H)$ and $T(x,0)$: The mathematical solution reflects the inversion simply by the specific form of arguments of the "sinh" function: $H-y$ versus just y, depending upon where we assign the non–homogeneous boundary condition (c.f. the results in §E.1).

We now must also consider the implications of the "zero–mode" eigen–value, $\zeta_0 = 0$, on Γ. Similar to the problem in Ψ above, this mode remakes the ordinary differential equation in Γ, i.e. in Eq. (F.1), into the independent equation

$$\Gamma_0\,''(y) \;=\; 0\,,$$

which again has its own *independent* solution

$$\Gamma_0\,(y) \;=\; \overline{\overline{C_1}}\, y \;+\; \overline{\overline{C_2}}\,,$$

that is readily derived by integrating twice. Again, employing the boundary condition $\Gamma_0(0) = 0$, we immediately find $\overline{\overline{C_2}} = 0$ and the solution is then

$$\Gamma_0(y) \;=\; \overline{\overline{C_1}}\, y\,.$$

Let us pause to summarize matters as they currently stand. The problem has been assumed separable using the form $\Psi(x)\,\Gamma(y)$, which has led to a conventional, countably–infinite set of eigen–solutions

$$\Psi_n(x)\,\Gamma_n(y) \;=\; C_n \cos(\zeta_n x)\, \sinh(\zeta_n y) \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L} \qquad n \in \{1,2,3,\dots\}$$

(where we have subsumed constants $\overline{A_n}$ into constants C_n) *and* an independent "zero–mode" eigen–solution

$$\Psi_0(x)\,\Gamma_0(y) \;=\; C_0\, y \qquad\qquad \zeta_0 \;=\; 0$$

(where we have subsumed constant $\overline{\overline{C_1}}$ into constant C_0). The linear superposition of all of these solutions is then the general solution for $T(x,y)$, i.e.

$$T(x,y) \;=\; C_0\, y \;+\; \sum_{n=1}^{\infty} C_n\, \cos(\zeta_n x)\, \sinh(\zeta_n y) \qquad\qquad \zeta_n \;=\; \frac{n\,\pi}{L}\,. \tag{F.5}$$

F.2. Determining Mode Coefficients

Up until now, we have exploited the orthogonality property directly to evaluate mode coefficients, e.g. as demonstrated for both sine and cosine expansions in footnotes B.5 on pp. 137 and C.2 on pp. 142, respectively. While the current solution consists of a family of trigonometric eigen–functions of the type that we are by now quite used to seeing (here being of the cosine type), the additional "zero–mode" and its associated extra term $C_0\, y$ in the general solution in Eq. (F.5) seem to complicate the process of finding the current mode coefficients. However, it is still conceptually the same as what we have seen for previous problems. The only procedural difference is that is that we must now perform two integrations for the respective different types of eigen–functions, rather than just one.

Evaluating the solution for the non–homogeneous boundary at $y = H$, we find[F.1]

$$T(x,H) \;=\; F(x) \;=\; C_0\, H \;+\; \sum_{n=1}^{\infty}\Big[C_n\, \sinh(\zeta_n H)\Big]\, \cos(\zeta_n x)\,. \tag{F.6}$$

Let us first process the more familiar $n \geq 1$ mode coefficients. From above, the eigen–function is $\cos(\zeta_n x)$, by which we multiply each term and then follow with integration

$$\begin{aligned}
\int_0^L F(x)\, \cos(\zeta_n x)\, dx \;&=\; \int_0^L C_0\, H\, \cos(\zeta_n x)\, dx \;+\; \int_0^L \sum_{m=1}^{\infty} C_m\, \sinh(\zeta_m H)\, \cos(\zeta_m x)\, \cos(\zeta_n x)\, dx \\
&=\; C_0\, H \int_0^L \cos(\zeta_n x)\, dx \;+\; \sum_{m=1}^{\infty} C_m\, \sinh(\zeta_m H) \int_0^L \cos(\zeta_m x)\, \cos(\zeta_n x)\, dx \\
&=\; C_0\, H \int_0^L \cos(\zeta_n x)\, dx \;+\; C_n\, \sinh(\zeta_n H) \int_0^L \cos^2(\zeta_n x)\, dx\,.
\end{aligned}$$

Here, we have already exploited the orthogonality property for the eigen–function $\cos(\zeta_n x)$, as described in footnote C.2 on pp. 142, whereby all terms except the n–th one in the summation vanish. We still must eliminate C_0, since we are trying to determine the values of C_n. Recalling that $\zeta_n = n\pi/L$,

$$\int_0^L \cos(\zeta_n x)\, dx \;=\; \frac{1}{\zeta_n}\, \sin(\zeta_n x)\,\Big|_0^L \;=\; \frac{L}{n\,\pi}\Big[\cancelto{0}{\sin(n\pi)} - \sin 0\Big] \;=\; 0\,, \tag{F.7}$$

so that the term containing C_0 vanishes without any extra effort. The normalization integral is

$$\begin{aligned}
N(\zeta_n) \;=\; \int_0^L \cos^2(\zeta_n\, x)\, dx \;&=\; \left[\frac{x}{2} \;+\; \frac{1}{4\,\zeta_n}\, \sin(2\,\zeta_n\, x)\right]\Bigg|_0^L \\
&=\; \left[\frac{L}{2} \;+\; \frac{\cancelto{0}{\sin(2\,n\,\pi)}}{4\,\zeta_n}\right] - \left[\frac{0}{2} \;+\; \frac{\sin 0}{4\,\zeta_n}\right] \;=\; \frac{L}{2}\,.
\end{aligned}$$

[F.1] This expression is a "half–range" expansion of $F(x)$ in a cosine series (Kreyszig, 1993). Although H in the first term and the $\sinh(\zeta_n H)$ term within the summation may initially seem not to belong, it must be remembered that these terms have *values*. That is, they are not a function of the independent variable, x. In this sense, they can properly be thought of as part of the coefficients of each term in the expansion. See more detailed discussion of this phenomenon in footnote E.1 for half–range sine expansion.

Note that in both of the above integral evaluations we have used the fact that sine vanishes when its argument is any multiple of π. Substituting these observations into the above integral equation yields

$$\int_0^L F(x)\ \cos(\zeta_n x)\, dx \ =\ C_0\, H \cdot 0 \ +\ C_n\ \sinh(\zeta_n H)\ \frac{L}{2}\,,$$

which is readily solved as

$$C_n \ =\ \frac{2}{L\ \sinh(\zeta_n H)} \int_0^L F(x)\ \cos(\zeta_n x)\, dx\,. \tag{F.8}$$

Now we examine the "zero–mode". Recall from above that the solution to this problem is a constant, which we can write in a useful way as $\Psi_0(x) = C_0 \cdot 1$, i.e. a mode coefficient C_0 multiplied by an eigen–function of unity. Once we have made this observation, the procedure is exactly as above: multiply each term in Eq. (F.1) by the eigen–function, integrate, and exploit orthogonality to solve for C_0. We find

$$\begin{aligned}
\int_0^L F(x)\cdot 1\, dx \ &=\ \int_0^L C_0\, H\cdot 1\, dx \ +\ \int_0^L \sum_{m=1}^{\infty} C_m\ \sinh(\zeta_m H)\ \cos(\zeta_m x)\cdot 1\, dx \\
&=\ C_0\, H \int_0^L dx \ +\ \sum_{m=1}^{\infty} C_m\ \sinh(\zeta_m H) \cancelto{0}{\int_0^L \cos(\zeta_m x)\, dx} \\
&=\ C_0\, H\, L \ +\ \sum_{m=1}^{\infty} C_m\ \sinh(\zeta_m H)\cdot 0\,,
\end{aligned}$$

where we have used the observation in Eq. (F.7) above that the integral of $\cos(\zeta_m x)$ over the domain vanishes identically. Consequently, every term in the series also vanishes. We then solve directly for C_0 as

$$C_0 \ =\ \frac{1}{H\, L} \int_0^L F(x)\, dx \tag{F.9}$$

In summary, the solution to the mixed Laplace problem posed at the beginning of this section is Eq. (F.5) with the zero–mode coefficient given by the integral in Eq. (F.9) and the mode coefficients for $n \geq 1$ given by Eq. (F.8). As usual, the actual difficulty of evaluating the mode coefficients for a specific problem depends on the complexity of $F(x)$.

F.3. Example: Piecewise Boundary Temperature Imposed by Local Heat Sources

In §5.4, we consider an example having a rather more complicated non–homogeneous boundary condition than what we have treated thus far. It is a domain $0 \leq x \leq 0.1$ and $0 \leq y \leq 0.05$ that models two separate devices, each creating heat and thereby imposing elevated temperatures at the top edge (see Fig. 5.8 on pp. 52). The overall boundary condition is given by a piecewise

specification: Eq. (5.11) on pp. 53:

$$F(x) \;=\; \begin{cases} 0 & : \quad 0 \le x < 0.02 \\ 50\,\sin\Big[10\,\pi(5\,x-0.1)\Big] & : \quad 0.02 \le x < 0.04 \\ 0 & : \quad 0.04 \le x < 0.05 \\ 20\,\sin\Big[50\,\pi(2\,x-0.1)\Big] & : \quad 0.05 \le x < 0.06 \\ 0 & : \quad 0.06 \le x \le 0.1\,. \end{cases}$$

Finishing the problem for this specific example merely involves the formality of evaluating the zero–mode coefficient, C_0 in Eq. (F.9), and the regular mode coefficients, C_n in Eq. (F.8). However, as we just commented, the difficulty of carrying out these operations is related to the complexity of $F(x)$ and this case is somewhat more involved than the simpler constant–temperature model we considered in §E.3 for the Dirichlet problem.

Let us start with C_0 in Eq. (F.9):

$$\begin{aligned} C_0 \;&=\; \frac{1}{0.05\cdot 0.1}\left(\int_0^{0.02} \cancelto{0}{0\,dx} + \int_{0.02}^{0.04} 50\,\sin\Big[10\,\pi(5\,x-0.1)\Big]\,dx \;+\; \int_{0.04}^{0.05} \cancelto{0}{0\,dx}\right. \\ &\qquad \left. +\int_{0.05}^{0.06} 20\,\sin\Big[50\,\pi(2\,x-0.1)\Big]\,dx \;+\; \int_{0.06}^{1} \cancelto{0}{0\,dx}\right) \\ &=\; 200\left(50\int_{0.02}^{0.04} \sin\Big[10\,\pi(5\,x-0.1)\Big]\,dx \;+\; 20\int_{0.05}^{0.06} \sin\Big[50\,\pi(2\,x-0.1)\Big]\,dx\right) \\ &=\; 200\left(-\,50\,\frac{1}{50\,\pi}\,\cos\Big[10\,\pi(5\,x-0.1)\Big]\,\Big|_{0.02}^{0.04} \;-\; 20\,\frac{1}{100\,\pi}\,\cos\Big[50\,\pi(2\,x-0.1)\Big]\,\Big|_{0.05}^{0.06}\right) \\ &=\; 200\left(-\,\frac{1}{\pi}\,\Big[\cos\pi \;-\; \cos 0\Big] \;-\; \frac{1}{5\,\pi}\,\Big[\cos\pi \;-\; \cos 0\Big]\right) \\ &=\; 200\left(-\,\frac{1}{\pi}\,\Big[-1-1\Big] \;-\; \frac{1}{5\,\pi}\,\Big[-1-1\Big]\right) \\ &=\; 200\left(\frac{2}{\pi} \;+\; \frac{2}{5\,\pi}\right) \\ &=\; \frac{480}{\pi}\,. \end{aligned}$$

The C_n in Eq. (F.8) are somewhat more complicated. We start with a similar piece–wise integration

$$\begin{aligned} C_n \;&=\; \frac{2}{0.1\cdot\sinh\left(\frac{n\,\pi}{0.1}\cdot 0.05\right)}\left(\int_0^{0.02} \cancelto{0}{0\cdot\cos\left[\frac{n\pi\,x}{0.1}\right]\,dx}\right. \\ &\qquad +\int_{0.02}^{0.04} 50\,\sin\Big[10\,\pi(5\,x-0.1)\Big]\,\cos\left[\frac{n\pi\,x}{0.1}\right]\,dx \;+\; \int_{0.04}^{0.05} \cancelto{0}{0\cdot\cos\left[\frac{n\pi\,x}{0.1}\right]\,dx} \\ &\qquad \left. +\int_{0.05}^{0.06} 20\,\sin\Big[50\,\pi(2\,x-0.1)\Big]\,\cos\left[\frac{n\pi\,x}{0.1}\right]\,dx \;+\; \int_{0.06}^{1} \cancelto{0}{0\cdot\cos\left[\frac{n\pi\,x}{0.1}\right]\,dx}\right)\,, \end{aligned}$$

but we will have to be more cautious here because of the form of the integrals. First, let us split the expression into $C_n = C_{n,1} + C_{n,2}$, where

$$\begin{aligned} C_{n,1} &= \frac{20}{\sinh(n\,\pi/2)}\,50\int_{0.02}^{0.04}\sin\Big[10\,\pi(5\,x-0.1)\Big]\,\cos(10\,n\,\pi\,x)\,dx \\ C_{n,2} &= \frac{20}{\sinh(n\,\pi/2)}\,20\int_{0.05}^{0.06}\sin\Big[50\,\pi(2\,x-0.1)\Big]\,\cos(10\,n\,\pi\,x)\,dx\,, \end{aligned}$$

which can be usefully simplified using the angle–difference formula for sine[F.2] as

$$\begin{aligned} C_{n,1} &= \frac{1000}{\sinh(n\,\pi/2)}\int_{0.02}^{0.04}\Big[\sin(50\,\pi\,x)\,\underbrace{\cos\pi}_{-1} - \cos(50\,\pi\,x)\,\cancelto{0}{\sin\pi}\Big]\,\cos(10\,n\,\pi\,x)\,dx \\ &= -\frac{1000}{\sinh(n\,\pi/2)}\int_{0.02}^{0.04}\sin(50\,\pi\,x)\,\cos(10\,n\,\pi\,x)\,dx \\ C_{n,2} &= \frac{400}{\sinh(n\,\pi/2)}\int_{0.05}^{0.06}\Big[\sin(100\,\pi\,x)\,\underbrace{\cos(5\,\pi)}_{-1} - \cos(100\,\pi\,x)\,\cancelto{0}{\sin(5\,\pi)}\Big]\,\cos(10\,n\,\pi\,x)\,dx \\ &= -\frac{400}{\sinh(n\,\pi/2)}\int_{0.05}^{0.06}\sin(100\,\pi\,x)\,\cos(10\,n\,\pi\,x)\,dx\,. \end{aligned}$$

We are now in a position to see a subtlety that must be handled properly. Each of these expressions has the form $\int \sin(\varphi x)\,\cos(\eta x)\,dx$, whose actual integration depends upon whether φ and η are equivalent (Beyer, 1984). Specifically,[F.3]

$$\int \sin(\varphi x)\,\cos(\eta x)\,dx = \begin{cases} \dfrac{1}{2\,\eta}\,\sin^2(\eta\,x) & : \quad \eta = \varphi \\ -\dfrac{\cos\big[(\varphi-\eta)x\big]}{2(\varphi-\eta)} - \dfrac{\cos\big[(\varphi+\eta)x\big]}{2(\varphi+\eta)} & : \quad \eta \neq \varphi \end{cases}$$

The salient variable here is the summation index, n, which for values of $n = 5$ and $n = 10$ invoke the equivalence condition for $C_{n,1}$ and $C_{n,2}$, respectively. We find, for these two special cases

$$\begin{aligned} C_{5,1} &= -\frac{1000}{\sinh(5\,\pi/2)}\int_{0.02}^{0.04}\sin(50\,\pi\,x)\,\cos(50\,\pi\,x)\,dx \\ &= -\frac{1000}{\sinh(5\,\pi/2)}\;\frac{1}{2\cdot 50\,\pi}\;\sin^2(50\,\pi\,x)\,\Big|_{0.02}^{0.04} \\ &= -\frac{10}{\sinh(5\,\pi/2)\,\pi}\Big(\sin^2(2\,\pi) - \sin^2\pi\Big) \\ &= 0 \end{aligned}$$

[F.2] The formula is $\sin(\varphi-\eta) = \sin\varphi\,\cos\eta - \cos\varphi\,\sin\eta$, see e.g. Beyer (1984).

[F.3] In the general case, the condition for the second result is $\eta^2 \neq \varphi^2$, which accounts for the case where either coefficient could be negative (Beyer, 1984), thus precluding *either* denominator from going to 0. Since this aspect of the solution is restricted to positive eigen–mode numbers, $n \geq 1$, implying $\eta \geq 10\pi$, that condition obviously cannot arise, so we simply use $\eta \neq \varphi$.

and

$$\begin{aligned} C_{10,2} &= -\frac{400}{\sinh(10\,\pi/2)} \int_{0.05}^{0.06} \sin(100\,\pi\,x)\,\cos(100\,\pi\,x)\,dx \\ &= -\frac{400}{\sinh(5\,\pi)}\;\frac{1}{2\cdot 100\,\pi}\;\sin^2(100\,\pi\,x)\Big|_{0.05}^{0.06} \\ &= -\frac{2}{\sinh(5\,\pi)\,\pi}\left(\sin^2(6\,\pi)\;-\;\sin^2(5\,\pi)\right) \\ &= 0\,. \end{aligned}$$

The general cases are

$$\begin{aligned} C_{n,1} &= -\frac{1000}{\sinh(n\,\pi/2)} \int_{0.02}^{0.04} \sin(50\,\pi\,x)\,\cos(10\,n\,\pi\,x)\,dx \\ &= -\frac{1000}{\sinh(n\,\pi/2)}\left[-\frac{\cos[(50\,\pi\;-\;10\,n\,\pi)\,x]}{2\,(50\,\pi\;-\;10\,n\,\pi)}\;-\;\frac{\cos[(50\,\pi\;+\;10\,n\,\pi)\,x]}{2\,(50\,\pi\;+\;10\,n\,\pi)}\right]\Bigg|_{0.02}^{0.04} \\ &= \frac{1000}{\sinh(n\,\pi/2)}\left[\frac{\cos[10\,\pi\,(5\;-\;n)\,x]}{20\,\pi\,(5\;-\;n)}\;+\;\frac{\cos[10\,\pi\,(5\;+\;n)\,x]}{20\,\pi\,(5\;+\;n)}\right]\Bigg|_{0.02}^{0.04} \\ &= \frac{50}{\pi\,\sinh(n\,\pi/2)}\left[\frac{\cos[0.4\,\pi\,(5\;-\;n)\,]}{5\;-\;n}\;+\;\frac{\cos[0.4\,\pi\,(5\;+\;n)\,]}{5\;+\;n}\right. \\ &\qquad\left.-\;\frac{\cos[0.2\,\pi\,(5\;-\;n)\,]}{5\;-\;n}\;-\;\frac{\cos[0.2\,\pi\,(5\;+\;n)\,]}{5\;+\;n}\right] \end{aligned}$$

where $n \neq 5$ and

$$\begin{aligned} C_{n,2} &= -\frac{400}{\sinh(n\,\pi/2)} \int_{0.05}^{0.06} \sin(100\,\pi\,x)\,\cos(10\,n\,\pi\,x)\,dx \\ &= -\frac{400}{\sinh(n\,\pi/2)}\left[-\frac{\cos[(100\,\pi\;-\;10\,n\,\pi)\,x]}{2\,(100\,\pi\;-\;10\,n\,\pi)}\;-\;\frac{\cos[(100\,\pi\;+\;10\,n\,\pi)\,x]}{2\,(100\,\pi\;+\;10\,n\,\pi)}\right]\Bigg|_{0.05}^{0.06} \\ &= \frac{400}{\sinh(n\,\pi/2)}\left[\frac{\cos[10\,\pi\,(10\;-\;n)\,x]}{20\,\pi\,(10\;-\;n)}\;+\;\frac{\cos[10\,\pi\,(10\;+\;n)\,x]}{20\,\pi\,(10\;+\;n)}\right]\Bigg|_{0.05}^{0.06} \\ &= \frac{20}{\pi\,\sinh(n\,\pi/2)}\left[\frac{\cos[0.6\,\pi\,(10\;-\;n)\,]}{10\;-\;n}\;+\;\frac{\cos[0.6\,\pi\,(10\;+\;n)\,]}{10\;+\;n}\right. \\ &\qquad\left.-\;\frac{\cos[0.5\,\pi\,(10\;-\;n)\,]}{10\;-\;n}\;-\;\frac{\cos[0.5\,\pi\,(10\;+\;n)\,]}{10\;+\;n}\right] \end{aligned}$$

where $n \neq 10$. Recall once again that $C_n = C_{n,1} + C_{n,2}$ for Eq. (F.8).

APPENDIX G

Green's Integral Theorems

THERE ARE USEFUL IDENTITIES IN VECTOR CALCULUS for transforming integrals variously from volume to area or area to contour representations. For example, we used Green's theorem in the plane in §E.4 on pp. 155 in demonstrating that steady 2–D Dirichlet conduction problem had only 1 solution. Here, we show short derivations of the generalized Green's theorems, aka Green's identities, which are useful in developing the eigen–function integral transform method in §6.5 on pp. 73 and §6.6 on pp. 78.

Consider the basic Gauss Divergence theorem for a vector $\mathbf{V}$

$$\iiint \nabla \cdot \mathbf{V} \, dv \;=\; \iint \mathbf{V} \cdot \mathbf{n} \, dA \,, \tag{G.1}$$

where $\mathbf{n}$ is the outward–facing unit normal that orients the bounding surface (Hildebrand, 1976).[G.1] Green's identities can be shown to be direct implications of Eq. (G.1).

Let φ_1 and φ_2 be scalar functions that depend only on spatial coordinates, e.g. $\varphi_1 = \varphi_1(x, y, z)$ for a rectangular coordinate system, and define $\mathbf{V} = \varphi_1 \nabla \varphi_2$. Then[G.2]

$$\nabla \cdot \mathbf{V} \;=\; \nabla \cdot \Big(\varphi_1 \nabla \varphi_2 \Big) \;=\; \varphi_1 \, \nabla^2 \varphi_2 \;+\; \nabla \varphi_2 \cdot \nabla \varphi_1$$

and

$$\mathbf{V} \cdot \mathbf{n} \;=\; \varphi_1 \nabla \varphi_2 \cdot \mathbf{n} \;=\; \varphi_1 \, \frac{\partial \varphi_2}{\partial n} \,,$$

where $\partial \varphi_2 / \partial n$ is the directional derivative in the outward–normal direction. Substituting these expressions into Eq. (G.1) yields *Green's first integral theorem*

$$\iiint \Big(\varphi_1 \, \nabla^2 \varphi_2 \;+\; \nabla \varphi_1 \cdot \nabla \varphi_2 \Big) \, dv \;=\; \iint \varphi_1 \, \frac{\partial \varphi_2}{\partial n} \, dA \,. \tag{G.2}$$

Given that φ_1 and φ_2 are arbitrary, they can be swapped yielding a very similar result

$$\iiint \Big(\varphi_2 \, \nabla^2 \varphi_1 \;+\; \nabla \varphi_2 \cdot \nabla \varphi_1 \Big) \, dv \;=\; \iint \varphi_2 \, \frac{\partial \varphi_1}{\partial n} \, dA \,,$$

which can be subtracted from Eq. (G.2) to obtain *Green's second integral theorem*

$$\iiint \Big(\varphi_1 \, \nabla^2 \varphi_2 \;-\; \varphi_2 \, \nabla^2 \varphi_1 \Big) \, dv \;=\; \iint \left(\varphi_1 \, \frac{\partial \varphi_2}{\partial n} \;-\; \varphi_2 \, \frac{\partial \varphi_1}{\partial n} \right) \, dA \,. \tag{G.3}$$

[G.1] We note that the Divergence theorem can be written in 2–D as well (Hildebrand, 1976), which can, in turn, be shown to lead to Green's theorem in the plane. We do not furnish its derivation here, but note that that result was used in §E.4 on pp. 155 in demonstrating that steady 2–D Dirichlet conduction problem had only 1 solution.

[G.2] The differentiation formula

$$\nabla \cdot \varphi \, \mathbf{V} \;=\; \varphi \, \nabla \cdot \mathbf{V} \;+\; \mathbf{V} \cdot \nabla \varphi$$

is useful here (see e.g. Hildebrand, 1976, equation (74a) pp. 284).

APPENDIX H

Integral Transform Solution of the Couette Problem

The general Couette problem is posed by Eqs. (8.7) through (8.10) starting on pp. 101 and the domain is represented by Fig. 8.6. Here, we shall solve both the momentum and energy components using the eigen–function integral transform method, largely as shown by Wendl and Agarwal (2002). By way of reference, we shall follow somewhat the details shown in §6.1 and §6.2 beginning on pp. 63 for the unsteady 1–D Dirichlet problem, as that is a similar problem in the sense that it:

- is 2–D, so only 1 coordinate transform is required, and
- has all homogeneous boundary conditions of the first kind.

We note that a transform pair very much like Eqs. (6.3) and (6.4) on pp. 64 in the z direction would work in this case, since its boundary conditions are homogeneous of the first kind, both for u and for T.

H.1. The Momentum Problem

The momentum problem is restated here from Eqs. (8.7) and (8.9) on pp. 101 for convenience as

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad u\Big|_{y=0} = u\Big|_{z=0} = u\Big|_{z=\phi} = 0 \qquad \text{and} \qquad u\Big|_{y=1} = 1\,.$$

Without the formalities detailed in §6.1 and §6.2, we simply state the transform pair

$$\overline{u}(\zeta_n, y) = \int_0^\phi u(y,z)\,\Psi_n(\zeta_n, z)\,dz \tag{H.1}$$

$$u(y,z) = \sum_{n=1}^{\infty} \frac{\overline{u}(\zeta_n, y)}{N(\zeta_n)}\,\Psi_n(\zeta_n, z)\,, \tag{H.2}$$

where $\Psi_n = \Psi_n(\zeta_n, z)$ is the solution of the auxiliary eigen–problem

$$\Psi_n'' + \zeta_n^2\,\Psi_n = 0 \qquad \Psi_n\Big|_{z=0} = \Psi_n\Big|_{z=\phi} = 0\,. \tag{H.3}$$

The transform step proceeds as usual

$$\int_0^\phi \frac{\partial^2 u}{\partial y^2}\,\Psi_n(\zeta_n, z)\,dz + \int_0^\phi \frac{\partial^2 u}{\partial z^2}\,\Psi_n(\zeta_n, z)\,dz = 0\,.$$

The order of operations can be swapped in the left term, i.e. switching integration in z with differentiation in y. For the right term, integration by parts (IBP) is again performed twice, as was done, for example, in §6.2. We pick similar assignments as described in footnote 6.8 on pp. 66, i.e.

such that the second derivative term is reduced by one order on each round. These developments yield

$$\begin{aligned}
\underbrace{\frac{\partial^2}{\partial y^2}\int_0^\phi u\,\Psi_n\,dz}_{\text{swap order}} + \underbrace{\cancelto{0}{\left(\Psi_n\frac{\partial u}{\partial z}\right)\Big|_0^\phi} - \int_0^\phi \frac{\partial u}{\partial z}\,\frac{d\Psi_n}{dz}\,dz}_{\text{first round IBP}} &= 0 \\
\frac{d^2\overline{u}}{d^2y^2} - \underbrace{\left[\cancelto{0}{\left(u\frac{d\Psi_n}{dz}\right)\Big|_0^\phi} - \int_0^\phi u\,\frac{d^2\Psi_n}{dz^2}\,dz\right]}_{\text{second round IBP}} &= 0 \\
\frac{d^2\overline{u}}{d^2y^2} - \zeta_n^2\int_0^\phi u\,\Psi_n\,dz &= 0 \\
\frac{d^2\overline{u}}{d^2y^2} - \zeta_n^2\,\overline{u} &= 0\,. \qquad \text{(H.4)}
\end{aligned}$$

Let us review these operations. In the first line, we exploited the auxiliary problem's homogeneous boundary conditions for Ψ_n and in the second we did the same with the physical velocity, which is also 0 at both boundaries. In the third line we have again applied the auxiliary problem in the form $\Psi_n'' = -\zeta_n^2\,\Psi_n$ and in the fourth we have again used the transform definition. The last equation, being essentially $\overline{u}'' - \zeta_n^2\,\overline{u} = 0$, is an ODE because it is not a function of z.

Let us now flesh–out the eigen–function itself. The auxiliary problem is essentially the same as those for earlier examples, e.g. in §B.2 (pp. 135) and §E.1 (pp. 151), so that we have, by inspection,

$$\Psi_n(\zeta_n,z) = \sin(\zeta_n z) \qquad \zeta_n = \frac{n\,\pi}{\phi} \qquad N(\zeta_n) = \frac{\phi}{2}\,,$$

where we also have the normalization integral by virtue of similar inference from Eq. (B.16) in footnote B.6 on pp. 138. Therefore, the transformed boundary conditions for the problem in y are

$$\begin{aligned}
\overline{u}(\zeta_n,0) &= \int_0^\phi u(0,z)\,\Psi_n(\zeta_n,z)\,dz = 0 \\
\overline{u}(\zeta_n,1) &= \int_0^\phi u(1,z)\,\Psi_n(\zeta_n,z)\,dz = \int_0^\phi 1\cdot\sin(\zeta_n z)\,dz \\
&= -\frac{1}{\zeta_n}\cos(\zeta_n z)\Big|_0^\phi = -\frac{1}{\zeta_n}\left[\cos\left(\frac{n\,\pi\,\phi}{\phi}\right) - \cos\left(\frac{n\,\pi\cdot 0}{\phi}\right)\right] = \frac{1-(-1)^n}{\zeta_n}\,.
\end{aligned}$$

Rather than rehash the entire solution of this problem, we recall that we have already solved a very similar one in §F.1 in the form of Eq. (F.4) on pp. 159. In particular, the general solution of $\overline{u}$, coupled with the homogeneous boundary condition $\overline{u} = 0$ at $y = 0$ implies

$$\overline{u}(\zeta_n,y) = C\,\sinh(\zeta_n y)\,,$$

from which the constant C is readily solved from the remaining boundary condition, whereby

$$\overline{u}(\zeta_n, y) \;=\; \frac{1-(-1)^n}{\zeta_n \,\sinh\zeta_n}\,\sinh(\zeta_n y)\,.$$

The transformed solution is simple to invert. Substituting $\overline{u}$, N, and Ψ_n from above into Eq. (H.2) above, we find

$$\begin{aligned}
u(y,z) &= \sum_{n=1}^{\infty} \frac{2}{\phi}\cdot\frac{1-(-1)^n}{\zeta_n\,\sinh\zeta_n}\,\sinh(\zeta_n y)\,\sin(\zeta_n z) \\
&= \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{\phi}\cdot\frac{\phi}{n\,\pi}\cdot\frac{2}{\sinh\zeta_n}\,\sinh(\zeta_n y)\,\sin(\zeta_n z) \\
&= \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\,\pi}\cdot\frac{\sinh(n\,\pi\,y/\phi)}{\sinh(n\,\pi/\phi)}\,\sin(n\,\pi\,z/\phi) \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2\,n-1}\,\frac{\sinh[(2\,n-1)\,\pi\,y/\phi]}{\sinh[(2\,n-1)\,\pi/\phi]}\,\sin[(2\,n-1)\,\pi\,z/\phi]\,. \qquad \text{(H.5)}
\end{aligned}$$

As with the 1–D unsteady Dirichlet problem in §B.4 on pp. 139 and the steady 2–D problem in §E.3 on pp. 154, we exploited the fact that the even modes vanish and rewrote the series only for the odd terms. We then re–purposed index n to write the odd modes using $2n-1$. Eq. (H.5) is the velocity profile in the flow domain.[H.1]

H.2. The Energy Problem

The energy (temperature) problem is restated here from Eqs. (8.8) and (8.10) on pp. 101 for convenience as

$$\frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = -\,Br\left[\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right] \qquad T\Big|_{y=0} = T\Big|_{z=0} = T\Big|_{z=\phi} = T\Big|_{y=1} = 0\,.$$

The first task is to take the proper derivatives on the right–hand–side from the momentum solution to get the actual equation that is to be solved. Differentiating term–by–term, we have

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\partial}{\partial y}\left(\frac{\sinh[(2\,n-1)\,\pi\,y/\phi]\,\sin[(2\,n-1)\,\pi\,z/\phi]}{(2\,n-1)\,\sinh[(2\,n-1)\,\pi/\phi]}\right) \\
&= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cosh[(2\,n-1)\,\pi\,y/\phi]\,\sin[(2\,n-1)\,\pi\,z/\phi]}{(2\,n-1)\,\sinh[(2\,n-1)\,\pi/\phi]}\cdot\frac{(2\,n-1)\,\pi}{\phi} \\
&= \frac{4}{\phi}\sum_{n=1}^{\infty}\frac{\cosh[(2\,n-1)\,\pi\,y/\phi]\,\sin[(2\,n-1)\,\pi\,z/\phi]}{\sinh[(2\,n-1)\,\pi/\phi]}
\end{aligned}$$

[H.1] The steady Couette flow problem has been solved several times in the literature. The most widely cited version is perhaps Berker (1963), who gave the general solution for the general problem of a sliding boundary *with* a pressure gradient. The latter aspect obviously does not apply to our problem if taken as a model of flow in a bearing.

and

$$\begin{aligned}\frac{\partial u}{\partial z} &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\partial}{\partial z}\left(\frac{\sinh[(2\,n-1)\,\pi\,y/\phi]\,\sin[(2\,n-1)\,\pi\,z/\phi]}{(2\,n-1)\,\sinh[(2\,n-1)\,\pi/\phi]}\right)\\ &= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sinh[(2\,n-1)\,\pi\,y/\phi]\,\cos[(2\,n-1)\,\pi\,z/\phi]}{(2\,n-1)\,\sinh[(2\,n-1)\,\pi/\phi]}\cdot\frac{(2\,n-1)\,\pi}{\phi}\\ &= \frac{4}{\phi}\sum_{n=1}^{\infty}\frac{\sinh[(2\,n-1)\,\pi\,y/\phi]\,\cos[(2\,n-1)\,\pi\,z/\phi]}{\sinh[(2\,n-1)\,\pi/\phi]}\end{aligned}$$

The squares of these expressions are

$$\left(\frac{\partial u}{\partial y}\right)^2 = \frac{16}{\phi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left[\frac{(2m-1)\pi z}{\phi}\right]\sin\left[\frac{(2n-1)\pi z}{\phi}\right]\cosh\left[\frac{(2m-1)\pi y}{\phi}\right]\cosh\left[\frac{(2n-1)\pi y}{\phi}\right]}{\sinh\left[\frac{(2m-1)\pi}{\phi}\right]\sinh\left[\frac{(2n-1)\pi}{\phi}\right]}$$

$$\left(\frac{\partial u}{\partial z}\right)^2 = \frac{16}{\phi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\cos\left[\frac{(2m-1)\pi z}{\phi}\right]\cos\left[\frac{(2n-1)\pi z}{\phi}\right]\sinh\left[\frac{(2m-1)\pi y}{\phi}\right]\sinh\left[\frac{(2n-1)\pi y}{\phi}\right]}{\sinh\left[\frac{(2m-1)\pi}{\phi}\right]\sinh\left[\frac{(2n-1)\pi}{\phi}\right]}$$

and they are summed as part of the right–hand–side of the equation. It will pay to take a moment to simplify this expression as much as possible. Here, we will use the so–called *function product relations* (Beyer, 1984),[H.2] where, if we let

$$\eta = \frac{(2\,m-1)\,\pi\,z}{\phi}\,,\quad \beta = \frac{(2\,n-1)\,\pi\,z}{\phi}\,,\quad \xi = \frac{(2\,m-1)\,\pi\,y}{\phi}\,,\quad \chi = \frac{(2\,n-1)\,\pi\,y}{\phi}\,,$$

then the numerator of the summed squared series becomes

$$\begin{aligned}\sin\eta\sin\beta\cosh\xi\cosh\chi\; + \;\cos\eta\cos\beta\sinh\xi\sinh\chi &= \left(\frac{\cos(\eta-\beta)}{2}-\frac{\cos(\eta+\beta)}{2}\right)\left(\frac{\cosh(\xi+\chi)}{2}+\frac{\cosh(\xi-\chi)}{2}\right)+\\ &\quad\left(\frac{\cos(\eta-\beta)}{2}+\frac{\cos(\eta+\beta)}{2}\right)\left(\frac{\cosh(\xi+\chi)}{2}-\frac{\cosh(\xi-\chi)}{2}\right)\\ &= \frac{\cos(\eta-\beta)\,\cosh(\xi+\chi)}{2}-\frac{\cos(\eta+\beta)\,\cosh(\xi-\chi)}{2}\end{aligned}$$

after some algebra. Note that 4 of the 8 product terms have canceled one another. Likewise, the denominator, a product of 2 sinh functions, can be seen directly in one of the relations in

[H.2] The specific relations of use here are:

$$\sin\eta\sin\beta = \frac{1}{2}\cos(\eta-\beta) - \frac{1}{2}\cos(\eta+\beta) \qquad \cos\eta\cos\beta = \frac{1}{2}\cos(\eta-\beta) + \frac{1}{2}\cos(\eta+\beta)$$

$$\sinh\eta\sinh\beta = \frac{1}{2}\cosh(\eta+\beta) - \frac{1}{2}\cosh(\eta-\beta) \qquad \cosh\eta\cosh\beta = \frac{1}{2}\cosh(\eta+\beta) + \frac{1}{2}\cosh(\eta-\beta)$$

footnote H.2. Noticing that a factor of 1/2 cancels globally, we can substitute values for η, β, ξ, and χ, whereby the squared velocity derivatives can be written, again after a little algebra, as

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 =$$

$$\frac{16}{\phi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\overbrace{\cos\left[\frac{2\pi z(m-n)}{\phi}\right]}^{\text{term 1(a)}}\cosh\left[\frac{2\pi y(m+n-1)}{\phi}\right] - \overbrace{\cos\left[\frac{2\pi z(m+n-1)}{\phi}\right]}^{\text{term 2(a)}}\cosh\left[\frac{2\pi y(m-n)}{\phi}\right]}{\cosh\left[\frac{2\pi(m+n-1)}{\phi}\right] - \cosh\left[\frac{2\pi(m-n)}{\phi}\right]}$$

and where we remind ourselves that this term is still multiplied by the negative of the Brinkman number, Br, to complete the right–hand–side.

We are now prepared to transform the energy equation, consisting of the Laplacian operator on the left and the product of $-Br$ with the source term above on the right. Importantly, we can re–use the transform and inverse in Eqs. (H.1) and (H.2) on pp. 167 directly, with T substituted for u of course, because the auxiliary problem is the same

$$\overline{T}(\zeta_j, y) \;=\; \int_0^{\phi} T(y,z)\,\Psi_j(\zeta_j, z)\,dz \tag{H.6}$$

$$T(y,z) \;=\; \sum_{j=1}^{\infty}\frac{\overline{T}(\zeta_j, y)}{N(\zeta_j)}\,\Psi_j(\zeta_j, z)\,, \tag{H.7}$$

where $\Psi_j = \Psi_j(\zeta_j, z)$ is the solution of the auxiliary eigen–problem

$$\Psi_j'' \,+\, \zeta_j^2\,\Psi_j \;=\; 0 \qquad\qquad \Psi_j\,\Big|_{z=0} = \;\Psi_j\,\Big|_{z=\phi} = \;0 \tag{H.8}$$

written again directly from above as

$$\Psi_j(\zeta_j, z) \;=\; \sin(\zeta_j z) \qquad\qquad \zeta_j \;=\; \frac{j\,\pi}{\phi} \qquad\qquad N(\zeta_j) \;=\; \frac{\phi}{2}\,\cdot$$

Note that we have adopted a new index j because we are already using the customary n (and m, as well) above in legacy roles from the momentum problem.

Note also since the boundary conditions in the z direction for T are identical to those for u, i.e. homogeneous conditions of the first kind, that the integral transform for the Laplacian on the left will identically follow the developments that led to Eq. (H.4) on pp. 168. We can state then, by inspection, that the left side of the equation transforms to $\overline{T}'' - \zeta_j^2\,\overline{T}$. The real challenge in this problem is the source term on the right. Our transform is in z and we note that there are only 2 components on the right, labeled "term 1(a)" and "term 2(a)" above, that are functions of z. Because we can apply the transform term–by–term in the double series, we will focus on these terms singly, applying Eq. (H.6) to each one. We will then reconstruct the entire transformed equation to proceed further.

For term 1(a), we find[H.3]

$$\int_0^\phi \cos\left[\frac{2\pi(m-n)z}{\phi}\right]\sin\left[\frac{j\pi z}{\phi}\right]dz = -\left[\frac{\cos\left(\dfrac{j\pi-2\pi(m-n)}{\phi}\,z\right)}{2\left(\dfrac{j\pi-2\pi(m-n)}{\phi}\right)}+\frac{\cos\left(\dfrac{j\pi+2\pi(m-n)}{\phi}\,z\right)}{2\left(\dfrac{j\pi+2\pi(m-n)}{\phi}\right)}\right]\Bigg|_0^\phi$$

$$= -\frac{\phi}{2\pi}\left[\frac{\cos\big[j\pi-2\pi(m-n)\big]\;-\;\cos 0}{j-2(m-n)}+\right.$$

$$\left.\frac{\cos\big[j\pi+2\pi(m-n)\big]\;-\;\cos 0}{j+2(m-n)}\right].$$

Let us pause once again to simplify. Clearly, $\cos 0 = 1$, but there is also a more subtle simplification in the other cosine terms. Since m and n are always integers, the quantity $2\pi(m-n)$ is nothing more than various multiples of a *full circle rotation*. This term does not change the value of cosine, since it is cyclical in 2π. In other words, $\cos\big[j\pi \pm 2\pi(m-n)\big] = \cos(j\pi) = (-1)^j$. Absorbing the negative sign, we continue the derivation as

$$\int_0^\phi \cos\left[\frac{2\pi(m-n)z}{\phi}\right]\sin\left[\frac{j\pi z}{\phi}\right]dz = \frac{\phi}{2\,\pi}\left[\frac{1-(-1)^j}{j-2(m-n)}\;+\;\frac{1-(-1)^j}{j+2(m-n)}\right]$$

$$= \frac{\phi\left[1-(-1)^j\right]}{2\,\pi}\left(\frac{j+\cancel{2(m-n)}+j-\cancel{2(m-n)}}{\big[j-2(m-n)\big]\cdot\big[j+2(m-n)\big]}\right)$$

$$= \frac{\phi\left[1-(-1)^j\right]j}{(j^2-a_{mn}^2)\,\pi},$$

where we are now using the shorthand $a_{mn} = 2(m-n)$. Note that $1-(-1)^j$ means that the even terms in j vanish for this integral. We note then that the condition of unequal squares for the integral in footnote H.3 is identically satisfied. That condition basically requires $4(m-n)^2 \neq j^2$, the former and latter being the relevant parts of the arguments of the cosine and sine, respectively, in the integral above. The left side is always even, while the right, being a square of an odd number, is itself always odd.[H.4] We can show by precisely the same operations and deductions that the second

[H.3] Both terms involve the integral of the product of a cosine and a sine function, for which the general expression is (Beyer, 1984)

$$\int \sin(mx)\,\cos(nx)\,dx = -\left[\frac{\cos[(m-n)\,x]}{2\,(m-n)}\;+\;\frac{\cos[(m+n)\,x]}{2\,(m+n)}\right],$$

where $m^2 \neq n^2$. We are not concerned about the case where the squares are equal. As we shall see, this contingency is not part of the physical solution.

[H.4] These two assertions about even and odd products can be proved by elementary deductions. An even number is any number divisible by 2. Since m and n are both integers, $(m-n)^2$ is also an integer, and the form of the left term can be written as $2\cdot 2\cdot(m-n)^2$, which is clearly divisible by 2, and therefore even. The above definition of an even number implies that we can represent any odd number as $j = 2a+1$, where a is an even or odd integer. Its square is $j^2 = (2a+1)^2 = 4a^2+4a+1 = 4a(a+1)+1$. Since $4a(a+1)$ is even by way of being divisible by 2, the addition of 1 makes this result odd.

term eventually resolves to

$$\int_0^{\phi} \cos\left[\frac{2\pi(m+n-1)z}{\phi}\right] \sin\left[\frac{j\pi z}{\phi}\right] dz \;=\; \frac{\phi\left[1-(-1)^j\right] j}{(j^2-b_{mn}^2)\,\pi}\,,$$

where the shorthand is $b_{mn} = 2(m+n-1)$. Given that even modes will vanish for this integral as well, it seems they will disappear for the general solution as a whole.

We can now reconstitute the entire governing energy equation in the transformed space by substituting the derived expressions for "term 1(a)" and "term 2(a)". We find

$$\begin{aligned}\overline{T}'' - \zeta_j^2\,\overline{T} &= -\frac{16\,Br}{\phi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\dfrac{\phi\left[1-(-1)^j\right] j}{(j^2-a_{mn}^2)\,\pi}\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right] - \dfrac{\phi\left[1-(-1)^j\right] j}{(j^2-b_{mn}^2)\,\pi}\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]}{\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right] - \cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]} \\ &= -\frac{16\,j\,Br\left[1-(-1)^j\right]}{\pi\,\phi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\dfrac{1}{j^2-a_{mn}^2}\overbrace{\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]}^{\text{term 1(b)}} - \dfrac{1}{j^2-b_{mn}^2}\overbrace{\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]}^{\text{term 2(b)}}}{\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right] - \cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]}\end{aligned} \tag{H.9}$$

This is an ODE in y only, albeit a non–homogeneous one having a rather complicated, non–constant source term. We also require the boundary conditions, and since the physical conditions are homogeneous of the first kind, they can be seen, hopefully by inspection of Eq. (H.6) on pp. 171, to be 0 in the transform domain, as well:

$$\overline{T}\,\Big|_{y=0} = \overline{T}\,\Big|_{y=1} = 0\,. \tag{H.10}$$

We might now recall that we have already been faced with non–homogeneous ODEs in the example problems we examined to help develop the integral transform method in chapter 6. Those were first–order involving time and we solved them handily using the method of the integrating factor (c.f. footnote 6.15 on pp. 71). Unfortunately, that method does not apply directly to second–order equations (Nelson et al., 1960). Instead, the best approach here will be the slightly more sophisticated *method of partial fractions* (Sokolnikoff and Sokolnikoff, 1941).

The general solution to this problem is the sum of an appropriate homogeneous solution, $\overline{T}_H$, and a particular solution, $\overline{T}_P$, i.e. $\overline{T} = \overline{T}_H + \overline{T}_P$. The homogeneous problem is obviously

$$\overline{T}_H'' \;-\; \zeta_j^2\,\overline{T}_H \;=\; 0$$

and we have addressed precisely this equation a number of times already (c.f. Eq. (E.1) on pp. 151 or Eq. (F.4) on pp. 159). Its solution is

$$\overline{T}_H \;=\; \overline{C_1}\,e^{\zeta_j\,y} \;+\; \overline{C_2}\,e^{-\zeta_j\,y}\,,$$

although it will be more useful to cast it as[H.5]

$$\overline{T}_H \;=\; C_1\,\cosh(\zeta_j\,y) \;+\; C_2\,\sinh(\zeta_j\,y)\,. \tag{H.11}$$

For the particular problem, let the source term in Eq. (H.9) be represented by $f(y)$. Then, if we let $D = d/dy$, we can write Eq. (H.9)

$$\begin{aligned} D^2\overline{T}_P \;-\; \zeta_j^2\,\overline{T}_P \;&=\; f(y) \\ (D^2 \;-\; \zeta_j^2)\,\overline{T}_P \;&= \\ (D \;-\; \zeta_j)\,(D \;+\; \zeta_j)\,\overline{T}_P \;&=\; f(j)\,, \end{aligned}$$

which can be solved for $\overline{T}_P$ as

$$\overline{T}_P \;=\; \frac{f(y)}{(D \;-\; \zeta_j)\,(D \;+\; \zeta_j)} \;=\; \left(\frac{1}{D \;-\; \zeta_j} \;-\; \frac{1}{D \;+\; \zeta_j}\right)\frac{f(y)}{2\,\zeta_j}\,.$$

Though the notation may at first be a little unfamiliar, this equation indicates a certain integration on the right–hand side.[H.6] The solution has the form

$$\overline{T}_P \;=\; \frac{1}{2\,\zeta_j}\,\left[e^{\zeta_j y}\int e^{-\zeta_j y}\,f(y)\,dy \;-\; e^{-\zeta_j y}\int e^{\zeta_j y}\,f(y)\,dy\right] \tag{H.12}$$

and it is clear from examining Eq. (H.9) that $f(y)$ contains precisely 2 terms in y, labeled "term 1(b)" and "term 2(b)", that must be integrated in *each* of these 2 integrals. Moreover, each of these resulting 4 integrals has the form of a product of an exponential and a hyperbolic cosine. Here, it will be more convenient to convert the latter over to exponential form (c.f. identities in footnote 3.6 on pp. 21). Let us develop the basic form in detail, where we use the shorthand η_{mn} to mean either

---

[H.5] We can see that the exponential form is readily transformed to the hyperbolic form using the identities in footnote 3.6 on pp. 21

$$\begin{aligned} C_1\,\cosh(\varphi) \;+\; C_2\,\sinh(\varphi) \;&=\; C_1\,\frac{e^{\varphi} \;+\; e^{-\varphi}}{2} \;+\; C_2\,\frac{e^{\varphi} \;-\; e^{-\varphi}}{2} \\ &=\; \frac{C_1 \;+\; C_2}{2}\,e^{\varphi} \;+\; \frac{C_1 \;-\; C_2}{2}\,e^{-\varphi}\,, \end{aligned}$$

so that $\overline{C_1} = (C_1 + C_2)/2$ and $\overline{C_2} = (C_1 - C_2)/2$ in this particular case.

[H.6] Consider, for example, the elementary problem $y' = g(x)$ that determines $y = y(x)$. It is obviously separable, such that $y = \int g(x)\,dx$, but it can also be written in the style $Dy = g(x)$, whereby $y = g(x)/D$, which implies (Sokolnikoff and Sokolnikoff, 1941)

$$\frac{g(x)}{D} \;=\; \int g(x)\,dx\,.$$

Although we do not derive the general theory here, the method of partial fractions dictates that an equation that can be written in the general form using the D operator, as

$$y \;=\; \left(\frac{C_1}{D \;-\; \eta_1} \;+\; \frac{C_2}{D \;-\; \eta_2} \;+\; \frac{C_3}{D \;-\; \eta_3} \;+\; \cdots\right)\,g(x)$$

has the solution

$$y \;=\; C_1\,e^{\eta_1 x}\int e^{-\eta_1 x}\,g(x)\,dx \;+\; C_2\,e^{\eta_2 x}\int e^{-\eta_2 x}\,g(x)\,dx \;+\; C_3\,e^{\eta_3 x}\int e^{-\eta_3 x}\,g(x)\,dx \;+\; \cdots$$

$a_{mn}\pi/\phi$ or $b_{mn}\pi/\phi$ and remind ourselves that $\zeta_j = j\pi/\phi$. We find

$$\begin{aligned} e^{\zeta_j y}\int e^{-\zeta_j y}\,\cosh(\eta_{mn}y)\,dy &= \int e^{-\zeta_j y}\,\frac{e^{\eta_{mn}y}\,+\,e^{-\eta_{mn}y}}{2}\,dy \\ &= \frac{1}{2}\,e^{\zeta_j y}\,\left[\int e^{-\zeta_j y}\,e^{\eta_{mn}y}\,dy\,+\,\int e^{-\zeta_j y}\,e^{-\eta_{mn}y}\,dy\right] \\ &= \frac{1}{2}\,e^{\zeta_j y}\,\left[\int e^{-(\zeta_j-\eta_{mn})y}\,dy\,+\,\int e^{-(\zeta_j+\eta_{mn})y}\,dy\right] \\ &= -\,\frac{1}{2}\,e^{\zeta_j y}\,\left[\frac{1}{\zeta_j-\eta_{mn}}\,e^{-(\zeta_j-\eta_{mn})y}\,+\,\frac{1}{\zeta_j+\eta_{mn}}\,e^{-(\zeta_j+\eta_{mn})y}\right]. \end{aligned}$$

The integrals are indefinite here, but we took the constant of integration to be zero. We already have 2 integration constants in the homogeneous solution in Eq. (H.11) on pp. 174 and these will be used to account for conditions at the two boundaries. This result can be simplified further as

$$e^{\zeta_j y}\int e^{-\zeta_j y}\,\cosh(\eta_{mn}y)\,dy\;=\;-\,\frac{1}{2}\,\left[\frac{1}{\zeta_j-\eta_{mn}}\,e^{\eta_{mn}y}\,+\,\frac{1}{\zeta_j+\eta_{mn}}\,e^{-\eta_{mn}y}\right]$$

and by similar operations we find

$$e^{-\zeta_j y}\int e^{\zeta_j y}\,\cosh(\eta_{mn}y)\,dy\;=\;\frac{1}{2}\,\left[\frac{1}{\zeta_j+\eta_{mn}}\,e^{\eta_{mn}y}\,+\,\frac{1}{\zeta_j-\eta_{mn}}\,e^{-\eta_{mn}y}\right],$$

so that

$$\begin{aligned} e^{\zeta_j y}\int e^{-\zeta_j y}\,\cosh(\eta_{mn}y)\,dy\;-\;e^{-\zeta_j y}\int e^{\zeta_j y}\,\cosh(\eta_{mn}y)\,dy &= -\,\frac{1}{2}\,\left[\left(\frac{1}{\zeta_j-\eta_{mn}}\,+\,\frac{1}{\zeta_j+\eta_{mn}}\right)e^{\eta_{mn}y}\right. \\ &\qquad \left.+\,\left(\frac{1}{\zeta_j-\eta_{mn}}\,+\,\frac{1}{\zeta_j+\eta_{mn}}\right)e^{-\eta_{mn}y}\right] \\ &= -\,\frac{1}{2}\,\left[\frac{2\,\zeta_j}{\zeta_j^2-\eta_{mn}^2}\,e^{\eta_{mn}y}\,+\,\frac{2\,\zeta_j}{\zeta_j^2-\eta_{mn}^2}\,e^{-\eta_{mn}y}\right] \\ &= -\,\frac{2\,\zeta_j}{\zeta_j^2-\eta_{mn}^2}\,\frac{e^{\eta_{mn}y}\,+\,e^{-\eta_{mn}y}}{2} \\ &= -\,\frac{2\,\zeta_j}{\zeta_j^2-\eta_{mn}^2}\,\cosh\Big(\eta_{mn}y\Big), \end{aligned} \tag{H.13}$$

the last step obtained by again applying the hyperbolic cosine identity in footnote 3.6 on pp. 21 to simplify. This result is the "general form" for the integrations via Eq. (H.12) on pp. 174 of both "term 1(b)" and "term 2(b)" in Eq. (H.9) on pp. 173.

Note also that the factor $2\,\zeta_j$ in the numerator in Eq. (H.13) cancels with the same factor in the denominator of Eq. (H.12). Let us restore the terms for which we have been using shorthand

$$\eta_{mn} = \begin{cases} \dfrac{a_{mn}\,\pi}{\phi} \;=\; \dfrac{2\,(m-n)\,\pi}{\phi} & : \quad \text{"term 2(b)"} \\ \dfrac{b_{mn}\,\pi}{\phi} \;=\; \dfrac{2\,(m+n-1)\,\pi}{\phi} & : \quad \text{"term 1(b)"}, \end{cases}$$

so that

$$\frac{1}{\zeta_j^2 - \eta_{mn}^2} = \begin{cases} \dfrac{\phi^2}{\pi^2}\,\dfrac{1}{j^2 - a_{mn}^2} & : \text{ “term 2(b)”} \\ \dfrac{\phi^2}{\pi^2}\,\dfrac{1}{j^2 - b_{mn}^2} & : \text{ “term 1(b)”} \end{cases}$$

We are now ready to expand Eq. (H.12) back into its infinite series form by recasting $f(y)$ within the context of the integral results, i.e. “partial fraction” solution of the “general form” given by Eq. (H.13). These are *substituted* for both both “term 1(b)” and “term 2(b)” in Eq. (H.9), because those are the only components that were integrated in that equation in y. We find

$$\overline{T}_P = -\frac{16\,j\,Br\left[1-(-1)^j\right]}{\pi\,\phi}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{\dfrac{1}{j^2-a_{mn}^2}\overbrace{\left(-\dfrac{\phi^2}{\pi^2}\cdot\dfrac{1}{j^2-b_{mn}^2}\cdot\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]\right)}^{\text{term 1(b)}}}{\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\,-\,\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]}\right.$$

$$\left. +\quad\frac{\dfrac{1}{j^2-b_{mn}^2}\overbrace{\left(\dfrac{\phi^2}{\pi^2}\cdot\dfrac{1}{j^2-a_{mn}^2}\cdot\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}^{\text{term 2(b)}}}{\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\,-\,\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]}\right]$$

$$= \frac{16\,j\,Br\left[1-(-1)^j\right]\,\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)\left(\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\,-\,\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]\right)}$$

Now that we have both $\overline{T}_H$ and $\overline{T}_P$, we can reconstruct the full solution in transform space by taking $\overline{T} = \overline{T}_H + \overline{T}_P$, the summands of which are Eq. (H.11) on pp. 174 and the above, respectively. We find

$$\overline{T} \;=\; C_1\,\cosh\left[\frac{j\,\pi\,y}{\phi}\right] \,+\, C_2\,\sinh\left[\frac{j\,\pi\,y}{\phi}\right] \,+$$

$$\frac{16\,j\,Br\left[1-(-1)^j\right]\,\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)\left(\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\,-\,\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]\right)}\,. \tag{H.14}$$

The integration constants, C_1 and C_2, are found using the transformed boundary conditions in Eq. (H.10) on pp. 173. At $y = 0$, we note that every term in the double infinite series in Eq. (H.14) vanishes because each numerator is effectively $\big[\cosh(0) - \cosh(0)\big]$, which leaves the boundary

equation

$$0 \;=\; C_1 \cdot 1 \;+\; C_2 \cdot 0\,,$$

from which which we immediately conclude that $C_1 = 0$. At $y = 1$, we rather note that the difference of cosh terms in the numerator is the same as that in the denominator, so that the boundary equation there is

$$0 \;=\; C_2 \sinh\left[\frac{j\,\pi}{\phi}\right] \;+\; \frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)}\,,$$

where we have already accounted for the fact that $C_1 = 0$ by omitting that term. Clearly

$$C_2 \;=\; -\,\frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3\,\sinh(j\,\pi/\phi)}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)}\,,$$

so that the transformed solution is

$$\begin{aligned}\overline{T} \;&=\; -\,\frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3}\cdot\frac{\sinh\left[\dfrac{j\,\pi\,y}{\phi}\right]}{\sinh\left[\dfrac{j\,\pi}{\phi}\right]}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)} \;+\\ &\frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\left(\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)\left(\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\;-\;\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]\right)}\,.\end{aligned}$$

Distributing the sinh term over the double series, along with a little bit of algebra shows the neater result to be

$$\begin{aligned}\overline{T} \;&=\; \frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)}\;\times\\ &\left[\;\frac{\left(\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{\left(\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\;-\;\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]\right)}\;-\;\frac{\sinh\left[\dfrac{j\,\pi\,y}{\phi}\right]}{\sinh\left[\dfrac{j\,\pi}{\phi}\right]}\;\right]\,.\end{aligned}$$

The substantive part of the problem is now finished and it only remains to apply the inverse transform in Eq. (H.7) on pp. 171, along with substitution of the eigen–function and normalization result shown there, to recover the physical solution. We can write this directly, and still somewhat naïvely as

$$\begin{aligned}T(y,z) \;&=\; \frac{2}{\phi}\sum_{j=1}^{\infty}\sin\left[\frac{j\,\pi\,z}{\phi}\right]\frac{16\,j\,Br\left[1-(-1)^j\right]\phi}{\pi^3}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)}\;\times\\ &\left[\;\frac{\left(\cosh\left[\dfrac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\dfrac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{\left(\cosh\left[\dfrac{\pi\,b_{mn}}{\phi}\right]\;-\;\cosh\left[\dfrac{\pi\,a_{mn}}{\phi}\right]\right)}\;-\;\frac{\sinh\left[\dfrac{j\,\pi\,y}{\phi}\right]}{\sinh\left[\dfrac{j\,\pi}{\phi}\right]}\;\right]\,.\end{aligned}$$

We now note several points in preparation for writing this expression in the most useful form for presentation, if not exactly the most efficient way for actual numerical evaluation.[H.7] First, there are various and obvious algebraic simplifications, including factoring and cancellation of some terms, like ϕ. Second, it is clear that once again the even modes in j vanish because of the $\left[1-(-1)^j\right]$ term.[H.8] Noting the extra factor of 2 that comes about from the odd terms in j, we can write

$$T(y,z) = \frac{64\,Br}{\pi^3}\sum_{j=1,3,5,\ldots}^{\infty} j\,\sin\left[\frac{j\,\pi\,z}{\phi}\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2-a_{mn}^2)\,(j^2-b_{mn}^2)}\,\times$$
$$\left[\frac{\left(\cosh\left[\frac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\frac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{\left(\cosh\left[\frac{\pi\,b_{mn}}{\phi}\right]-\cosh\left[\frac{\pi\,a_{mn}}{\phi}\right]\right)}-\frac{\sinh\left[\frac{j\,\pi\,y}{\phi}\right]}{\sinh\left[\frac{j\,\pi}{\phi}\right]}\right]$$

and, finally, restoring consecutive numbering in the j series by setting $c_j = 2j-1$, we have

$$T(y,z) = \frac{64\,Br}{\pi^3}\sum_{j=1}^{\infty} c_j\,\sin\left[\frac{c_j\,\pi\,z}{\phi}\right]\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(c_j^2-a_{mn}^2)\,(c_j^2-b_{mn}^2)}\,\times$$
$$\left[\frac{\left(\cosh\left[\frac{\pi\,b_{mn}\,y}{\phi}\right]-\cosh\left[\frac{\pi\,a_{mn}\,y}{\phi}\right]\right)}{\left(\cosh\left[\frac{\pi\,b_{mn}}{\phi}\right]-\cosh\left[\frac{\pi\,a_{mn}}{\phi}\right]\right)}-\frac{\sinh\left[\frac{c_j\,\pi\,y}{\phi}\right]}{\sinh\left[\frac{c_j\,\pi}{\phi}\right]}\right] \tag{H.15}$$

as the final solution for the temperature distribution, where we repeat the definitions of the coefficients

$$\begin{aligned} a_{mn} &= 2(m-n) \\ b_{mn} &= 2(m+n-1) \\ c_j &= 2j-1\,. \end{aligned}$$

This is the exact solution for the energy problem.

H.3. Evaluating the Triple–Infinite Series Temperature Profile

The velocity profile for $u(y,z)$ in Eq. (H.5) on pp. 169 is, by now, a routine expression to evaluate numerically. That is, it is a single infinite series having the usual pitfalls of hyperbolic and cyclical terms with their respective numerical problems related to floating–point arithmetic and pre–mature convergence first discussed in §5.2 on pp. 43.

Conversely, the temperature profile, $T(y,z)$, in Eq. (H.15) raises some new issues. Most obviously, it is a *triple* infinite series, raising the important question of just how many terms in each sub–series should be evaluated. That is, this expression seems to be basically different for numerical purposes from the single infinite series, for example Eq. (H.5), where we simply continue the

[H.7] That is, we will leave the sinh term inside of the double summation, even though it is a function neither of m or n, but only of j. While this term could have remained outside (factored), the form we use here is more visually pleasing, for example, one can see immediately that conditions are satisfied on all 4 boundaries.

[H.8] We first saw this phenomenon in §B.4 on pp. 139 for the 1–D Dirichlet conduction problem.

summing procedure until some convergence criterion is met. Here, the danger is that we proceed too far down the path of one of the sub–series, exhausting most of our computational effort on accumulating terms that do not actually contribute importantly to the final result. Naïvely, we might pick, for example 1000 terms for both the interior series in m and n, but there is no easy way to determine whether all of those *one million* sub–terms for *each* term in j are important. In looking at the equation, one of the non–trivial complications seems to be that terms involving all of j, m, and n are inseparably mixed within the most interior part of the summation. There is no way, for example, to sum terms isolated in, say m in the most interior part until some local convergence is reached, then multiply by the current term in n and accumulate that to the sum, etc.

We have to be more clever here. Let us observe that all the hyperbolic functions within the "large" square brackets are configured collectively as differences. That is, they will *tend* to cancel one another, of course doing so identically at boundaries, but do not appear otherwise to necessarily become monotonically smaller as m and n become large. However, the interior coefficient

$$\left| \frac{1}{(c_j^2 - a_{mn}^2)\,(c_j^2 - b_{mn}^2)} \right| = \left| \frac{1}{\left[(2j-1)^2 - 4(m-n)^2\right]\left[(2j-1)^2 - 4(m+n-1)^2\right]} \right| \tag{H.16}$$

does become smaller. This component directly affects the degree to which a given term contributes to the overall solution. The key observation is *how* this coefficient becomes smaller as a function of the *order* in which we perform the summation.[H.9]

Let us first examine the naïve approach mentioned above, which is to perform the summation exactly as Eq. (H.15) is written. For a given j, say $j = 1$, take $n = 1$ and start summing terms for $m = 1, 2, 3, \ldots$, in which case "early terms" will contribute significantly because m and n are both small in the denominator. Later terms will be progressively smaller because m will be large, even though n remains small. Eventually we halt this procedure, go back to set $n = 2$, then start summing $m = 1, 2, 3, \ldots$ over again, so that the same behavior is realized. Early terms contribute much more than later terms. This process continues for $n = 3, 4, \ldots$, so that if we were to graphically represent the relative contribution of the terms in the order that we are summing them, this procedure would resemble something what is shown in Fig. H.1. Clearly, much of our computational effort is spent on calculating terms that contribute little to the overall solution. Although the exact solution is represented elegantly by Eq. (H.15), the numerical approach it implies is very inefficient.

We would rather develop an ordering of terms that gives a summation representation more like what is shown in Fig. H.2. Here, the important terms are all accumulated much earlier in the double–series and we can then truncate in such a way that valuable computational effort is not wasted on insignificant terms. The key observation here is then that we design the summation to proceed along *counter–diagonals* of the pair of indices, much like what is shown on pp. 74 from our discussion of the general integral transform method. In essence, we re–order the terms so that we

[H.9] In discussing numerical implementation, we ordinarily would be conscious of checking for "divide by zero" exceptions in an expression like Eq. (H.16). However, recalling footnote H.4 on pp. 172, we note that $(2j-1)^2$ will always be odd, while $4(m-n)^2$ and $4(m+n-1)^2$ will both always be even. Therefore, the differences in the denominator of Eq. (H.16) will never be zero because odds and evens are mutually exclusive. This clever mathematical observation obviously saves the enormously repetitive testing we would have otherwise done as routine "good programming practice" to handle floating–point exceptions.

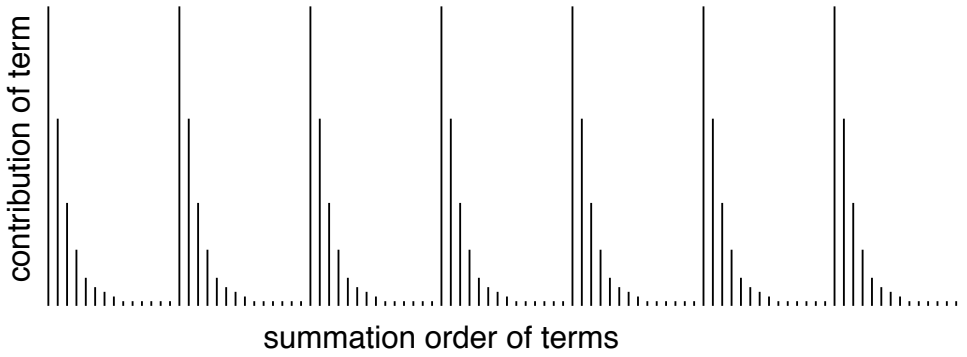


FIGURE H.1. *Rough representation of the relative contribution of terms as quantified by the interior coefficient in Eq.* (H.16) *if summing in "naïve order".*

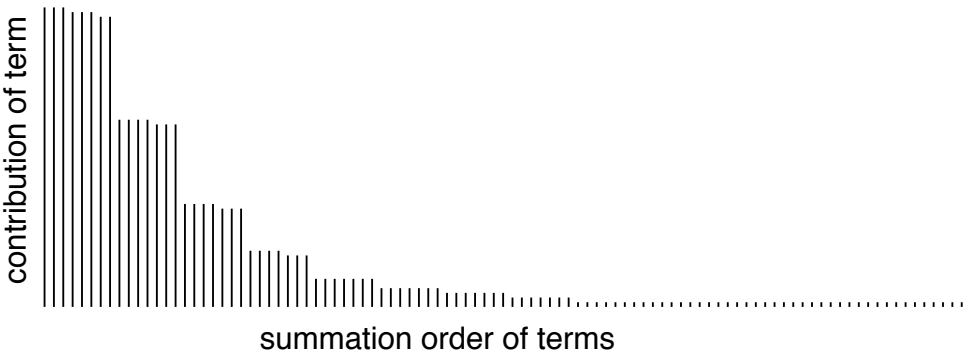


FIGURE H.2. *Rough representation of the relative contribution of terms as quantified by the interior coefficient in Eq.* (H.16) *if summing in "clever order". Large terms are "front–loaded" in the series.*

sum in increasing order of $m + n$, for example arrange the (m, n) as

$$\underbrace{(1,1)}_{2} \quad \underbrace{(2,1)\,(1,2)}_{3} \quad \underbrace{(3,1)\,(2,2)\,(1,3)}_{4} \quad \underbrace{(4,1)\,(3,2)\,(2,3)\,(1,4)}_{5} \quad \underbrace{(5,1)\,(4,2)\,(3,3)\,(2,4)\,(1,5)}_{6} \quad \ldots$$

where the sums are labeled below each group of terms.[H.10] This idea is shown in algorithm H.1, which can be implemented in any suitable language. Note that we presume that $f(j, m, n, y, \phi)$ in the algorithm would be implemented with proper numerical precautions against floating–point anomalies for the hyperbolic functions, as discussed in §5.2 on pp. 43, as would the sine function.

[H.10] Note that we do not *exactly* follow the ordering shown in the matrix on pp. 74, which alternates directions through the counter–diagonals of the matrix. Rather, here we are always proceeding from "south–west" to "north–east", which is slightly easier than alternation to implement computationally (see algorithm H.1). This subtlety is not particularly important to the actual algorithm, though. Its crux is that, by proceeding along diagonals, the interior coefficient values in Eq. (H.16) will *roughly* be ordered from large to small.

Algorithm H.1 Evaluation of $T(y, z)$ in Eq. (H.15), Emphasizing the Ordering of the Terms

input values of ϕ, Br, y, and z
define $f(j, m, n, y, \phi)$ as the complete term within the inner double summation
set $(j, T) = (1, 0)$
set convergence threshold for the iteration, ϵ, suitably near 0
while true **do**
  set $(S, s) = (2, 0)$
  while true **do**
    for $n = 1, 2, 3, \ldots S - 1$ **do**
      $m = S - n$
      $s = s + f(j, m, n, y, \phi)$
      go to CONTINUE if s converged within ϵ
    end for
    $S = S + 1$
  end while
  CONTINUE
  set T_j equal to the product of c_j, the sine term, and s
  $T = T + T_j$
  last if T converged within ϵ
  $j = j + 1$
end while
$T = 64 \cdot T \cdot Br/\pi^3$
print T

APPENDIX I

Convection Equations in Cylindrical Coordinates

In §7.1 we derived the conservation laws for mass, energy, and momentum and quoted their forms in general vector representation, respectively, in Eqs. (7.3) on pp. 82, (7.5) on pp. 84, and (7.15) on pp. 88. Here, we report these equations in the cylindrical coordinate system (Fig. I.1) without derivation, but rather with a reference to texts such as Panton (1984) or Bejan (1984) for

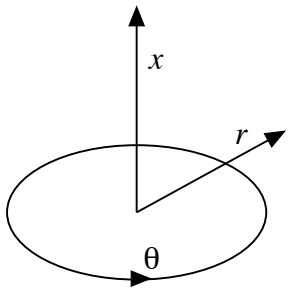


FIGURE I.1. *Cylindrical coordinate system in* (x, r, θ).

further details. Let the velocity components for the (x, r, θ) directions be (u, v, w), respectively. Then, conservation of mass is

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\,v\right) + \frac{1}{r}\frac{\partial w}{\partial \theta} = 0\,, \tag{I.1}$$

conservation of momentum in the respective (x, r, θ) directions is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} + \frac{w}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right] \tag{I.2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r} + \frac{w}{r}\frac{\partial v}{\partial \theta} - \frac{w^2}{r} = \tag{I.3}$$

$$-\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,v\right)\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial w}{\partial \theta}\right]$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial r} + \frac{w}{r}\frac{\partial w}{\partial \theta} + \frac{v\,w}{r} = \tag{I.4}$$

$$-\frac{1}{\rho\,r}\frac{\partial P}{\partial \theta} + \nu\left[\frac{\partial^2 w}{\partial x^2} + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\,w\right)\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v}{\partial \theta}\right],$$

where ρ and ν are again the density and kinematic viscosity, respectively. The energy equation for temperature, T, is

$$\begin{aligned}\rho\, c_p \left[\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} + \frac{w}{r} \frac{\partial T}{\partial \theta} \right] &= \dot{q} + k \left[\frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] \\ &\quad + 2\,\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial w}{\partial \theta} + \frac{v}{r} \right)^2 \right] + \\ &\quad + \mu \left[\left(\frac{\partial w}{\partial r} - \frac{w}{r} + \frac{1}{r} \frac{\partial v}{\partial \theta} \right)^2 + \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial r} \right)^2 \right], \end{aligned} \tag{I.5}$$

where k, c_p, and μ are the thermal conductivity, heat capacity, and dynamic viscosity, respectively. This equation also specifies an additional generic volumetric heat generation rate $\dot{q}$.

APPENDIX J

Fully Developed Laminar Convection in Pipes

THIS APPENDIX FURNISHES SOME OF THE CALCULATION DETAILS of the steady, fully–developed, laminar convection in a circular pipe for both the constant heat flux and constant temperature boundary configurations discussed in §8.5 and §8.6, respectively.

J.1. Integral Solution for the Case of Constant Heat Flux Boundary

This section describes the solution procedure for Eq. (8.21) on pp. 110, which is repeated here for convenience

$$\frac{d}{dr}\left(r\,\frac{dT^*}{dr}\right) \;=\; \varphi\, r\left[1-\left(\frac{r}{R}\right)^2\right] \qquad \text{where} \qquad \varphi \;=\; \frac{4\,q''_R}{k\,R\,(T_m\;-\;T_R)} \quad \text{is constant.}$$

This equation can be integrated twice directly to get

$$T^* \;=\; \frac{\varphi\, r^2}{4} \;-\; \frac{\varphi\, r^4}{16\,R^2} \;+\; C_1\,\ln(r) \;+\; C_2\,,$$

where C_1 and C_2 are integration constants. Boundary conditions for this problem are a little subtle in the sense that we have already "used" the obvious heat flux specification, q''_R, in casting the problem itself. However, we have already invoked a boundary temperature, T_R, and can use that at the pipe wall at $r = R$, i.e.

$$T^*\Big|_{r=R} \;=\; \frac{T\;-\;T_R}{T_m\;-\;T_R}\bigg|_{r=R} \;=\; \frac{T_R\;-\;T_R}{T_m\;-\;T_R} \;=\; 0\,,$$

as alluded to in footnote 8.22 on pp. 107. The second boundary condition at $r = 0$ is similar in a sense to that for the velocity problem discussed in footnote 8.19 on pp. 106, being deducible from

- the temperature is well–behaved, so that the natural log term must remain finite at $r = 0$
- because of symmetry of the overall problem, the temperature profile must also be symmetric about $r = 0$

either of which yields the same result that $C_1 = 0$. Applying the remaining condition, we find

$$C_2 \;=\; -\,\frac{3}{16}\,\varphi\, R^2\,,$$

which enables us to write T^* as

$$T^* \;=\; \frac{\varphi\, r^2}{4} \;-\; \frac{\varphi\, r^4}{16\,R^2} \;-\; \frac{3}{16}\,\varphi\, R^2 \;=\; -\,\varphi\, R^2\left[-\,\frac{1}{4}\left(\frac{r}{R}\right)^2 \;+\; \frac{1}{16}\left(\frac{r}{R}\right)^4 \;+\; \frac{3}{16}\right].$$

We now observe that we can actually eliminate φ entirely by a clever use of the definition of the mean temperature, Eq. (8.15) on pp. 107, as follows

$$
\begin{aligned}
T_m &= \frac{2}{\overline{u}\,R^2}\int_0^R u\,T\,r\,dr \\
&= \frac{2}{\overline{u}\,R^2}\int_0^R 2\,\overline{u}\left[1-\left(\frac{r}{R}\right)^2\right]\left(-\,\varphi\,R^2\left[\frac{1}{16}\left(\frac{r}{R}\right)^4-\ \frac{1}{4}\left(\frac{r}{R}\right)^2+\ \frac{3}{16}\right](T_m-T_R)+T_R\right) r\,dr \\
&= \frac{4}{R^2}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right](-\,\varphi\,R^2)\left[\frac{1}{16}\left(\frac{r}{R}\right)^4-\ \frac{1}{4}\left(\frac{r}{R}\right)^2+\ \frac{3}{16}\right](T_m-T_R)\,r\,dr \\
&\qquad\qquad +\frac{4}{R^2}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]T_R\,r\,dr \\
&= -\,4\,\varphi\,(T_m-T_R)\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]\left[\frac{1}{16}\left(\frac{r}{R}\right)^4-\ \frac{1}{4}\left(\frac{r}{R}\right)^2+\ \frac{3}{16}\right]\,r\,dr \\
&\qquad\qquad +\frac{4\,T_R}{R^2}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]r\,dr \\
&= -\,4\,\varphi\,(T_m-T_R)\int_0^R\left[-\,\frac{r^7}{16\,R^6}\ +\ \frac{5\,r^5}{16\,R^4}\ -\ \frac{7\,r^3}{16\,R^2}\ +\ \frac{3\,r}{16}\right]dr \\
&\qquad\qquad +\frac{4\,T_R}{R^2}\int_0^R\left[r\ -\ \frac{r^3}{R^2}\right]dr \\
&= -\,4\,\varphi\,(T_m-T_R)\left[-\,\frac{r^8}{128\,R^6}\ +\ \frac{5\,r^6}{96\,R^4}\ -\ \frac{7\,r^4}{64\,R^2}\ +\ \frac{3\,r^2}{32}\right]\Bigg|_0^R+\ \frac{4\,T_R}{R^2}\left[\frac{r^2}{2}\ -\ \frac{r^4}{4\,R^2}\right]\Bigg|_0^R \\
&= -\,4\,\varphi\,(T_m-T_R)\,R^2\left[-\,\frac{1}{128}\ +\ \frac{5}{96}\ -\ \frac{7}{64}\ +\ \frac{3}{32}\right]+\ \frac{4\,T_R\,R^2}{R^2}\left[\frac{1}{2}\ -\ \frac{1}{4}\right] \\
&= -\,4\,\varphi\,(T_m-T_R)\,R^2\left[-\,\frac{3}{384}\ +\ \frac{20}{384}\ -\ \frac{42}{384}\ +\ \frac{36}{384}\right]\ +\ 4\,T_R\left[\frac{1}{4}\right] \\
&= -\,\frac{11}{96}\,\varphi\,(T_m-T_R)\,R^2\ +\ T_R\,.
\end{aligned}
$$

A little more algebra shows that the quantity $T_m - T_R$ can actually be eliminated, leaving

$$\varphi\,R^2\ =\ -\,\frac{96}{11}\,,$$

so that

$$T^*\ =\ \frac{96}{11}\left[-\,\frac{1}{4}\left(\frac{r}{R}\right)^2\ +\ \frac{1}{16}\left(\frac{r}{R}\right)^4\ +\ \frac{3}{16}\right]\,.$$

J.2. Axial Conduction for the Case of Constant Temperature Boundary

In §8.6 on pp. 112, we claim that axial conduction, though routinely neglected in established analyses, does not actually vanish. This claim can be demonstrated by a simple energy balance,

analogous to what is shown in Fig. 8.11 on pp. 109, except where the constant flux boundary condition at $r = R$ is replaced by a constant temperature boundary condition. The heat entering the differential element from the periphery is then given by Newton's Law based on $T_R - T_m$, whereby the energy balance is

$$h\,dA\,(T_R - T_m) \;+\; \dot{m}\,c_p\,T_m \;=\; \dot{m}\,c_p\,\left(T_m \;+\; \frac{dT_m}{dx}\,dx\right)\,.$$

Given that $dA = P\,dx$, where P is again the perimeter, a little algebra shows that, instead of a constant value for dT_m/dx, we find the differential equation

$$\dot{m}\,c_p\,\frac{dT_m}{dx} \;=\; h\,P\,(T_R \;-\; T_m)\,.$$

The equation is separable and can be solved by direct integration, as

$$\begin{aligned}
\frac{dT_m}{T_R \;-\; T_m} &= \frac{h\,P}{\dot{m}\,c_p}\,dx \\
\int_{T_m(0)}^{T_m(x)} \frac{dT'_m}{T_R \;-\; T'_m} &= \int_0^x \frac{h\,P}{\dot{m}\,c_p}\,dx' \\
-\,\ln(T_R \;-\; T'_m)\Big|_{T_m(0)}^{T_m(x)} &= \frac{h\,P}{\dot{m}\,c_p}\,x'\,\Big|_0^x \\
-\,\ln\big[T_R \;-\; T_m(x)\big] \;+\; \ln\big[T_R \;-\; T_m(0)\big] &= \frac{h\,P}{\dot{m}\,c_p}\,(x \;-\; 0) \\
\ln\left[\frac{T_m(x) \;-\; T_R}{T_m(0) \;-\; T_R}\right] &= -\,\frac{h\,P}{\dot{m}\,c_p}\,x \\
\frac{T_m(x) \;-\; T_R}{T_m(0) \;-\; T_R} &= e^{-\frac{h\,P}{\dot{m}\,c_p}\,x}
\end{aligned}$$

$$T_m(x) \;=\; \big[T_m(0) \;-\; T_R\big]\,e^{-\frac{h\,P}{\dot{m}\,c_p}\,x} \;+\; T_R \tag{J.1}$$

$$\frac{dT_m}{dx} \;=\; \frac{\big[T_R \;-\; T_m(0)\big]\,h\,P}{\dot{m}\,c_p}\,e^{-\frac{h\,P}{\dot{m}\,c_p}\,x}\,. \tag{J.2}$$

Given Eq. (8.25) on pp. 112 and the condition that T^* is independent of x, the second derivative is then

$$\frac{\partial^2 T}{\partial x^2} \;=\; T^*\,\frac{d^2 T_m}{dx^2} \;=\; \big[T_m(0) \;-\; T_R\big]\,\left(\frac{h\,P}{\dot{m}\,c_p}\right)^2\,e^{-\frac{h\,P}{\dot{m}\,c_p}\,x}\,. \tag{J.3}$$

APPENDIX K

Kármán–Pohlhausen Method for Prandtl's Boundary Layer Flow

THE BOUNDARY LAYER EQUATIONS, Eqs. (9.1) through (9.3), can be solved by a variety of approaches. Here, we develop the details of the classical Kármán–Pohlhausen approximation discussed in §9.2 on pp. 118. This approximation takes the form of an integral equation, which itself furnishes the boundary layer growth law for any "trial" velocity one might use to model the boundary layer. The method is straightforward in that it uses nothing more than basic calculus techniques. We cast the problem according to Fig. 9.1 on pp. 115. In particular, (x, y) are the streamwise and normal coordinates, respectively, and δ is the assigned, but *unknown* boundary layer thickness.[K.1]

K.1. Development of the Kármán–Pohlhausen Integral Momentum Equation

We start by integrating the conservation of mass statement, Eq. (9.1) on pp. 118, across the boundary layer:

$$\int_0^\delta \frac{\partial u}{\partial x}\, dy \,+\, \int_0^\delta \frac{\partial v}{\partial y}\, dy \;=\; 0\,.$$

Note that the integrand in the second term is simply dv by Chain Rule. Moreover, $v = 0$ at $y = 0$ because of the no–slip boundary condition. These observations yield

$$v\,\Big|_{y=\delta} \;=\; -\int_0^\delta \frac{\partial u}{\partial x}\, dy\,, \tag{K.1}$$

which we set aside to be used momentarily. Now, integrate the conservation of momentum statement, Eq. (9.2) on pp. 118, likewise over the boundary layer thickness

$$\int_0^\delta u\,\frac{\partial u}{\partial x}\, dy \,+\, \int_0^\delta v\,\frac{\partial u}{\partial y}\, dy \;=\; \nu \int_0^\delta \frac{\partial^2 u}{\partial y^2}\, dy\,,$$

which gives, after integrating the second term on the left hand side by parts[K.2] and recognizing the right hand side can be integrated directly,

$$\int_0^\delta u\,\frac{\partial u}{\partial x}\, dy \,+\, \big[u\,v\big]\Big|_0^\delta \,-\, \int_0^\delta u\,\frac{\partial v}{\partial y}\, dy \;=\; \nu\,\frac{\partial u}{\partial y}\,\Big|_0^\delta\,. \tag{K.2}$$

[K.1] This is synonymous with what we will refer to as the "boundary layer growth law", $\delta = \delta(x)$.

[K.2] We can consider $\partial u/\partial y \cdot dy$ in the second term as simply du by Chain Rule, which means we are evaluating $\int v\, du$. Applying the method of integration by parts, we see $\int v\, du = u\, v - \int u\, dv$. However, $\partial v/\partial y \cdot dy = dv$, again by Chain Rule. That is

$$\int_0^\delta v\,\frac{\partial u}{\partial y}\, dy \;=\; \int_0^\delta v\, du \;=\; \big[u\,v\big]\Big|_0^\delta \,-\, \int_0^\delta u\, dv \;=\; \big[u\,v\big]\Big|_0^\delta \,-\, \int_0^\delta u\,\frac{\partial v}{\partial y}\, dy\,.$$

This development gives the second and third terms on the left hand side of Eq. (K.2).

Note that the first term has not changed at all. To further develop Eq. (K.2), we make the following observations. The second term can be evaluated outright using Eq. (K.1) for v and observing that $u = 0$ at $y = 0$ and $u = u_\infty$ at $y = \delta$. In the third term, we utilize the original continuity statement, Eq. (9.1) on pp. 118, to swap $\partial v/\partial y = -\,\partial u/\partial x$. The fourth term is evaluated using the fact that $\partial u/\partial y$ is zero at the edge of the boundary layer, $y = \delta$. These modifications give

$$\int_0^\delta u\,\frac{\partial u}{\partial x}\,dy \;-\; u_\infty \int_0^\delta \frac{\partial u}{\partial x}\,dy \;+\; \int_0^\delta u\,\frac{\partial u}{\partial x}\,dy \;=\; -\,\nu\,\frac{\partial u}{\partial y}\bigg|_{y=0} ,$$

which is readily simplified to

$$u_\infty \int_0^\delta \frac{\partial u}{\partial x}\,dy \;-\; \int_0^\delta 2\,u\,\frac{\partial u}{\partial x}\,dy \;=\; \nu\,\frac{\partial u}{\partial y}\,\bigg|_{y=0} .$$

Being a constant, u_∞ can be taken directly under the integral sign and u can be "integrated" in,[K.3] which yields

$$\int_0^\delta \frac{\partial}{\partial x}\Big(u_\infty \cdot u - u \cdot u\Big)\,dy \;=\; \nu\,\frac{\partial u}{\partial y}\,\bigg|_{y=0} .$$

Now, the right hand side is basically a term that is *evaluated*, specifically, it is the shear stress at the plate surface divided by density. On the left, we assume the order of differentiation can be swapped. Moreover, we notice that the integral in y is evaluated over the given limits, meaning the result is not a function of y. This result is also the argument of the differential operator in x. Precisely because it is then at most a function of x, we change $\partial/\partial x$ to d/dx, which yields

$$\frac{d}{dx}\int_0^\delta \big(u_\infty \;-\; u\big)\,u\,dy \;=\; \nu\,\frac{\partial u}{\partial y}\,\bigg|_{y=0} . \tag{K.3}$$

Eq. (K.3) is the Kármán–Pohlhausen integral momentum equation.

K.2. Sketch of Companion Integral Energy Equation Derivation

A similar integral relationship can be derived for the conservation of energy statement, Eq. (9.3) on pp. 118. This process relies on an equation very similar to Eq. (K.1) above for v at the boundary layer edge, except here the integration is taken across the *thermal* boundary layer of thickness δ_t. It then proceeds by integrating the conservation of energy statement over δ_t and following very similar procedures to those above

$$\begin{aligned}
\int_0^{\delta_t} u\,\frac{\partial T}{\partial x}\,dy \;+\; \int_0^{\delta_t} v\,\frac{\partial T}{\partial y}\,dy \;&=\; \alpha\int_0^{\delta_t}\frac{\partial^2 T}{\partial y^2}\,dy \\
\int_0^{\delta_t} u\,\frac{\partial T}{\partial x}\,dy \;+\; \big[v\,T\big]\Big|_0^{\delta_t} \;-\; \int_0^{\delta_t} T\,\frac{\partial v}{\partial y}\,dy \;&=\; \alpha\,\frac{\partial T}{\partial y}\,\bigg|_0^{\delta_t} \\
\int_0^{\delta_t} u\,\frac{\partial T}{\partial x}\,dy \;-\; T_\infty\int_0^{\delta_t}\frac{\partial u}{\partial x}\,dy \;+\; \int_0^{\delta_t} T\,\frac{\partial u}{\partial x}\,dy \;&=\; -\,\alpha\,\frac{\partial T}{\partial y}\bigg|_{y=0} .
\end{aligned}$$

[K.3] That is we essentially follow the rule of differentiation, except in reverse, to find

$$2\,u\,\frac{\partial u}{\partial x} \;=\; \frac{\partial\left(u^2\right)}{\partial x} .$$

Noticing that the first and the third terms on the left–hand side can be combined because of their relationship

$$u\,\frac{\partial T}{\partial x}\;+\;T\,\frac{\partial u}{\partial x}\;=\;\frac{\partial\,(u\,T)}{\partial x}$$

the derivation can be completed with a few more steps similar to those used above for the momentum equation to give

$$\frac{d}{dx}\int_0^{\delta_t}\big(T_\infty\;-\;T\big)\,u\,dy\;=\;\alpha\,\frac{\partial T}{\partial y}\bigg|_{y=0}\,. \tag{K.4}$$

Eq. (K.4) is the companion Kármán–Pohlhausen integral equation for energy.

K.3. Example Boundary Layer Profiles: Third–Order Polynomials

Eqs. (K.3) and (K.4) enforce relationships between the streamwise development of the u and T profiles with their respective gradients at the plate surface, $y = 0$. One can now assume various "trial profiles" to model the boundary layer, ultimately using the Kármán–Pohlhausen integrals to derive the associated growth laws δ and δ_t as closures. In particular, there is a symmetry in the streamwise direction that can be exploited to write the general forms as[K.4]

$$u(x,y)\;=\;u\big(\delta(x),y\big)\;=\;u\left(\frac{y}{\delta(x)}\right)\qquad\text{and}\qquad T(x,y)\;=\;T\big(\delta_t(x),y\big)\;=\;T\left(\frac{y}{\delta_t(x)}\right)\,.$$

Let us illustrate this process with an example that describes u and T as third–order polynomials.[K.5] For instance, take u as

$$\frac{u}{u_\infty}\;=\;a_1\;+\;a_2\,\left(\frac{y}{\delta}\right)\;+\;a_3\,\left(\frac{y}{\delta}\right)^2\;+\;a_4\,\left(\frac{y}{\delta}\right)^3\,,$$

where we can invoke the following boundary conditions to evaluate $a_1 \ldots a_4$

- Recall that u is prescribed explicitly at the wall and the edge of the boundary layer, i.e. Eqs. (9.4) and (9.5) on pp. 118

$$u\,\Big|_{y=0}=\;0\qquad\text{and}\qquad u\,\Big|_{y=\delta}=\;u_\infty$$

- Matching between the boundary layer and the freestream is asymptotic, i.e. "smooth", implying the gradient of velocity vanishes at the edge of the boundary layer

$$\frac{\partial u}{\partial y}\,\bigg|_{y=\delta}=\;0\,.$$

- Take the conservation of momentum statement, Eq. (9.2) on pp. 118, and *evaluate* it at the plate surface, $y = 0$, under the restriction of no–slip. Plugging in $u = v = 0$ at $y = 0$, we find

$$\frac{\partial^2 u}{\partial y^2}\,\bigg|_{y=0}=\;0\,.$$

[K.4] We exploit this aspect once again in the "similarity technique" for this problem, discussed in detail in appendix L starting on pp. 197.

[K.5] Here is where the idea of various degrees of accuracy enters into the problem. Specifically, lower–order polynomials will, in general, yield less accurate results as compared to higher–order polynomials. The order is limited by how many boundary conditions we can identify because these must be used to evaluate the polynomial coefficients.

We now have 4 conditions to evaluate the 4 unknowns. Carrying this out, we find $a_1 = a_3 = 0$, $a_2 = 1.5$, and $a_4 = -0.5$, so that

$$u^* \;=\; \frac{u}{u_\infty} \;=\; \frac{3}{2}\,\left(\frac{y}{\delta}\right) \;-\; \frac{1}{2}\,\left(\frac{y}{\delta}\right)^3 . \tag{K.5}$$

We now know the *form* of u/u_∞, but since we do not know how δ varies, we still do not know the explicit solution for the profile u. Here is where we will now apply the Kármán–Pohlhausen integral condition, Eq. (K.3), to derive the boundary layer growth law as a closure to the system. Specifically, solve Eq. (K.5) for u as

$$u \;=\; \frac{3\,u_\infty}{2\,\delta}\,y \;-\; \frac{u_\infty}{2\,\delta^3}\,y^3$$

and substitute this expression into Eq. (K.3), whereby

$$\begin{aligned}
\frac{d}{dx}\int_0^\delta \left(u_\infty \;-\; \frac{3\,u_\infty}{2\,\delta}\,y \;+\; \frac{u_\infty}{2\,\delta^3}\,y^3\right)\left(\frac{3\,u_\infty}{2\,\delta}\,y \;-\; \frac{u_\infty}{2\,\delta^3}\,y^3\right) dy \;&=\; \nu\left(\frac{3\,u_\infty}{2\,\delta} \;-\; \frac{3\,u_\infty}{2\,\delta^3}\,y^2\right)\Bigg|_{y=0} \\
u_\infty^2\,\frac{d}{dx}\int_0^\delta \left(\frac{3\,y}{2\,\delta} \;-\; \frac{9\,y^2}{4\,\delta^2} \;-\; \frac{y^3}{2\,\delta^3} \;+\; \frac{3\,y^4}{2\,\delta^4} \;-\; \frac{y^6}{4\,\delta^6}\right) dy \;&=\; \frac{3\,\nu\,u_\infty}{2\,\delta} \\
u_\infty^2\,\frac{d}{dx}\left(\frac{3\,y^2}{4\,\delta} \;-\; \frac{3\,y^3}{4\,\delta^2} \;-\; \frac{y^4}{8\,\delta^3} \;+\; \frac{3\,y^5}{10\,\delta^4} \;-\; \frac{y^7}{28\,\delta^6}\right)\Bigg|_0^\delta \;&= \\
u_\infty^2\,\frac{d}{dx}\,\delta\left(\frac{3}{4} \;-\; \frac{3}{4} \;-\; \frac{1}{8} \;+\; \frac{3}{10} \;-\; \frac{1}{28}\right) \;&= \\
\frac{39\,u_\infty^2}{280}\,\frac{d\,\delta}{dx} \;&=\; \frac{3\,\nu\,u_\infty}{2\,\delta} .
\end{aligned}$$

Notice that we have been careful to retain δ under d/dx because $\delta = \delta(x)$. This is now a differential equation that describes the growth of δ as a function of x. It is separable

$$\delta\,d\delta \;=\; \frac{140\,\nu}{13\,u_\infty}\,dx\,,$$

and therefore easily solved by integrating in the streamwise direction as

$$\int \delta\,d\delta \;=\; \int \frac{140\,\nu}{13\,u_\infty}\,dx \qquad \text{from which we find} \qquad \frac{\delta^2}{2} \;=\; \frac{140\,\nu}{13\,u_\infty}\,x \;+\; C_0\,,$$

where C_0 is a constant of integration. If we take the boundary condition as the boundary layer thickness being zero at the leading edge of the plate, according to Fig. 9.1 on pp. 115, that is $\delta(0) = 0$, we find $C_0 = 0$ so that

$$\delta \;=\; \sqrt{\frac{280}{13}\;\frac{\nu\,x}{u_\infty}} \;\approx\; 4.641\,\sqrt{\frac{\nu}{u_\infty\,x}}\;\;x \;=\; \frac{4.641\,x}{\sqrt{Re_x}}\,, \tag{K.6}$$

where Re_x is the Reynolds number based on x. This is an explicit description of how the boundary layer grows along the plate in x and "closes" the system. That is, Eqs. (K.5) and (K.6) taken together specify the boundary layer flow over the plate for the 3–rd order polynomial approximation. Many other approximations are possible.

For the energy problem, we will assume a boundary condition of a constant temperature at the plate surface, T_S. Define a dimensionless temperature and take its profile as a third–order polynomial[K.6]

$$\frac{T - T_S}{T_\infty - T_S} = b_1 + b_2 \left(\frac{y}{\delta_t}\right) + b_3 \left(\frac{y}{\delta_t}\right)^2 + b_4 \left(\frac{y}{\delta_t}\right)^3 , \tag{K.7}$$

where we can invoke very similar boundary conditions as those for the momentum problem to evaluate $b_1 \dots b_4$

- For a constant temperature plate with a specified freestream temperature, i.e. at the edge of the thermal boundary layer, we have
$$T\Big|_{y=0} = T_S \qquad \text{and} \qquad T\Big|_{y=\delta_t} = T_\infty$$
- Matching between the boundary layer and the freestream is once again asymptotic, i.e. "smooth", implying a vanishing gradient
$$\frac{\partial T}{\partial y}\Big|_{y=\delta_t} = 0 .$$
- Take the conservation of energy statement, Eq. (9.3) on pp. 118, and *evaluate* it at the plate surface, $y = 0$, under the restriction of no–slip. Plugging in $u = v = 0$ at $y = 0$, we find
$$\frac{\partial^2 T}{\partial y^2}\Big|_{y=0} = 0 .$$

We now have 4 conditions to evaluate the 4 unknowns. Carrying this out, we find $b_1 = b_3 = 0$, $b_2 = 1.5$, and $b_4 = -0.5$. These are identical to the corresponding values for $a_1 \dots a_4$ for the momentum problem, which is little surprise since the boundary conditions are identical. The temperature profile is then

$$T^* = \frac{T - T_S}{T_\infty - T_S} = \frac{3}{2}\left(\frac{y}{\delta_t}\right) - \frac{1}{2}\left(\frac{y}{\delta_t}\right)^3 . \tag{K.8}$$

Here we will now apply the Kármán–Pohlhausen integral condition for the energy statement, Eq. (K.4), to derive the thermal boundary layer growth law as a closure to the system. Specifically, solve Eq. (K.8) for T as

$$T = \left(\frac{3}{2\,\delta_t}\,y - \frac{1}{2\,\delta_t^3}\,y^3\right)(T_\infty - T_S) + T_S$$

and substitute both this expression and the velocity profile into Eq. (K.4) on pp. 189, whereby

$$\frac{d}{dx}\int_0^{\delta_t} \left(T_\infty - \left[\left(\frac{3}{2\,\delta_t}\,y - \frac{1}{2\,\delta_t^3}\,y^3\right)(T_\infty - T_S) + T_S\right]\right)\left(\frac{3\,u_\infty}{2\,\delta}\,y - \frac{u_\infty}{2\,\delta^3}\,y^3\right) dy$$

$$= \alpha\left(\frac{3}{2\,\delta_t} - \frac{3}{2\,\delta_t^3}\,y^2\right)(T_\infty - T_S)\Big|_{y=0} .$$

This is not an entirely trivial equation to solve, but let us notice immediately that $T_\infty - T_S$ factors on both the left and the right sides and can therefore be canceled.

[K.6] This form of dimensionless temperature is similar to that defined for the pipe convection problem in Eq. (8.16) on pp. 107. It will be used again when applying the iterative integral method, i.e. in Eq. (L.14) on pp. 206.

We will also factor $u_\infty/2$ on the left hand side, after which the remaining operations simplify the equation as

$$\begin{aligned}
\frac{u_\infty}{2}\,\frac{d}{dx}\int_0^{\delta_t}\left(1\,-\,\frac{3}{2\,\delta_t}\,y\,+\,\frac{1}{2\,\delta_t^3}\,y^3\right)\left(\frac{3}{\delta}\,y\,-\,\frac{1}{\delta^3}\,y^3\right)dy &= \frac{3\,\alpha}{2\,\delta_t}\\
\frac{d}{dx}\int_0^{\delta_t}\left(\frac{3\,y}{\delta}\,-\,\frac{y^3}{\delta^3}\,-\,\frac{9\,y^2}{2\,\delta_t\,\delta}\,+\,\frac{3\,y^4}{2\,\delta_t\,\delta^3}\,+\,\frac{3\,y^4}{2\,\delta_t^3\,\delta}\,-\,\frac{y^6}{2\,\delta_t^3\,\delta^3}\right)dy &= \frac{3\,\alpha}{u_\infty\,\delta_t}\\
\frac{d}{dx}\left(\frac{3\,y^2}{2\,\delta}\,-\,\frac{y^4}{4\,\delta^3}\,-\,\frac{3\,y^3}{2\,\delta_t\,\delta}\,+\,\frac{3\,y^5}{10\,\delta_t\,\delta^3}\,+\,\frac{3\,y^5}{10\,\delta_t^3\,\delta}\,-\,\frac{y^7}{14\,\delta_t^3\,\delta^3}\right)\Bigg|_0^{\delta_t} &=\\
\frac{d}{dx}\left(\frac{3\,\delta_t^2}{2\,\delta}\,-\,\frac{\delta_t^4}{4\,\delta^3}\,-\,\frac{3\,\delta_t^3}{2\,\delta_t\,\delta}\,+\,\frac{3\,\delta_t^5}{10\,\delta_t\,\delta^3}\,+\,\frac{3\,\delta_t^5}{10\,\delta_t^3\,\delta}\,-\,\frac{\delta_t^7}{14\,\delta_t^3\,\delta^3}\right) &=\\
\frac{d}{dx}\left(\frac{3\,\delta_t^2}{2\,\delta}\,-\,\frac{\delta_t^4}{4\,\delta^3}\,-\,\frac{3\,\delta_t^2}{2\,\delta}\,+\,\frac{3\,\delta_t^4}{10\,\delta^3}\,+\,\frac{3\,\delta_t^2}{10\,\delta}\,-\,\frac{\delta_t^4}{14\,\delta^3}\right) &=\\
\frac{d}{dx}\left(\frac{3\,\delta_t^2}{10\,\delta}\,-\,\frac{3\,\delta_t^4}{140\,\delta^3}\right) &= \frac{3\,\alpha}{u_\infty\,\delta_t}\,.
\end{aligned}$$

It seems as if this will become a little unwieldy to deal with δ_t/δ, so let us define

$$\xi\;=\;\xi(x)\;=\;\frac{\delta_t(x)}{\delta(x)}\,,$$

so that, with some factoring, the equation can be written in a slightly more maneuverable form

$$\frac{d}{dx}\left[\delta\left(\xi^2\,-\,\frac{\xi^4}{14}\right)\right]\;=\;\frac{10\,\alpha}{u_\infty\,\delta\,\xi}\,. \tag{K.9}$$

At this point it becomes convenient for the purposes of completing the problem to invoke the restriction $\delta_t < \delta$, i.e. the thermal boundary layer is smaller than the momentum boundary layer. This means $\xi < 1$, implying that the solution will now be valid only for fluids having $Pr > 1$. Under this condition $\xi^4/14 \ll \xi^2$, meaning the former can be neglected. We can write the simplified equation, omitting the $\xi^4/14$ and then differentiate, finding

$$\begin{aligned}
\delta\,\xi\,\frac{d}{dx}\Big(\delta\,\xi^2\Big) &= \frac{10\,\alpha}{u_\infty}\\
2\,\delta^2\,\underbrace{\xi^2\,\frac{d\xi}{dx}}_{\text{recast}}\,+\,\delta\,\xi^3\,\frac{d\delta}{dx} &= \frac{10\,\alpha}{u_\infty}\,,
\end{aligned}$$

from which it still seems we will still have some awkwardness in developing the solution. Let us use the observation

$$\frac{1}{3}\,\frac{d}{dx}\Big(\xi^3\Big)\;=\;\xi^2\,\frac{d\xi}{dx}$$

in order to recast the marked term above, so that the equation is now

$$\frac{2}{3}\,\delta^2\,\frac{d}{dx}\Big(\xi^3\Big)\;+\;\delta\,\xi^3\,\frac{d\delta}{dx}\;=\;\frac{10\,\alpha}{u_\infty}\,.$$

Now recall that we already determined δ above in Eq. (K.6) on pp. 190, from which we can explicitly write both terms containing δ in the above equation as

$$\delta^2 \;=\; \frac{280\,\nu}{13\,u_\infty}\,x \qquad \text{and} \qquad 2\,\delta\,\frac{d\delta}{dx} \;=\; \frac{280\,\nu}{13\,u_\infty} \qquad \rightarrow \qquad \delta\,\frac{d\delta}{dx} \;=\; \frac{140\,\nu}{13\,u_\infty}$$

so that substitution yields

$$\begin{aligned} \frac{2}{3}\cdot\frac{280\,\nu}{13\,u_\infty}\cdot x\cdot\frac{d}{dx}\Big(\xi^3\Big) \;+\; \xi^3\cdot\frac{140\,\nu}{13\,u_\infty} \;&=\; \frac{10\,\alpha}{u_\infty} \\ x\,\frac{d}{dx}\Big(\xi^3\Big) \;+\; \frac{3}{4}\,\xi^3 \;&=\; \frac{39}{56}\,\frac{\alpha}{\nu}\,. \end{aligned}$$

This appears to be a strange equation to solve, but if we use the substitution $y = \xi^3$ as a device for simplification and write the constant term on the right hand side as C_0 then this equation can be written

$$x\,\frac{dy}{dx} \;+\; \frac{3}{4}\,y \;=\; C_0 \qquad \text{or} \qquad \frac{dy}{dx} \;+\; \frac{3}{4\,x}\,y \;=\; \frac{C_0}{x}\,,$$

the latter being "standard form". Let us solve this using the method of variation of parameters (Martin and Reissner, 1956). First, we solve the homogeneous form of the problem. This is separable, whereby

$$\begin{aligned} \frac{dy_h}{dx} \;&=\; -\,\frac{3}{4\,x}\,y_h \\ \frac{dy_h}{y_h} \;&=\; -\,\frac{3\,dx}{4\,x} \\ \ln(y_h) \;&=\; -\,\frac{3}{4}\,\ln(x) \;+ \\ e^{\ln(y_h)} \;&=\; e^{-\frac{3}{4}\,\ln(x)\,+\,C_1} \;=\; \Big(e^{\ln(x)}\Big)^{-3/4}\cdot e^{C_1} \\ y_h \;&=\; C_2\,x^{-3/4}\,. \end{aligned}$$

Now we assume the solution to the complete problem by replacing the integration constant C_2 with an as–of–yet unknown function of x, call it $f(x)$, i.e. $y = f(x)\cdot x^{-3/4}$. We take its first derivative, so as to be able to substitute both y and y' into the differential equation, finding

$$\begin{aligned} f'\cdot x^{-3/4} \;-\; \cancel{\frac{3}{4}\,f\cdot x^{-7/4}} \;+\; \cancel{\frac{3}{4\,x}\,f\cdot x^{-3/4}} \;&=\; \frac{C_0}{x} \\ f' \;&=\; C_0\,x^{-1/4} \\ f(x) \;&=\; \frac{4}{3}\,C_0\,x^{3/4} \;+\; C_3\,, \end{aligned}$$

so that the complete solution is re–constituted as

$$y \;=\; f\cdot x^{-3/4} \;=\; \left(\frac{4}{3}\,C_0\,x^{3/4} \;+\; C_3\right)x^{-3/4} \;=\; \frac{4\,C_0}{3} \;+\; C_3\,x^{-3/4}\,,$$

where we remind ourselves that C_3 is the integration constant, while C_0 is shorthand for the group of constants defined above. If we now replace ξ^3 for y and the re–insert the full expression for C_0,

we can write the result as

$$\xi^3 \;=\; \frac{4\cdot 39}{3\cdot 56}\,\frac{\alpha}{\nu} \;+\; C_3\,x^{-3/4}$$

$$\left(\frac{\delta_t}{\delta}\right)^3 \;=\; \frac{13}{14\,Pr} \;+\; C_3\,x^{-3/4}\,, \tag{K.10}$$

with an integration constant C_3 that remains to be determined. Ordinarily, we would do this exactly as we did for the momentum boundary layer thickness, i.e. from a condition at the leading edge. However, the situation here is somewhat more subtle.

At first glance, it *appears* from Eq. (K.10) that we should simply set $C_3 = 0$, so as to avoid the implied singularity at the leading edge.[K.7] However, it is not hard to show that there is actually no such singularity at $x = 0$ using a simple limiting argument. We observe from Eq. (K.6) that the momentum thickness is $\delta = \varphi_1\sqrt{x}$, where φ_1 is a constant. Also, as we approach the leading edge "from the right", i.e. $x \to 0^+$, the second term in Eq. (K.10) is much larger than the first, so that $\xi^3 = (\delta_t/\delta)^3 \to C_3\,x^{-3/4}$. Substituting, we find

$$\delta_t \;\sim\; \left(C_3\,x^{-3/4}\right)^{1/3}\!\cdot\varphi_1\,x^{1/2} \;=\; C_3^{1/3}\varphi_1\,x^{-1/4}\,x^{1/2} \;=\; C_3^{1/3}\varphi_1\,x^{1/4} \qquad \text{implying} \qquad \delta_t\Big|_{x=0} \;=\; 0\,.$$

In other words, the boundary condition at the leading edge is satisfied, *regardless* of the value assigned to C_3, so the constant of integration in Eq. (K.10) is actually *indeterminate* with respect strictly to the leading edge boundary condition.[K.8] This is not just a consequence of our earlier discarding of the higher–order term ξ^4 in Eq. (K.9) on pp. 192 in order to expedite the solution. For example, we find the same contingency if using only linear approximations for u and T, and

[K.7] Unfortunately, the few texts that actually give substantive details this far into the solution process allow rigor to lapse at this point. For example Özişik (1985, pp. 368) argues a boundary condition of $\xi = 0$ at a finite standoff distance $x = x_0$, from which it is readily shown that

$$\left(\frac{\delta_t}{\delta}\right)^3 \;=\; \frac{13}{14\,Pr}\left[1-\left(\frac{x_0}{x}\right)^{3/4}\right]\,,$$

from whence it is then argued that if we *take* $x_0 = 0$ then $(\delta_t/\delta)^3 = 13\,Pr^{-1}/14$. Of course, this subtly obscures the fact that if $x = x_0$ and $x_0 = 0$, then $x = 0$ and this result is still indeterminate!

[K.8] The observation can also be proved formally, i.e. not just asymptotically, but as an equivalence, using L'Hospital's Rule. Again, write $\delta = \varphi_1\sqrt{x}$ from Eq. (K.6), where φ_1 is a constant, and let $\varphi_2 = 13\,Pr^{-1}/14$ in Eq. (K.10). Then Eq. (K.10) can be written in the convenient way

$$\delta_t \;=\; \frac{\varphi_1\,x^{1/2}}{\left(\varphi_2 + C_3\,x^{-3/4}\right)^{-1/3}}\,,$$

which is still indeterminate at $x = 0$. To apply the Rule, take d/dx of both the numerator and the denominator, after which some algebra shows

$$\delta_t\Big|_{x=0} = \frac{\dfrac{\varphi_1\,x^{-1/2}}{2}}{\dfrac{-\left(\varphi_2 + C_3\,x^{-3/4}\right)^{-4/3}}{3}\cdot\dfrac{-\,3\,C_3\,x^{-7/4}}{4}} \;=\; \cdots \;=\; \frac{2\,\varphi_1\left(\varphi_2\,x + C_3\,x^{1/4}\right)^{4/3}}{C_3\,x^{1/12}}\,,$$

which is still indeterminate. Taking d/dx of both the numerator and the denominator once again, we find

$$\delta_t\Big|_{x=0} = \frac{2\,\varphi_1\,\frac{4}{3}\left(\varphi_2\,x + C_3\,x^{1/4}\right)^{1/3}\left(\varphi_2 + \frac{1}{4}\,C_3\,x^{-3/4}\right)}{\frac{1}{12}\,C_3\,x^{-11/12}} \;=\; \frac{96}{3}\,\varphi_1\,\left(\varphi_2\,x + C_3\,x^{1/4}\right)^{1/3}\left(\frac{x^{11/12}\,\varphi_2}{C_3} + \frac{x^{1/6}}{4}\right)\,,$$

which obviously vanishes at $x = 0$.

for which no such simplification need be invoked. It is very significant because we see that we now have to infer C_3 in some other way besides the strict mathematical boundary condition.

Recall from Eq. (K.9) on pp. 192 that our simplification basically limits us to $Pr > 1$, implying $\delta_t < \delta$. In other words, for $Pr > 1$ momentum diffuses faster than heat, so δ *must* be larger than δ_t. Eq. (K.10) suggests that if $C_3 > 0$, the thermal boundary layer will be *thicker* than the momentum boundary layer, which violates our Prandtl number assumption. For example, Fig. K.1 shows 3 values of C_3 plotted when $Pr = 5$. Even the seemingly very small case of $C_3 = 0.01$ shows

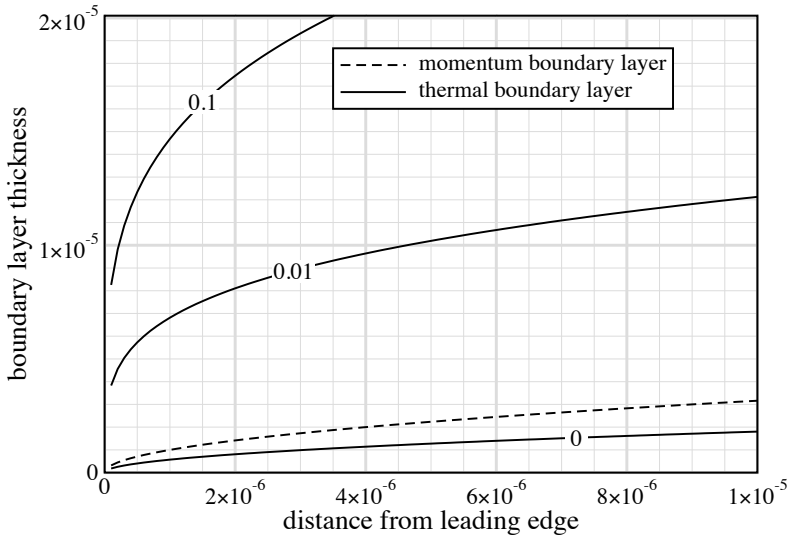


FIGURE K.1. *Momentum boundary layer thickness for* $\delta = 0.001\sqrt{x}$ *and thermal boundary layer thickness plotted from Eq.* (K.10) *for* $Pr = 5$ *and values of* $C_3 \in \{0, 0.01, 0.1\}$. *Curves for all 3 values of* C_3 *satisfy the leading edge boundary condition of* $\delta_t = 0$ *at* $x = 0$.

significant violation of the requirement that the thermal layer be thinner than the momentum layer. We can confirm this behavior by revisiting the asymptotic argument made above, which implies that when we are arbitrarily close to the leading edge, Eq. (K.10) becomes

$$\left.\frac{\delta_t}{\delta}\right|_{x\to 0} \to \left(C_3\, x^{-3/4}\right)^{1/3} = C_3^{1/3}\, x^{-1/4}\,,$$

from which δ_t will clearly (and erroneously) be larger than δ, *unless* $C_3 = 0$. In the end we see that Eq. (K.10) is ultimately resolved as

$$\frac{\delta_t}{\delta} \;=\; \left(\frac{13}{14\,Pr}\right)^{1/3} \approx\; 0.976\, Pr^{-1/3}\,,$$

which had to be inferred from observing the relative size of δ_t vs. δ very near the leading edge. It could not be calculated from the boundary condition at the leading edge, which is indeterminate. Substituting the explicit result for δ in Eq. (K.6) on pp. 190 into this equation, we find the growth law for the thermal boundary layer to be

$$\delta_t \;=\; \delta\,\left(\frac{13}{14\,Pr}\right)^{1/3} \;=\; \sqrt{\frac{280}{13}}\;\frac{x}{\sqrt{Re_x}}\;\left(\frac{13}{14\,Pr}\right)^{1/3} \;\approx\; \frac{4.528\,x}{Re_x^{1/2}\,Pr^{1/3}}\,. \tag{K.11}$$

As with the momentum solution, the temperature profile in Eq. (K.8) on pp. 191, coupled with the thermal boundary layer growth law here in Eq. (K.11) specify the complete solution.

Ultimately, we want to state the practical answer to this problem in the form of the Nusselt number, which we can calculate following the principles in §7.4 on pp. 92. Equating conduction and convection at the no–slip wall, we can use the temperature solution in Eq. (K.8) to specify $\partial T/\partial y$, finding

$$
\begin{aligned}
h(T_S \,-\, T_\infty) &= -\,k\,\frac{3}{2\,\delta_t}\,(T_\infty \,-\, T_S) \\
\frac{h}{k} &= \frac{3}{2\,\delta_t} \,=\, \frac{3\,Re_x^{1/2}\,Pr^{1/3}}{2\cdot 4.528\,x} \\
Nu_x &= \frac{h\,x}{k} \,\approx\, 0.3313\,Re_x^{1/2}\,Pr^{1/3}\,. \qquad \text{(K.12)}
\end{aligned}
$$

APPENDIX L

Similarity Technique Applied to Prandtl's Boundary Layer Flow

As we mentioned in appendix K, the boundary layer equations, Eqs. (9.1) through (9.3) on pp. 118, can be solved by a variety of approaches. Here, we develop the classical similarity procedure that reduces the conservation of mass and momentum PDEs to a single ODE of a derived function. This transformation is possible because there is no obvious finite length scale, since we take the plate to be semi–infinite. We followed this procedure for the Rayleigh problem, detailed in appendix D starting on pp. 146, and will follow a similar order of operations here.

L.1. Derivation of the Similarity Parameter

As with the Rayleigh problem in §D.1 on pp. 146, we opt for the "stretching variables" method to determine the similarity parameter (Hansen, 1967). Define stretched–variable versions of the physical variables as

$$\overline{u} \,=\, \xi^a \, u \,, \qquad \overline{v} \,=\, \xi^b \, v \,, \qquad \overline{x} \,=\, \xi^c \, x \,, \qquad \overline{y} \,=\, \xi^d \, y \,,$$

where ξ is the parameter for the group and variables a through d are unknown. To obtain equations in the "bar" (coordinate stretched) system, we apply Chain Rule in much the same way as we use it for non–dimensionalizing an equation (c.f. Eqs. (4.7) and (4.8) on pp. 34). For conservation of mass, Eq. (9.1) on pp. 118, repeated here for convenience as

$$\frac{\partial u}{\partial x} \,+\, \frac{\partial v}{\partial y} \,=\, 0 \,,$$

we find

$$\xi^{c-a} \, \frac{\partial \overline{u}}{\partial \overline{x}} \,+\, \xi^{d-b} \, \frac{\partial \overline{v}}{\partial \overline{y}} \,=\, 0 \,.$$

To maintain invariance between the original and the stretched system, the ξ must be able to be canceled, implying $c-a = d-b$, which cannot be simplified further, at the moment. For conservation of momentum, Eq. (9.2) on pp. 118, again repeated for convenience as

$$u \, \frac{\partial u}{\partial x} \,+\, v \, \frac{\partial u}{\partial y} \,=\, \nu \, \frac{\partial^2 u}{\partial y^2} \,,$$

similar operations yield

$$\xi^{c-2a} \, \overline{u} \, \frac{\partial \overline{u}}{\partial \overline{x}} \,+\, \xi^{d-a-b} \, \overline{v} \, \frac{\partial \overline{u}}{\partial \overline{y}} \,=\, \xi^{2d-a} \, \nu \, \frac{\partial^2 \overline{u}}{\partial \overline{y}^2} \,,$$

so that by similar argument we have $c - 2a = d - a - b = 2d - a$.

We immediately see that $a = 0$, otherwise the upstream boundary condition in the stretched system, $\xi^{-a} \, \overline{u} = u_\infty$, would not be invariant. Substituting for the relations obtained from mass and momentum, respectively, we find $c = d - b$ and $c = d - b = 2d$.[L.1] Next, we must find a parameter,

[L.1] Note that the equation from conservation of mass is now superfluous because $c = d - b$ is repeated.

φ, that is invariant under transformations of the independent variables. We have two independent variables here, x and y, so we choose a power expression of their product of the form

$$\varphi \;=\; x^e\, y\,,$$

noting that there is no need to assume a power of y other than unity because e would be determined commensurately (Hansen, 1967). The invariance condition just mentioned for φ requires

$$x^e\, y \;=\; \overline{x}^e\, \overline{y} \;=\; \Big(\xi^c\, x\Big)^e\, \Big(\xi^d\, y\Big) \;=\; \xi^{ce+d}\, x^e\, y\,,$$

where $c\,e + d = 0$ for invariance to hold. Given $c = 2\,d$ from above, a little algebra shows

$$e \;=\; -\,\frac{d}{c} \;=\; -\,\frac{d}{2\,d} \;=\; -\,\frac{1}{2}\,,$$

so that a valid similarity parameter is

$$\varphi \;=\; \frac{y}{\sqrt{x}}\,.$$

Recall from footnote D.1 on pp. 147 that the successful similarity parameter is not necessarily unique to a specific problem. The form $\varphi = y/\sqrt{x}$ is the salient similarity relationship, but it will be more convenient in the resulting ODE if we multiply this expression by the constant value $\sqrt{u_\infty/\nu}$, so that the final similarity parameter is[L.2]

$$\varphi \;=\; y\,\sqrt{\frac{u_\infty}{\nu\,x}}\,. \tag{L.1}$$

This combination will allow reduction of the PDE system into a very tidy ODE.

Let us preemptively work out the partial derivatives that will be needed in the solution procedure. As usual, these are determined by simple application of Chain Rule, specifically

$$\begin{aligned}
\frac{\partial}{\partial x} \;&=\; \frac{\partial\varphi}{\partial x}\,\frac{d}{d\varphi} \;=\; -\,\frac{1}{2}\,x^{-3/2}\,y\,\sqrt{\frac{u_\infty}{\nu}}\;\frac{d}{d\varphi} \;=\; -\,\frac{\varphi}{2\,x}\;\frac{d}{d\varphi} \\
\frac{\partial}{\partial y} \;&=\; \frac{\partial\varphi}{\partial y}\,\frac{d}{d\varphi} \;=\; \sqrt{\frac{u_\infty}{\nu\,x}}\;\frac{d}{d\varphi} \\
\frac{\partial^2}{\partial y^2} \;&=\; \frac{\partial}{\partial y}\left(\sqrt{\frac{u_\infty}{\nu\,x}}\;\frac{d}{d\varphi}\right) \;=\; \cdots \;=\; \frac{u_\infty}{\nu\,x}\;\frac{d^2}{d\varphi^2}\,.
\end{aligned}$$

L.2. Reduction of the Momentum PDE System to an ODE

The previous problem for which we used the similarity approach, Rayleigh conduction in appendix D starting on pp. 146, was relatively easy to transform. The boundary layer problem is slightly more complicated because we are dealing with conservation of both mass and momentum. Recall the clever mathematical device of the stream function, ψ, defined relative to the velocity distribution as

$$u \;=\; \frac{\partial\psi}{\partial y} \qquad \text{and} \qquad v \;=\; -\,\frac{\partial\psi}{\partial x}\,, \tag{L.2}$$

which we can use to eliminate conservation of mass outright, since ψ automatically satisfies that law. The first real step is then to *deduce* the relationship of ψ to the univariate function $f = f(\varphi)$, for which we want to fashion our ODE. The former has physical units of length squared per unit time,

[L.2] Recall from footnote D.1 on pp. 147 that a successful similarity parameter is not necessarily unique.

while the latter is unitless. That is, if we take $\psi(x,y) \propto f(\varphi)$, we must determine the appropriate multiplier such that physical dimensions are consistent. Given the stream function definition, we observe

$$u \,=\, \frac{\partial \psi}{\partial y} \qquad : \qquad \frac{\partial \psi}{\partial y} \,\to\, \sqrt{\frac{u_\infty}{\nu\, x}}\; \frac{df}{d\varphi} \,=\, \sqrt{\frac{u_\infty}{\nu\, x}}\; f' \,,$$

but the square root term has physical units of length^{-1}. We can remedy this artifact by simply multiplying by $\sqrt{u_\infty\, \nu\, x}$, so that

$$\text{(L.3)} \quad \frac{\partial \psi}{\partial y} \,=\, u \,=\, \sqrt{\frac{u_\infty}{\nu\, x}}\; \sqrt{u_\infty\, \nu\, x}\; f' \,=\, u_\infty\, f' \qquad \text{and consequently} \qquad \psi \,=\, \sqrt{u_\infty\, \nu\, x}\; f \,.$$

Using this definition of ψ and the partial derivatives above, let us flesh–out various derivatives of ψ that will be needed in the subsequent form of the boundary layer momentum equation

$$\begin{aligned}
\frac{\partial^2 \psi}{\partial y^2} \;&=\; \frac{\partial}{\partial y}\Big(u_\infty\, f'\Big) \,=\, \sqrt{\frac{u_\infty}{\nu\, x}}\; u_\infty\, \frac{df'}{d\varphi} \,=\, \sqrt{\frac{u_\infty}{\nu\, x}}\; u_\infty\, f'' \\
\frac{\partial^3 \psi}{\partial y^3} \;&=\; \frac{\partial}{\partial y}\left(\frac{\partial^2 \psi}{\partial y^2}\right) \,=\, \sqrt{\frac{u_\infty}{\nu\, x}}\; \sqrt{\frac{u_\infty}{\nu\, x}}\; u_\infty\, \frac{df''}{d\varphi} \,=\, \frac{u_\infty^2}{\nu\, x}\, f''' \\
\frac{\partial \psi}{\partial x} \;&=\; \frac{1}{2}\, \sqrt{u_\infty\, \nu}\; x^{-1/2}\, f \,+\, \sqrt{u_\infty\, \nu\, x}\; f'\, \Big(-\, \frac{\varphi}{2\, x}\Big) \\
&=\; \frac{1}{2}\, \sqrt{\frac{u_\infty\, \nu}{x}}\, \Big(f \,-\, \varphi\, f'\Big) \qquad \text{(L.4)} \\
\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) \;&=\; -\, \frac{\varphi}{2\, x}\, \frac{d}{d\varphi}\Big(u_\infty\, f'\Big) \,=\, -\, \frac{\varphi\, u_\infty}{2\, x}\, f'' \,.
\end{aligned}$$

It is now a straightforward task, starting with the momentum equation, to write it in terms of the stream function, ψ, then to substitute the above quantities with the goal of ultimately writing it in terms of f. We find

$$\begin{aligned}
u\, \frac{\partial u}{\partial x} \,+\, v\, \frac{\partial u}{\partial y} \;&=\; \nu\, \frac{\partial^2 u}{\partial y^2} \\
\frac{\partial \psi}{\partial y} \cdot \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) \,-\, \frac{\partial \psi}{\partial x} \cdot \frac{\partial^2 \psi}{\partial y^2} \;&=\; \nu\, \frac{\partial^3 \psi}{\partial y^3} \\
\underbrace{\Big(u_\infty\, f'\Big)}_{u}\; \underbrace{\Big(-\, \frac{\varphi\, u_\infty}{2\, x}\, f''\Big)}_{\partial u/\partial x} \,-\, \underbrace{\frac{1}{2}\, \sqrt{\frac{u_\infty\, \nu}{x}}\, \Big(f \,-\, \varphi\, f'\Big)}_{-v}\; \underbrace{\sqrt{\frac{u_\infty}{\nu\, x}}\; u_\infty\, f''}_{\partial u/\partial y} \;&=\; \nu\, \frac{u_\infty^2}{\nu\, x}\, f''' \\
-\, \frac{\varphi\, u_\infty^2}{2\, x}\, f'\, f'' \,-\, \frac{u_\infty^2}{2\, x}\, \Big(f \,-\, \varphi\, f'\Big) f'' \;&=\; \frac{u_\infty^2}{x}\, f''' \\
-\, \frac{f\, f''}{2} \;&=\; f''' \\
f''' \,+\, \frac{f\, f''}{2} \;&=\; 0 \,, \qquad \text{(L.5)}
\end{aligned}$$

which is now an ODE in $f(\varphi)$ only. The boundary conditions in Eqs. (9.4) and (9.5) on pp. 118 must also be transformed to the similarity domain. Noting that $y = 0$ implies $\varphi = 0$ from the definition in Eq. (L.1), the no–slip condition for u is readily found as

$$u\Big|_{y=0} = 0 \qquad \text{implies} \qquad f'(0) = 0 \tag{L.6}$$

and the no–slip condition in v is also found, with slightly more effort, to be

$$v\Big|_{y=0} = -\left.\frac{\partial\psi}{\partial x}\right|_{y=0} = 0 \quad \text{implies} \quad -\frac{1}{2}\sqrt{\frac{u_\infty\,\nu}{x}}\left(f - 0\cdot f'\right) = 0 \quad \rightarrow \quad f(0) = 0\,. \tag{L.7}$$

The freestream boundary condition is a little bit more subtle. Recall that Eq. (9.5) specifies "freestream" at the edge of the boundary layer, however we do not know δ. In fact, we have made no mention of any finite length scale, including δ, nor even any notion of an explicit division between the boundary layer and the freestream. That is, we have no basis for making any sort of statement about what happens at the defined length of δ. Instead, we can only observe that freestream conditions are attained "very far away from the plate's surface", whereby $y \rightarrow \infty$ implies $\varphi \rightarrow \infty$, again from the definition in Eq. (L.1), so that

$$u\Big|_{y\rightarrow\infty} = u_\infty \qquad \text{implies} \qquad f'(\infty) = 1 \tag{L.8}$$

from Eq. (L.3). The PDE system has now been successfully recast as a 3–rd order ODE, Eq. (L.5), having 3 boundary conditions given by Eqs. (L.6) through (L.8). Once f is solved, we readily recover u from its first derivative, per Eq. (L.3).

L.3. Piercy–Preston Iterative Integration Procedure for Velocity

Quick inspection immediately reveals that the similarity procedure only converted the PDE system to an ODE, but did *not* linearize the problem. Eq. (L.5) remains non–linear, suggesting the need for an *ad hoc* solution procedure.[L.3] Here, we will apply a clever iterative technique that renders progressively improved specifications for f (Piercy and Preston, 1936; Watson and Preston, 1951). In principle, we could obtain the solution exactly. In practice, each step becomes increasingly difficult, however the procedure converges very quickly. We will only perform sufficient iterations to demonstrate the basic method.

The Piercy–Preston concept is to develop an increasingly accurate set of solutions $f_1, f_2, \ldots$ toward the exact solution $f(\varphi)$ and this procedure is framed by *splitting* Eq. (L.5) as

$$f_i''' + \frac{f_{i-1}\cdot f_i''}{2} = 0\,,$$

where level $i-1$ represents the inferior solution from the previous iteration and level i represents the superior solution from the current iteration. In other words, in solving for level i, the "solution" for level $i-1$ is already *known*, meaning we can integrate this equation directly. For convenience, let us introduce the following notation

$$y_{i,1} = \frac{df_i}{d\varphi} = f_i' \qquad \text{and} \qquad y_{i,2} = \frac{dy_{i,1}}{d\varphi}\,,$$

[L.3] We already examined an approximate procedure in the form of the Kármán–Pohlhausen method in appendix K on pp. 187.

so that the split equation can be written as

$$\begin{aligned}
\frac{d}{d\varphi}\left[\frac{d}{d\varphi}\left(\frac{df_i}{d\varphi}\right)\right] &= -\frac{1}{2}\,f_{i-1}(\varphi)\,\frac{d}{d\varphi}\left(\frac{df_i}{d\varphi}\right) \\
\frac{d}{d\varphi}\left(\frac{dy_{i,1}}{d\varphi}\right) &= -\frac{1}{2}\,f_{i-1}(\varphi)\,\frac{dy_{i,1}}{d\varphi} \\
\frac{dy_{i,2}}{d\varphi} &= -\frac{1}{2}\,f_{i-1}(\varphi)\,y_{i,2}\,.
\end{aligned}$$

This equation is clearly separable, whereby it can be integrated directly as

$$\begin{aligned}
\int_{y_{i,2}(0)}^{y_{i,2}(\varphi)}\frac{dy_{i,2}}{y_{i,2}} &= -\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi \\
\ln\left(y_{i,2}\right)\Big|_{y_{i,2}(0)}^{y_{i,2}(\varphi)} = \ln\big[y_{i,2}(\varphi)\big] - \underbrace{\ln\big[y_{i,2}(0)\big]}_{\text{set to } C_{i,2}} &= \\
\ln\big[y_{i,2}(\varphi)\big] &= -\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi \,+\, C_{i,2} \\
y_{i,2}(\varphi) &= \exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi \,+\, C_{i,2}\right) \\
&= \exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi\right)\cdot\exp\big(C_{i,2}\big) \\
\frac{dy_{i,1}}{d\varphi} &= C'_{i,2}\,\exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi\right)\,,
\end{aligned}$$

where we have observed that $C_{i,2} = \ln\big[y_{i,2}(0)\big]$ is a yet–to–be–determined constant, given that the function is *evaluated* at $\varphi = 0$. Notice that in the last step we have also replaced $y_{i,2}$ with the equivalent derivative term that is a function of the predecessor $y_{i,1}$. This equation is once again separable and therefore again directly integrable

$$\begin{aligned}
\int_{y_{i,1}(0)}^{y_{i,1}(\varphi)} dy_{i,1} &= \int_0^{\varphi} C'_{i,2}\,\exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi\right)d\varphi \\
y_{i,1}(\varphi) &= C'_{i,2}\int_0^{\varphi}\exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi\right)d\varphi \,+\, y_{i,1}(0) \\
f'_i(\varphi) &= C'_{i,2}\int_0^{\varphi}\exp\left(-\frac{1}{2}\int_0^{\varphi} f_{i-1}(\varphi)\,d\varphi\right)d\varphi \,+\, \cancelto{0}{f'_i(0)}\,.
\end{aligned}$$

In the second line, we moved the constant $C_{i,2}$ outside the integral and in the third line we replaced $y_{i,1}$ with its definition of f'_i. We must now evaluate $C_{i,2}$ and $f'_i(0)$ from the boundary conditions of the problem. Eq. (L.6) states identically that $f'(0) = 0$, which implies likewise that $f'_i(0) = 0$, as shown in the equation above. Eq. (L.8) specifies $f'(\infty) = 1$, from which similar interpretation for

the i–th solution we find

$$1 \;=\; C'_{i,2} \int_0^\infty \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi \,,$$

so that finally

$$f'_i(\varphi) \;=\; \frac{\displaystyle\int_0^\varphi \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi}{\displaystyle\int_0^\infty \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi} \,. \tag{L.9}$$

This equation allows computation of a new f'_i given a previous function f_{i-1}, but does not yet lead to an improved version of f_i. For that, we have to integrate Eq. (L.9) one final time. Observing that $f'_i = df_i/d\varphi$, this expression is again separable and directly integrable. Reminding ourselves that the denominator in Eq. (L.9) is a constant and omitting formalities that are by now familiar, we find

$$f_i(\varphi) \;-\; \cancelto{0}{f_i(0)} \;=\; \frac{\displaystyle\int_0^\varphi \left[\int_0^\varphi \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi \right] d\varphi}{\displaystyle\int_0^\infty \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi} \,,$$

where we have now invoked the third and last boundary condition from Eq. (L.7), that being $f(0) = 0$. Consequently, we find

$$f_i(\varphi) \quad = \; \frac{\displaystyle\int_0^\varphi \left[\int_0^\varphi \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi \right] d\varphi}{\displaystyle\int_0^\infty \exp\left(-\,\frac{1}{2} \int_0^\varphi f_{i-1}(\varphi)\, d\varphi \right) d\varphi} \,. \tag{L.10}$$

Eqs. (L.9) and (L.10) are the Piercy–Preston integral equations that can now be used for iterating a solution toward $f(\varphi)$. In particular, the latter furnishes updated f_i from previous f_{i-1}.

L.4. Piercy–Preston Example: Uniform Flow Initial Profile

Conceptually speaking, the Piercy–Preston solution method now seems to be no more complicated than choosing an initial profile, f_1, then obtaining increasingly better functions by repeatedly using Eq. (L.10). As is usually the case with iterative methods, convergence will be promoted to the degree that the initial "guess" for f_1 is as good as possible. In their original paper, Piercy and Preston (1936) simply used the freestream profile itself

$$f_1 \;=\; \varphi \qquad \text{implying} \qquad f'_1 \;=\; 1 \qquad \text{implying from Eq. (L.3)} \qquad u \;=\; u_\infty \,.$$

In a later paper, Watson and Preston (1951) showed that taking f'_1 as an undetermined constant instead of 1 is a better choice. It is this example we will illustrate. We show only a single iteration, as f'_2 will be quite close to the exact profile and because succeeding iterations will be a little too involved, even for our considerable tolerance for minute details.

Defining κ as an undetermined constant, take $f_1 = \kappa\,\varphi$, so that $f'_1 = \kappa$. This implies that our initial guess is $u = \kappa\, u_\infty$, and, for the moment, we will defer discussion of how this constant is to

be determined. Note that the inner–most integral in Eqs. (L.9) and (L.10), i.e. the one within the parenthesis, is the same in the numerator and denominator, that expression for $i = 2$ now being

$$\int_0^\varphi f_1 \, d\varphi \;=\; \int_0^\varphi \kappa \, \varphi \, d\varphi \;=\; \kappa \int_0^\varphi \varphi \, d\varphi \;=\; \frac{\kappa \, \varphi^2}{2}$$

for our choice of f_1. We note here that, because we intend to go no further than $i = 2$ on this particular occasion, we will not actually be using Eq. (L.10). Instead, we evaluate Eq. (L.9), taking that as the solution for the velocity profile. The *form* of the integrals is the same for its numerator and denominator, though the upper limits are different. If we determine the numerator integral, the denominator is then simply a matter of an extra evaluation. The numerator of Eq. (L.9) using the $\kappa\varphi^2/2$ kernel we just derived is then

$$\begin{aligned}
\int_0^\varphi \exp\left(-\frac{1}{2} \int_0^\varphi f_1(\varphi) \, d\varphi \right) d\varphi \;&=\; \int_0^\varphi \exp\left(-\frac{1}{2} \cdot \frac{\kappa \, \varphi^2}{2} \right) d\varphi \\
&=\; \int_0^\varphi e^{-\kappa \, \varphi^2/4} \, d\varphi \\
&=\; \int_0^\varphi e^{-(\sqrt{\kappa} \, \varphi/2)^2} \, d\varphi \, .
\end{aligned}$$

We encountered a similar integral in the course of solving the Rayleigh conduction problem in §D.3 on pp. 148. There, we mentioned that this expression is not integrable in terms of elementary functions, but rather is a form of the so–called error function (Andrews, 1985).[L.4] Like the Rayleigh problem, our integral here will benefit from a quick change of variables to render it in the precise form to be integrated

$$\eta \;=\; \frac{\sqrt{\kappa} \, \varphi}{2} \qquad\qquad d\eta \;=\; \frac{\sqrt{\kappa}}{2} \, d\varphi \, ,$$

however, we are doing a definite integral here, rather than an indefinite one, as we did for the Rayleigh problem. That means changing the old limits $(0, \varphi)$ to new limits $(0, \eta)$. We rewrite the equation and execute the integration as

$$\begin{aligned}
\int_0^\varphi \exp\left(-\frac{1}{2} \int_0^\varphi f_1(\varphi) \, d\varphi \right) d\varphi \;&=\; \int_0^\eta e^{-\eta^2} \, \frac{2}{\sqrt{\kappa}} \, d\eta \\
&=\; \frac{2}{\sqrt{\kappa}} \, \frac{\sqrt{\pi}}{2} \, \underbrace{\left(\frac{2}{\sqrt{\pi}} \int_0^{\sqrt{\kappa} \, \varphi/2} e^{-\eta^2} \, d\eta \right)}_{\operatorname{erf}(\eta)} \\
&=\; \sqrt{\frac{\pi}{\kappa}} \, \operatorname{erf}(\eta) \, \Big|_0^{\sqrt{\kappa} \, \varphi/2} \\
&=\; \sqrt{\frac{\pi}{\kappa}} \, \operatorname{erf}\left(\frac{\sqrt{\kappa} \, \varphi}{2} \right) \, ,
\end{aligned}$$

where we made a few additional modifications in the second line so as to be able to identify the error function, as it is defined in Eq. (D.5) on pp. 149, and where we have invoked the fact that

[L.4] Some useful properties of the error function are summarized in Eqs. (D.5) through (D.8) on pp. 149.

$\text{erf}(0) = 0$ from Eq. (D.7) in our evaluation. As mentioned above, we also need the evaluation of this quantity itself at $\varphi \to \infty$, which is the denominator of Eq. (L.9) on pp. 202. Invoking the fact that $\text{erf}(+\infty) = 1$ from Eq. (D.8) on pp. 149, we have

$$\int_0^\infty \exp\left(-\frac{1}{2}\int_0^\varphi f_1(\varphi)\,d\varphi\right) d\varphi \;=\; \sqrt{\frac{\pi}{\kappa}}\;\text{erf}\big(\infty\big) \;=\; \sqrt{\frac{\pi}{\kappa}}\;,$$

so that Eq. (L.9) finally takes the form

$$f_2'(\varphi) \;=\; \text{erf}\left(\frac{\sqrt{\kappa}\;\varphi}{2}\right)\,. \tag{L.11}$$

As we mentioned above, Piercy and Preston (1936) simply took $\kappa = 1$ in their original paper, meaning the original solution is then immediately seen to be

$$\frac{u}{u_\infty} \;\approx\; f_2'(\varphi) \;=\; \text{erf}\Big(\frac{\varphi}{2}\Big) \;=\; \text{erf}\left(\frac{y}{2}\sqrt{\frac{u_\infty}{\nu\,x}}\;\right)\,, \tag{L.12}$$

where we have reconstituted the similarity parameter from its definition in Eq. (L.1) on pp. 198. However, if allowing for κ as a degree–of–freedom, some sort of additional constraint must be invoked to determine its value. In their later work, Watson and Preston (1951) cleverly re–purposed the Kármán–Pohlhausen integral condition in Eq. (K.3) on pp. 188 as a *closure* for this task. Before using this equation, we must recall that its original derivation was based on the assumption of a well–defined border, denoted by the length δ, between the boundary layer and the outer inviscid region. The similarity approach recognizes no such construction, nor any other finite length scale for that matter. It only allows for quantities defined at the plate surface, $y = 0$, and the freestream, $y \to \infty$. Consequently, in order to use the Kármán–Pohlhausen equation, one simply makes the observation that, for all practical purposes, the boundary layer edge δ is actually *at* ∞, whereby we just switch the limit.

Another relevant consideration relates to mathematical practicality. The Kármán–Pohlhausen equation itself is integrated in y, but the argument of the error function in Eq. (L.11) is $\sqrt{\kappa}\,\varphi/2$. Operations will obviously be significantly more straightforward if we could integrate directly in terms of the latter, so we will also make a change of variables during the course of the derivation, as well, specifically, let

$$\eta \;=\; \frac{\sqrt{\kappa}\;\varphi}{2} \;=\; y\,\frac{\sqrt{\kappa}}{2}\sqrt{\frac{u_\infty}{\nu\,x}} \qquad\qquad d\eta \;=\; \frac{1}{2}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;dy \qquad\qquad \eta \;=\; \infty \quad\text{at}\quad y \;=\; \infty\,.$$

Given our now–accepted approximate solution $u = u_\infty f_2'$, where f_2' is given by Eq. (L.11), we then find

$$\begin{aligned}
\frac{d}{dx}\int_{y=0}^{y=\infty}\big(u_\infty \;-\; u\big)\,u\,dy \;&=\; \nu\,\frac{\partial u}{\partial y}\bigg|_{y=0} \\
\frac{d}{dx}\int_{\eta=0}^{\eta=\infty}\Big[u_\infty \;-\; u_\infty\,\text{erf}(\eta)\Big]\,u_\infty\,\text{erf}(\eta)\left[2\sqrt{\frac{\nu\,x}{\kappa\,u_\infty}}\;d\eta\right] \;&=\; \nu\,\frac{\partial}{\partial y}\left[u_\infty\,\text{erf}\left(\frac{y}{2}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;\right)\right]\bigg|_{y=0} \\
2\,u_\infty^2\sqrt{\frac{\nu}{\kappa\,u_\infty}}\cdot\frac{d}{dx}\Bigg(\sqrt{x}\underbrace{\int_0^\infty\Big[1\;-\;\text{erf}(\eta)\Big]\text{erf}(\eta)\,d\eta}_{\text{Eq. (M.8) on pp. 217}}\Bigg) \;&=\; \nu\,u_\infty\,\frac{\partial}{\partial y}\,\text{erf}\left(\frac{y}{2}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;\right)\bigg|_{y=0}\,.
\end{aligned}$$

Note that the labeled integral has an identity in the form of Eq. (M.8) on pp. 217, being equal to $\left(\sqrt{2}-1\right)/\sqrt{\pi}$. Further development indicates

$$
\begin{aligned}
2\sqrt{\frac{u_\infty}{\kappa\,\nu}}\;\frac{d}{dx}\left(\sqrt{x}\;\frac{\sqrt{2}\;-\;1}{\sqrt{\pi}}\right) \;&=\; \frac{2}{\sqrt{\pi}}\;\exp\left[-\left(\frac{y}{2}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;\right)^2\right]\frac{1}{2}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;\Bigg|_{y=0} \\
2\sqrt{\frac{u_\infty}{\kappa\,\nu}}\;\frac{\sqrt{2}\;-\;1}{\sqrt{\pi}}\;\frac{d}{dx}\left(\sqrt{x}\;\right) \;&=\; \frac{1}{\sqrt{\pi}}\sqrt{\frac{\kappa\,u_\infty}{\nu\,x}}\;\underbrace{e^{-0^2}}_{1} \\
2\sqrt{\frac{1}{\kappa}}\;\left(\sqrt{2}\;-\;1\right)\frac{1}{2}\sqrt{\frac{1}{x}} \;&=\; \sqrt{\frac{\kappa}{x}} \\
\frac{1}{\sqrt{\kappa}}\left(\sqrt{2}\;-\;1\right) \;&=\; \sqrt{\kappa} \\
\sqrt{2}\;-\;1 \;&=\; \kappa\,,
\end{aligned}
$$

so that the general form of the Watson and Preston (1951) solution in Eq. (L.11) takes the exact form

$$
\frac{u}{u_\infty} \;\approx\; f_2'(\varphi) \;=\; \operatorname{erf}\left(\frac{\sqrt{\sqrt{2}\;-\;1}\;\varphi}{2}\right) \;=\; \operatorname{erf}\left(\frac{\sqrt{\sqrt{2}\;-\;1}\;y}{2}\sqrt{\frac{u_\infty}{\nu\,x}}\;\right) \tag{L.13}
$$

where we have again reconstituted the similarity parameter from its definition in Eq. (L.1) on pp. 198 as we likewise did in Eq. (L.12).

Note that the vertical velocity component, v, is not yet resolved in the treatment we have given here. Recall that v is defined in the context of the stream function, Eq. (L.2) on pp. 198, which takes the particular form given by Eq. (L.4) on pp. 199. In other words, resolving v still requires that must obtain f_2 from f_2', i.e. by applying the second Piercy–Preston integral in Eq. (L.10) on pp. 202. We will defer this step to the next section because it is a by–product of the determination of f for solving the energy equation.

L.5. Companion General Solution for the Energy Equation

In the "constant property" form of the boundary layer equations, Eqs. (9.1) through (9.3) on pp. 118, the flow problem, which consists of the conservation of mass and momentum equations, can be solved separately from the energy equation. This has now been completed. It only remains for the energy equation, Eq. (9.3), to be resolved. Importantly, this equation is linear because u and v are now known. There are 2 steps in framing the solution. First, we recognize that $T(x,y)$ can be written in terms of the similarity variable itself,[L.5] i.e. as $T(\varphi)$. This development is often

[L.5] We have actually already used this observation, stated at the beginning of §K.3 on pp. 189, when we observed that $T(x,y)$ could be recast in form that depended upon only a single variable that somehow *combined* x and y. In that instance, we wrote $T(y/\delta)$, where $\delta=\delta(x)$ was the yet–to–be–determined boundary layer growth law. Here, $T=T(\varphi)$, where φ is the already–known similarity parameter in Eq. (L.1) on pp. 198. We also used this same type of observation in the Rayleigh problem, specifically in §D.2 on pp. 147, although we did not actually call this issue out.

presented in a dimensionless form (e.g. Bejan, 1984) and we will follow suit here, i.e. reverting briefly to the "starred" notation (*) to define the usual form[L.6]

$$T^* \,=\, \frac{T \,-\, T_S}{T_\infty \,-\, T_S}\,, \tag{L.14}$$

but where we will drop the "star" notation and subsequently take T itself as the dimensionless temperature. The second step is to substitute the various similarity–related quantities into the energy equation, including the forms for u and v in Eqs. (L.3) and (L.4), respectively, on pp. 199 and the partial derivatives defined from Chain–ruling the similarity parameter on pp. 198. We find

$$\boxed{T \text{ still dimensional}} \qquad u\,\frac{\partial T}{\partial x} \,+\, v\,\frac{\partial T}{\partial y} \;=\; \alpha\,\frac{\partial^2 T}{\partial y^2}\,,$$

$$\underbrace{\left(u_\infty\, f'\right)}_{u}\,\underbrace{\left(-\,\frac{\varphi\,(T_\infty \,-\, T_S)}{2\,x}\,\frac{dT}{d\varphi}\right)}_{\partial T/\partial x} + \qquad \boxed{T \text{ now dimensionless}}$$

$$\underbrace{\left(-\,\frac{1}{2}\,\sqrt{\frac{u_\infty\,\nu}{x}}\left(f \,-\, \varphi\, f'\right)\right)}_{v}\,\underbrace{\left((T_\infty \,-\, T_S)\,\sqrt{\frac{u_\infty}{\nu\,x}}\,\frac{dT}{d\varphi}\right)}_{\partial T/\partial y} \;=\; \alpha\,\underbrace{\left(\frac{u_\infty\,(T_\infty \,-\, T_S)}{\nu\,x}\,\frac{d^2T}{d\varphi^2}\right)}_{\partial^2 T/\partial y^2}$$

$$-\,\frac{\varphi}{2}\,f'\,\frac{dT}{d\varphi} \,-\, \frac{1}{2}\left(f \,-\, \varphi\, f'\right)\frac{dT}{d\varphi} \;=\; \frac{\alpha}{\nu}\,\frac{d^2T}{d\varphi^2}$$

$$-\,\frac{f}{2}\,\frac{dT}{d\varphi} \;=\; \frac{1}{Pr}\,\frac{d^2T}{d\varphi^2}$$

$$\frac{d^2T}{d\varphi^2} \,+\, \frac{f\cdot Pr}{2}\,\frac{dT}{d\varphi} \;=\; 0\,,$$

where again we presume to know f from the momentum problem.

This equation is very similar to the Piercy–Preston *split* version of the momentum ODE discussed in §L.3 and it can be solved in precisely the same fashion. In particular, let

$$y_1 \,=\, \frac{dT}{d\varphi}\,,$$

whereby we are able to write the above statement in a more appropriate form as

$$\frac{dy_1}{d\varphi} \;=\; -\,\frac{Pr}{2}\,f(\varphi)\,y_1\,.$$

Once again, we see an equation that is clearly separable, whereby it can be integrated directly. This process follows the Piercy–Preston method in §L.3 essentially verbatim. We find

[L.6]This was the dimensionless form used for the Kármán–Pohlhausen method, as well, c.f. Eq. (K.7) on pp. 191. Here again, T_∞ and T_S are both prescribed constants.

$$\int_{y_1(0)}^{y_1(\varphi)} \frac{dy_1}{y_1} = -\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi$$

$$\ln(y_1)\Big|_{y_1(0)}^{y_1(\varphi)} = \ln\big[y_1(\varphi)\big] - \underbrace{\ln\big[y_1(0)\big]}_{\text{set to } C_1} =$$

$$\ln\big[y_1(\varphi)\big] = -\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi + C_1$$

$$y_1(\varphi) = \exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi + C_1\right)$$

$$= \exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right)\cdot\exp\big(C_1\big)$$

$$\frac{dT}{d\varphi} = C_1'\exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right),$$

where we have observed that $C_1 = \ln\big[y_1(0)\big]$ is a yet–to–be–determined constant, given that the function is *evaluated* at $\varphi = 0$ and where we have also replaced y_1 with its equivalent derivative term. This equation is once again separable and therefore again directly integrable

$$\int_{T(0)}^{T(\varphi)} dT = \int_0^{\varphi} C_1'\exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right)d\varphi$$

$$T(\varphi) = C_1'\int_0^{\varphi}\exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right)d\varphi + \cancelto{0}{T(0)}.$$

Note that we moved the constant C_1' outside the integral. We must now evaluate both C_1' and $T(0)$ from the boundary conditions of the problem. Given our dimensionless definition of temperature in in Eq. (L.14) and the similarity parameter in Eq. (L.1) on pp. 198, the boundary conditions of T_S at $y = 0$ and T_∞ at $y \to \infty$ translate to

$$T(0) = 0 \qquad \text{and} \qquad T(\infty) = 1\,.$$

The former is already reflected above and C_1' is determined from the latter, for which a little deduction then implies the solution

$$T(\varphi) = \frac{\int_0^{\varphi}\exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right)d\varphi}{\int_0^{\infty}\exp\left(-\frac{Pr}{2}\int_0^{\varphi} f(\varphi)\,d\varphi\right)d\varphi}\,. \tag{L.15}$$

Eq. (L.15) describes the determination of T once a solution for the momentum problem in the form of f is in–hand.

Of course, the Nusselt number is also a direct consequence of this equation, although it takes a little bit more development to find it. In light of the fact that no–slip holds at the plate surface, conduction and convection are equal at $y \to 0^+$, c.f. Eq. (7.22) on pp. 92, so that this holds as well for $\varphi \to 0^+$ in light of our similarity parameter definition in Eq. (L.1) on pp. 198. Here, it is a

little awkward that we have dropped the "starred" notation (*), so we will be careful to note that we can rewrite the *dimensional* partial derivative, $\partial T/\partial y$, in terms of the *dimensionless* derivative, itself given in terms of the similarity parameter, $dT/d\varphi$, defined from Chain–ruling the similarity parameter on pp. 198. Conversion between dimensionless and dimensional temperature is defined in Eq. (L.14) on pp. 206.

Taking all these factors into consideration, our statement of conduction–convection equivalence at the plate surface takes the form

$$\begin{aligned}
&\boxed{T \text{ still dimensional}} && -k\,\frac{\partial T}{\partial y}\bigg|_{y=0} &&= h\,(T_S - T_\infty)\\
&\boxed{T \text{ now dimensionless}} && -k\,(T_\infty - T_S)\,\sqrt{\frac{u_\infty}{\nu\,x}}\,\frac{dT}{d\varphi}\bigg|_{\varphi=0} &&= h\,(T_S - T_\infty)\\
&\boxed{\text{multiply each side by } x} && \sqrt{\frac{u_\infty}{\nu\,x}}\;x\,\frac{dT}{d\varphi}\bigg|_{\varphi=0} &&= \frac{h}{k}\;x\\
&\text{(L.16)} && \sqrt{Re_x}\;\frac{dT}{d\varphi}\bigg|_{\varphi=0} &&= Nu_x\,,
\end{aligned}$$

where we have used the fact that $Re_x = u_\infty x/\nu$. It therefore seems that obtaining the Nusselt number for any solution method based on the similarity approach reduces to the task of using Eq. (L.15) to evaluate

$$\text{(L.17)}\qquad \frac{dT}{d\varphi}\bigg|_{\varphi=0} = \frac{\exp\left(-\frac{Pr}{2}\int_0^0 f(\varphi)\,d\varphi\right)}{\int_0^\infty \exp\left(-\frac{Pr}{2}\int_0^\varphi f(\varphi)\,d\varphi\right)d\varphi} = \left[\int_0^\infty \exp\left(-\frac{Pr}{2}\int_0^\varphi f(\varphi)\,d\varphi\right)d\varphi\right]^{-1}$$

from knowledge of f and then substituting this result into Eq. (L.16). Note that we presume at this point to have at least f', having already solved for u. If having solved for v as well, we would be slightly further in having f itself. This observation implies there are still at least 2 levels of integration needed to arrive at the Nusselt number.

L.6. Completion of Piercy–Preston Example for Heat Transfer

Let us return to the Piercy–Preston method we have been examining for the flat plate problem. We have developed 2 versions of the momentum profile using the iterative integration method in the forms of Eqs. (L.12) on pp. 204 and (L.13) on pp. 205.[L.7] We shall now push toward the Nusselt number for the corresponding convection problem. Let us proceed using a general expression[L.8]

$$f' = \text{erf}(C\,\varphi)\,,$$

[L.7] These solutions are plotted in Fig. 9.2 on pp. 122 in comparison to the Kármán–Pohlhausen 3–rd order polynomial and Blasius profiles.

[L.8] We remind that this expression is still only an approximation obtained from just a single Piercy–Preston iteration.

where C is the solution–appropriate constant for Eqs. (L.12) and (L.13). Integral Eq. (M.3) on pp. 214 indicates that function f itself is then

$$\begin{aligned}
\int_0^{\varphi} df &= \int_0^{\varphi} f' \, d\varphi = \int_0^{\varphi} \mathrm{erf}(C\,\varphi)\, d\varphi = \left(\varphi\, \mathrm{erf}(C\,\varphi) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} \right) \Bigg|_0^{\varphi} \\
&= \left(\varphi\, \mathrm{erf}(C\,\varphi) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} \right) - \left(0 \cdot \mathrm{erf}(0 \cdot C) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(0 \cdot C)^2} \right) \\
f(\varphi) &= \varphi\, \mathrm{erf}(C\,\varphi) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} - \frac{1}{\sqrt{\pi}\, C} \, .
\end{aligned}$$

We are now poised to execute the *inner* integration in Eq. (L.17), i.e.

$$\begin{aligned}
\int_0^{\varphi} f(\varphi)\, d\varphi &= \int_0^{\varphi} \left(\varphi\, \mathrm{erf}(C\,\varphi) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} - \frac{1}{\sqrt{\pi}\, C} \right) d\varphi \\
&= \left[\frac{\varphi^2}{2}\, \mathrm{erf}(C\,\varphi) + \frac{\varphi}{2\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} - \frac{1}{4\, C^2}\, \mathrm{erf}(C\,\varphi) \right] \\
&\qquad \frac{1}{\sqrt{\pi}\, C}\, \frac{\sqrt{\pi}}{2\, C}\, \mathrm{erf}(C\,\varphi) - \frac{\varphi}{\sqrt{\pi}\, C} \\
&= \frac{\varphi^2}{2}\, \mathrm{erf}(C\,\varphi) + \frac{\varphi}{2\sqrt{\pi}\, C}\, e^{-(C\,\varphi)^2} + \frac{1}{4\, C^2}\, \mathrm{erf}(C\,\varphi) - \frac{\varphi}{\sqrt{\pi}\, C} \, .
\end{aligned}$$

where the integral of $\varphi\, \mathrm{erf}(C\,\varphi)$ comes directly from Eq. (M.5) on pp. 215, the integral of the exponential *is* an error function, c.f. Eq. (M.1) on pp. 213, and the third term is trivial. This, in turn, is substituted back into Eq. (L.17) to obtain[L.9]

(L.18)

$$\left. \frac{dT}{d\varphi} \right|_{\varphi=0} = \left(\int_0^{\infty} \exp \left[- \frac{Pr}{2} \left(\frac{\varphi^2\, \mathrm{erf}(C\,\varphi)}{2} + \frac{\varphi\, e^{-(C\,\varphi)^2}}{2\sqrt{\pi}\, C} + \frac{\mathrm{erf}(C\,\varphi)}{4\, C^2} - \frac{\varphi}{\sqrt{\pi}\, C} \right) \right] d\varphi \right)^{-1} ,$$

which is the final operation to be performed for quantifying Nu_x, according to Eq. (L.16).

As we advised previously, the mathematical difficulty tends to accelerate with successive integrations and Eq. (L.18) does not appear to be readily integrable. That is, the integrand is an exponentiation involving various special functions. One might try to *fit* a curve to the exponential argument (the term within square brackets) in the hopes of recasting into a simpler form that could then be integrated. However, it is usual more pragmatic to simply resort to *numerical integration* in cases such as these. There are two important factors here. First, floating–point arithmetic

[L.9] The specific forms for the two different momentum solutions in Eqs. (L.12) and (L.13), where C is, respectively, $1/2$ and $\sqrt{\kappa}/2$, are

$$\left. \frac{dT}{d\varphi} \right|_{\varphi=0} = \left(\int_0^{\infty} \exp \left[- \frac{Pr}{2} \left(\frac{\varphi^2\, \mathrm{erf}(\varphi/2)}{2} + \frac{\varphi\, e^{-(\varphi/2)^2}}{\sqrt{\pi}} + \mathrm{erf}(\varphi/2) - \frac{2\,\varphi}{\sqrt{\pi}} \right) \right] d\varphi \right)^{-1}$$

$$\left. \frac{dT}{d\varphi} \right|_{\varphi=0} = \left(\int_0^{\infty} \exp \left[- \frac{Pr}{2} \left(\frac{\varphi^2\, \mathrm{erf}(\sqrt{\kappa}\,\varphi/2)}{2} + \frac{\varphi\, e^{-(\sqrt{\kappa}\,\varphi/2)^2}}{\sqrt{\pi\,\kappa}} + \frac{\mathrm{erf}(\sqrt{\kappa}\,\varphi/2)}{\kappa} - \frac{2\,\varphi}{\sqrt{\pi\,\kappa}} \right) \right] d\varphi \right)^{-1}$$

obviously cannot represent the upper limit, ∞, so we must determine a suitable, finite substitute that does compromise accuracy. Second, we must decide which of the many standard numerical methods to use for this operation.

The first issue is generally straightforward and it is so too in this case. It is routine to simply *plot* the integrand over a wide range of the variable whose limits are in question and then to find where an acceptable truncation of the domain for that variable can be made. Fig. L.1 shows such a plot for the integrand of Eq. (L.18), specifically for the Watson and Preston (1951) momentum model.

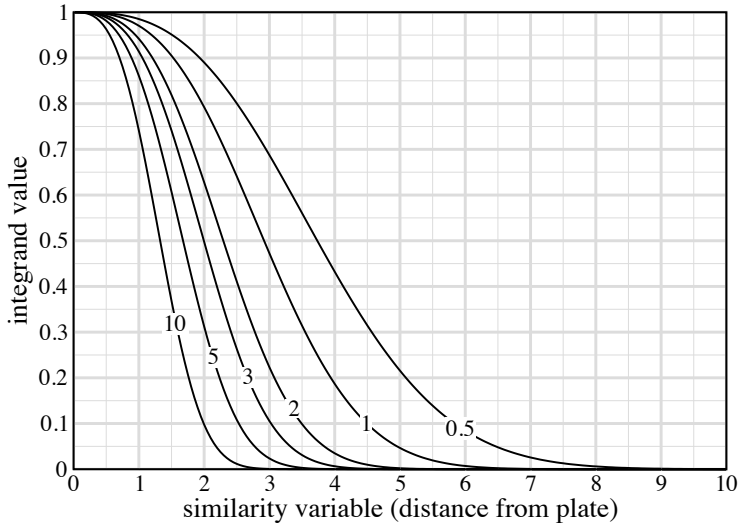


FIGURE L.1. *The integrand of Eq.* (L.18) *for the Watson and Preston (1951) model at representative values of the Prandtl number as functions of the similarity variable, φ. Evidently, there is little contribution for $\varphi > 10$.*

For the range of moderate Pr we are interested in, it seems that there is negligible contribution for $\varphi > 10$. One can therefore replace the limit of ∞ in the actual integral of Eq. (L.18) with a finite value, say 10 or 15, in the numerical model of Eq. (L.18).

The second matter is selecting a numerical algorithm. Numerical integration is a somewhat vast topic and there is a significant literature (e.g. Hamming, 1962; Conte, 1965; Forsythe et al., 1977). However, it appears from Fig. L.1, that, despite its mathematical complications, the function in Eq. (L.18) is "simple" in the sense that it does not have any unusual properties that would pose difficulty for numerical treatment.[L.10] We will therefore use an algorithm known as *Simpson's Rule*, which provides a good trade–off of accuracy versus ease of implementation. The principle of Simpson's Rule is that it patches together quadratic approximations of the integrand over the discretized variable of integration, finding that

$$\int_a^b f(\varphi)\,d\varphi \;\approx\; \frac{\Delta\varphi}{3}\Big[f\big(0\cdot\Delta\varphi\big) \,+\, 4\,f\big(1\cdot\Delta\varphi\big) \,+\, 2\,f\big(2\cdot\Delta\varphi\big) \,+\, 4\,f\big(3\cdot\Delta\varphi\big) \,+\, 2\,f\big(4\cdot\Delta\varphi\big) \\ +\,\cdots\,+\,2\,f\big((m-2)\cdot\Delta\varphi\big) \,+\, 4\,f\big((m-1)\cdot\Delta\varphi\big) \,+\, f\big(m\cdot\Delta\varphi\big)\Big]$$

[L.10] That is, it seems to be a smooth, monotonically decreasing function. Because Pr is a simple multiplicative coefficient in Eq. (L.18), the basic nature of this curve is independent of the actual value of Pr.

where m is the number of discrete segments that model the integration domain, having $m + 1$ point–values of the function, and $\Delta\varphi = (b - a)/m$ is the consequent segment size. Derivation of the algorithm is shown in numerous reference texts, including those referenced above. Also, while accuracy properties of this and other algorithms can be studied formally, we will take the more usual pragmatic approach of choosing m based simply on empirical observations of convergence. The method is not difficult to code, as shown in algorithm L.1, which can be implemented in any suitable language.

Algorithm L.1 Simpson's Rule for Numerical Integration of Eq. (L.18)

```
input values of m, Δφ, Pr, and C
define function f(Pr, φ, C) as integrand of Eq. (L.18)
initialize the integral F = 0
for n = 0, 1, 2, 3, ... m do
  if n = 0 or n = m then
    F = F + f(Pr, φ, C)
  else
    if n is odd then
      F = F + 4 · f(Pr, φ, C)
    else
      F = F + 2 · f(Pr, φ, C)
    end if
  end if
end for
F = F · Δφ/3
print 1/F
```

Note that the choices of m and $\Delta\varphi$ also imply the truncated upper–limit, $m \cdot \Delta\varphi$, that replaces the true limit of ∞. Based on Fig. L.1, a value of 15 is more than sufficient. For the particular function in Eq. (L.18) choices of 0.05 and 0.02 for $\Delta\varphi$ gave very comparable results.

Fig. L.2 then plots the integration specified by Eq. (L.18) for both the Piercy and Preston (1936) and Watson and Preston (1951) momentum models in the range of moderate Prandtl number $0.5 \leq Pr \leq 20$. These results are compared to the traditional "Blasius curve" (Bejan, 1984)

$$\left.\frac{dT}{d\varphi}\right|_{\varphi=0} = 0.332\, Pr^{1/3}\,. \tag{L.19}$$

As with the actual momentum solution plotted in Fig. 9.2 on pp. 122, the accuracy of the Watson and Preston (1951) model exceeds that of the Piercy and Preston (1936) result and is quite close to the theoretical Blasius curve. Again, it is rather remarkable to obtain such good results merely using the single–iteration momentum solutions.

Because the Nusselt number is proportional to these curves, as specified by Eq. (L.16) on pp. 208, and because these curves evidently cannot easily be expressed in terms of elementary functions, it is a common practice to quote results as *correlations*. Indeed, the Blasius curve in Eq. (L.19) is itself just such a correlation. The presumptive model is that the results should correlate in the

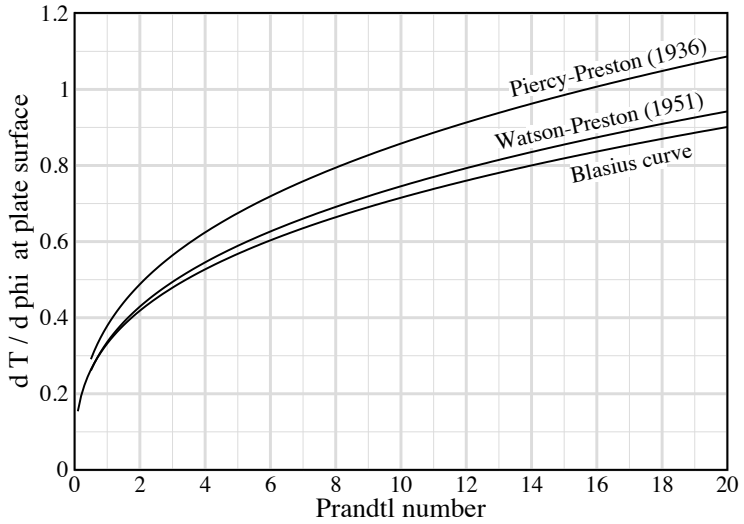


FIGURE L.2. *Eq.* (L.18) *for both the Piercy and Preston (1936) and Watson and Preston (1951) models plotted as functions of the Prandtl number. The conventional "Blasius curve" (Bejan, 1984) is shown for comparison.*

form of

$$\left.\frac{dT}{d\varphi}\right|_{\varphi=0} = a\, Pr^b\,,$$

where a and b are yet–to–be–determined constants. Here, it is much more convenient to transform this model into a form for which the numerical technique of *linear regression* can be used to determine the constants.[L.11] Taking the log, we find

$$\ln\left(\left.\frac{dT}{d\varphi}\right|_{\varphi=0}\right) = \ln\big(a\big) + b\,\ln\big(Pr\big)\,, \tag{L.20}$$

which is in the form of a line equation, i.e. one that looks like "$y = n\,x + c$", where b is the "slope" and $\ln\big(a\big)$ is the "y–intercept". Thus, it is the *logs* of the data points that are actually regressed, whereby b is determined directly, while a is the exponentiation of the "y–intercept". From the results in Fig. L.2, we find

$$\left.\frac{dT}{d\varphi}\right|_{\varphi=0} = 0.383\, Pr^{0.348} \qquad \text{Piercy and Preston (1936)} \tag{L.21}$$

$$\left.\frac{dT}{d\varphi}\right|_{\varphi=0} = 0.339\, Pr^{0.342} \qquad \text{Watson and Preston (1951)}\,. \tag{L.22}$$

The correlation coefficients for both models are extremely high, > 0.9998 and > 0.9999, respectively, indicating that the basic presumptive model is very good.

[L.11] Like numerical integration, regression techniques have a large literature base. We do not discuss the technique here at all, see e.g. Hamming (1962); Conte (1965); Forsythe et al. (1977). Neither do we give an algorithm for it because, aside from the transform in Eq. (L.20), there are no special circumstances for applying regression to this problem and most modern programming languages have native functionality or optional libraries that implement regression directly.

APPENDIX M

Some Useful Non–Standard Integrals

MANY DIFFERENT INTEGRALS are required in the course of developing the theories of conduction and convection heat transfer. A large subset of these involve elementary functions and are consequently to be found in standard handbooks (e.g. Gradshteyn and Ryzhik, 1980; Beyer, 1984), c.f. those in footnotes B.5 on pp. 137 and B.6 on pp. 138. Others are apparently not sufficiently common to appear in such sources, but are nevertheless important in problems related to heat transfer. Though these cases usually involve so–called "special function" (Andrews, 1985), integrals can usually be derived using the standard techniques, especially integration by parts (IBP), if sufficient care is taken. Recall, IBP is defined as

$$\int \xi_1 \, d\xi_2 \;=\; \left(\xi_1 \, \xi_2\right)\Big| \;-\; \int \xi_2 \, d\xi_1$$

Evaluation of the results over given limits is also sometimes non–trivial and we examine cases as they are relevant to the problems discussed herein.

M.1. Integrals of the Error Function

The Piercy–Preston example developed in §L.4 starting on pp. 202 uses several different integrals involving the error function, erf. Here, we develop anti–derivatives and definite integrals, as cases in the text may require. We note that some properties of erf are listed in Eqs. (D.5) through (D.8) on pp. 149.

M.1.1. Pre–Cursor of the Error Function. The exponential function that integrates *to* the error function is found widely in the literature. We only include it here for completeness. If C is a constant, then the change–of–variables method shows

$$\int_0^{\varphi} e^{-(C\,\eta)^2} \, d\eta \;=\; \int_0^{C\,\varphi} e^{-\chi^2} \, \frac{1}{C} \, d\chi \;=\; \frac{1}{C} \, \frac{\sqrt{\pi}}{2} \left(\frac{2}{\sqrt{\pi}} \int_0^{C\,\varphi} e^{-\chi^2} \, d\chi \right) \;=\; \frac{\sqrt{\pi}}{2\,C} \, \mathrm{erf}(\chi) \Big|_0^{C\,\varphi}$$

$$= \; \frac{\sqrt{\pi}}{2\,C} \, \mathrm{erf}(C\,\varphi) \; . \tag{M.1}$$

M.1.2. Error Function. The integral of erf itself can be derived using IBP by defining

$$\xi_1 \;=\; \mathrm{erf}(\eta) \qquad\qquad d\xi_2 \;=\; d\eta$$

$$d\xi_1 \;=\; \frac{2}{\sqrt{\pi}} \, e^{-\eta^2} \, d\eta \qquad\qquad \xi_2 \;=\; \eta \; ,$$

from which we find

$$\begin{aligned}\int \mathrm{erf}(\eta)\, d\eta &= \eta\, \mathrm{erf}(\eta) - \int \frac{2}{\sqrt{\pi}}\, \eta\, e^{-\eta^2}\, d\eta \\ &= \eta\, \mathrm{erf}(\eta) + \frac{1}{\sqrt{\pi}}\, e^{-\eta^2} \end{aligned} \tag{M.2}$$

If there is an accompanying constant, C, in the argument then

$$\begin{aligned} \xi_1 &= \mathrm{erf}(C\,\eta) & d\xi_2 &= d\eta \\ d\xi_1 &= \frac{2}{\sqrt{\pi}}\, e^{-(C\,\eta)^2}\, C\, d\eta & \xi_2 &= \eta\,, \end{aligned}$$

from which we find

$$\begin{aligned}\int \mathrm{erf}(C\,\eta)\, d\eta &= \eta\, \mathrm{erf}(C\,\eta) - \int \frac{2}{\sqrt{\pi}}\, \eta\, e^{-(C\,\eta)^2}\, C\, d\eta \\ &= \eta\, \mathrm{erf}(C\,\eta) + \frac{1}{\sqrt{\pi}\, C}\, e^{-(C\,\eta)^2} \end{aligned} \tag{M.3}$$

M.1.3. Function: $\mathbf{erf}^{\mathbf{2}}(\boldsymbol{C\,\eta})$. The integral of the square of erf requires 2 rounds of IBP. For the first round, take

$$\begin{aligned} \xi_1 &= \mathrm{erf}^{\,2}(\eta) & d\xi_2 &= d\eta \\ d\xi_1 &= 2\,\mathrm{erf}(\eta)\, \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}\, d\eta & \xi_2 &= \eta\,, \end{aligned}$$

where $d\xi_1$ can be thought of in the context of executing the derivative of some function f^2 to get $2 \cdot f \cdot df/d\xi$. From this, we find

$$\begin{aligned}\int \mathrm{erf}^{\,2}(\eta)\, d\eta &= \eta\, \mathrm{erf}^{\,2}(\eta) - \int \eta \cdot 2\,\mathrm{erf}(\eta) \cdot \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}\, d\eta \\ &= \eta\, \mathrm{erf}^{\,2}(\eta) - \frac{4}{\sqrt{\pi}} \int \eta\, e^{-\eta^2}\, \mathrm{erf}(\eta)\, d\eta\,, \end{aligned}$$

which now requires an additional step of IBP for the second term. Now take

$$\begin{aligned} \xi_1 &= \mathrm{erf}(\eta) & d\xi_2 &= \eta\, e^{-\eta^2}\, d\eta \\ d\xi_1 &= \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}\, d\eta & \xi_2 &= -\,\frac{1}{2}\, e^{-\eta^2}\,, \end{aligned}$$

so that we can proceed as

$$\begin{aligned}\int \mathrm{erf}^{\,2}(\eta)\, d\eta &= \eta\, \mathrm{erf}^{\,2}(\eta) - \frac{4}{\sqrt{\pi}} \left[-\frac{1}{2}\, e^{-\eta^2}\, \mathrm{erf}(\eta) + \int \frac{1}{2}\, e^{-\eta^2} \cdot \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}\, d\eta \right] \\ &= \eta\, \mathrm{erf}^{\,2}(\eta) + \frac{2}{\sqrt{\pi}}\, e^{-\eta^2}\, \mathrm{erf}(\eta) - \frac{4}{\pi} \int e^{-2\,\eta^2}\, d\eta\,. \end{aligned}$$

The last integral is another *form* of the error function, which we can resolve by a quick change of variables $\xi = \sqrt{2}\,\eta$ and $d\xi = \sqrt{2}\;d\eta$, so that

$$\int e^{-(\sqrt{2}\,\eta)^2}\,d\eta \;=\; \int e^{-\xi^2}\cdot\frac{1}{\sqrt{2}}\,d\xi \;=\; \frac{1}{\sqrt{2}}\,\frac{\sqrt{\pi}}{2}\left(\frac{2}{\sqrt{\pi}}\int e^{-\xi^2}\,d\xi\right) \;=\; \frac{\sqrt{\pi}}{2\,\sqrt{2}}\,\mathrm{erf}(\xi)\,,$$

which can be substituted back into the desired expression to find

$$\begin{aligned}\int \mathrm{erf}^{\,2}(\eta)\,d\eta \;&=\; \eta\,\mathrm{erf}^{\,2}(\eta) \;+\; \frac{2}{\sqrt{\pi}}\,e^{-\eta^2}\,\mathrm{erf}(\eta) \;-\; \frac{4}{\pi}\cdot\frac{\sqrt{\pi}}{2\,\sqrt{2}}\,\mathrm{erf}\big(\sqrt{2}\;\eta\big)\\ &=\; \eta\,\mathrm{erf}^{\,2}(\eta) \;+\; \frac{2}{\sqrt{\pi}}\,e^{-\eta^2}\,\mathrm{erf}(\eta) \;-\; \frac{2}{\sqrt{2\,\pi}}\,\mathrm{erf}\big(\sqrt{2}\;\eta\big)\,. \qquad \text{(M.4)}\end{aligned}$$

M.1.4. Function: $\eta\,\mathrm{erf}(C\,\eta)$. If C is a constant, then the integral of $\eta\,\mathrm{erf}(C\,\eta)$ is derived, as follows. Define

$$\begin{aligned}\xi_1 \;&=\; \mathrm{erf}(C\,\eta) & d\xi_2 \;&=\; \eta\,d\eta\\ d\xi_1 \;&=\; \frac{2}{\sqrt{\pi}}\,e^{-(C\,\eta)^2}\,C\,d\eta & \xi_2 \;&=\; \frac{\eta^2}{2}\,,\end{aligned}$$

from which a first pass of IBP yields

$$\int_0^{\varphi}\eta\,\mathrm{erf}(C\,\eta)\,d\eta \;=\; \left(\frac{\eta^2}{2}\,\mathrm{erf}(C\,\eta)\right)\Big|_0^{\varphi} \;-\; \frac{C}{\sqrt{\pi}}\int_0^{\varphi}\eta^2\,e^{-(C\,\eta)^2}\,d\eta\,,$$

but this form still requires another round of IBP. We set

$$\begin{aligned}\xi_1 \;&=\; \eta & d\xi_2 \;&=\; \eta\,e^{-(C\,\eta)^2}\,d\eta\\ d\xi_1 \;&=\; d\eta & \xi_2 \;&=\; -\,\frac{1}{2\,C^2}\,e^{-(C\,\eta)^2}\,,\end{aligned}$$

from which

$$\begin{aligned}\int_0^{\varphi}\eta\,\mathrm{erf}(C\,\eta)\,d\eta \;&=\; \frac{\varphi^2}{2}\,\mathrm{erf}(C\,\varphi) \;-\; \frac{C}{\sqrt{\pi}}\left[\Big(-\,\frac{\eta}{2\,C^2}\,e^{-(C\,\eta)^2}\Big)\Big|_0^{\varphi} \;+\; \int_0^{\varphi}\frac{1}{2\,C^2}\,e^{-(C\,\eta)^2}\,d\eta\right]\\ &=\; \frac{\varphi^2}{2}\,\mathrm{erf}(C\,\varphi) \;+\; \frac{\varphi}{2\,\sqrt{\pi}\,C}\,e^{-(C\,\varphi)^2} \;-\; \frac{C}{\sqrt{\pi}}\,\frac{1}{2\,C^2}\int_0^{\varphi}e^{-(C\,\eta)^2}\,d\eta\\ &=\; \frac{\varphi^2}{2}\,\mathrm{erf}(C\,\varphi) \;+\; \frac{\varphi}{2\,\sqrt{\pi}\,C}\,e^{-(C\,\varphi)^2} \;-\; \frac{1}{2\,\sqrt{\pi}\,C}\,\frac{\sqrt{\pi}}{2\,C}\,\mathrm{erf}(C\,\varphi)\\ &=\; \frac{\varphi^2}{2}\,\mathrm{erf}(C\,\varphi) \;+\; \frac{\varphi}{2\,\sqrt{\pi}\,C}\,e^{-(C\,\varphi)^2} \;-\; \frac{1}{4\,C^2}\,\mathrm{erf}(C\,\varphi)\,, \qquad \text{(M.5)}\end{aligned}$$

where we used Eq. (M.1) directly for the last integral.

M.1.5. Function: $\big[1-\mathrm{erf}(\eta)\big]\,\mathrm{erf}(\eta)$. The integral $\int\big[1-\mathrm{erf}(\eta)\big]\,\mathrm{erf}(\eta)\,d\eta$ is readily constructed by using Eqs. (M.2) and (M.4) directly, i.e.

$$\begin{aligned}\int\big[\mathrm{erf}(\eta)-\mathrm{erf}^{\,2}(\eta)\big]\,d\eta \;&=\; \eta\,\mathrm{erf}(\eta)\;+\;\frac{1}{\sqrt{\pi}}\,e^{-\eta^2}\\ &\qquad-\;\Big[\eta\,\mathrm{erf}^{\,2}(\eta)\;+\;\frac{2}{\sqrt{\pi}}\,e^{-\eta^2}\,\mathrm{erf}(\eta)\;-\;\frac{2}{\sqrt{2\,\pi}}\,\mathrm{erf}\big(\sqrt{2}\;\eta\big)\Big]\\ &=\; \eta\,\mathrm{erf}(\eta)\,\Big[1-\mathrm{erf}(\eta)\Big]\;+\;\frac{1}{\sqrt{\pi}}\,e^{-\eta^2}\Big[1-2\,\mathrm{erf}(\eta)\Big]\;+\;\frac{2}{\sqrt{2\,\pi}}\,\mathrm{erf}\big(\sqrt{2}\;\eta\big)\end{aligned}\tag{M.6}$$

M.2. Evaluations for Indeterminate Cases

The Piercy–Preston example developed in §L.4 starting on pp. 202 depends upon evaluating Eq. (M.6) over limits of $(0,\infty)$. This seems to be straightforward for the second and third terms, while the first is clearly indeterminate. Specifically, since $\mathrm{erf}(+\infty)=1$, according to Eq. (D.8) on pp. 149, the product $\eta\,\big[1-\mathrm{erf}(\eta)\big]$ gives the appearance of $\infty\cdot 0$. The plot in Fig. D.1 on pp. 150 suggests that $\big[1-\mathrm{erf}(\eta)\big]\to 0$ much faster than η grows, but let us demonstrate rigorously that this term does indeed vanish using L'Hospital's Rule. To form a ratio, let us take η into the denominator. Then, following the procedure of differentiating both numerator and denominator, we find

$$\lim_{\eta\to\infty}\eta\,\Big[1-\mathrm{erf}(\eta)\Big]\;=\;\frac{\dfrac{d}{d\eta}\Big[1-\mathrm{erf}(\eta)\Big]}{\dfrac{d}{d\eta}\Big[\eta^{-1}\Big]}\;=\;\frac{0\;-\;\dfrac{2}{\sqrt{\pi}}\,e^{-\eta^2}}{-\,\eta^{-2}}\;=\;\frac{2}{\sqrt{\pi}}\,\frac{\eta^2}{e^{\eta^2}}\,,$$

which, although again *very* suggestive, is not yet conclusive. Omitting the leading constant, one more round of L'Hospital's Rule shows

$$\lim_{\eta\to\infty}\eta\,\Big[1-\mathrm{erf}(\eta)\Big]\;=\;\frac{\dfrac{d}{d\eta}\Big(\eta^2\Big)}{\dfrac{d}{d\eta}\Big(e^{\eta^2}\Big)}\;=\;\frac{2\,\eta}{2\,\eta\,e^{\eta^2}}\;=\;e^{-\eta^2}\,,$$

which clearly vanishes as $\eta\to\infty$, consequently

$$\lim_{\eta\to\infty}\eta\,\Big[1-\mathrm{erf}(\eta)\Big]\;=\;0\,.\tag{M.7}$$

We can now evaluate Eq. (M.6) over limits of $(0, \infty)$ as

$$\begin{aligned}
\int_0^\infty \left[\mathrm{erf}(\eta) \,-\, \mathrm{erf}^{\,2}(\eta)\right] \, d\eta \;\; &= \;\; \Bigg(\underbrace{\infty \cdot \mathrm{erf}(\infty) \, \Big[1 - \mathrm{erf}(\infty)\Big]}_{0 \text{ by Eq. (M.7)}} + \frac{1}{\sqrt{\pi}} \, \underbrace{e^{-\infty^2}}_{0} \Big[1 - 2 \cdot \mathrm{erf}(\infty)\Big] + \\
&\qquad \frac{2}{\sqrt{2\,\pi}} \, \mathrm{erf}\big(\sqrt{2} \cdot \infty\big) \Bigg) \;-\; \Bigg(\underbrace{0 \cdot \mathrm{erf}(0) \, \Big[1 - \mathrm{erf}(0)\Big]}_{0} + \\
&\qquad\qquad \frac{1}{\sqrt{\pi}} \, \underbrace{e^{-0^2}}_{1} \Big[1 - \underbrace{2 \cdot \mathrm{erf}(0)}_{0}\Big] \;+\; \underbrace{\frac{2}{\sqrt{2\,\pi}} \, \mathrm{erf}\big(\sqrt{2} \cdot 0\big)}_{0} \Bigg) \\
&= \;\; \frac{2}{\sqrt{2\,\pi}} \;-\; \frac{1}{\sqrt{\pi}} \\
&= \;\; \frac{\sqrt{2} \;-\; 1}{\sqrt{\pi}} \, . \qquad \text{(M.8)}
\end{aligned}$$

About the Author

Michael Wendl completed his undergraduate and graduate studies at Washington University, where he is now on the faculties of genetics, engineering, and mathematics. Dr. Wendl is broadly interested in the mathematical theory underlying pressing problems in the biomedical and engineering sciences. He has made a number of contributions in these areas, including the general solution for Couette flow first examined by G. I. Taylor in 1923. He has also participated in many of the pioneering "big–biology" efforts of our time, including the Human Genome Project (HGP), and is lately part of a research group at the Washington University Genome Institute focusing on mathematical aspects of cancer genomics and biophysics. Dr. Wendl is an enthusiastic proponent of mathematical and scientific literacy for the general public, and is active in policy issues related to research, science, technology, and education.

Creative Commons Public License

Attribution–NoDerivs–NonCommercial 1.0

THE WORK (AS DEFINED BELOW) IS PROVIDED UNDER THE TERMS OF THIS CREATIVE COMMONS PUBLIC LICENSE ("CCPL" OR "LICENSE"). THE WORK IS PROTECTED BY COPYRIGHT AND/OR OTHER APPLICABLE LAW. ANY USE OF THE WORK OTHER THAN AS AUTHORIZED UNDER THIS LICENSE IS PROHIBITED.

BY EXERCISING ANY RIGHTS TO THE WORK PROVIDED HERE, YOU ACCEPT AND AGREE TO BE BOUND BY THE TERMS OF THIS LICENSE. THE LICENSOR GRANTS YOU THE RIGHTS CONTAINED HERE IN CONSIDERATION OF YOUR ACCEPTANCE OF SUCH TERMS AND CONDITIONS.

1. Definitions.

a. "Collective Work" means a work, such as a periodical issue, anthology or encyclopedia, in which the Work in its entirety in unmodified form, along with a number of other contributions, constituting separate and independent works in themselves, are assembled into a collective whole. A work that constitutes a Collective Work will not be considered a Derivative Work (as defined below) for the purposes of this License.

b. "Derivative Work" means a work based upon the Work or upon the Work and other pre-existing works, such as a translation, musical arrangement, dramatization, fictionalization, motion picture version, sound recording, art reproduction, abridgment, condensation, or any other form in which the Work may be recast, transformed, or adapted, except that a work that constitutes a Collective Work will not be considered a Derivative Work for the purpose of this License.

c. "Licensor" means the individual or entity that offers the Work under the terms of this License.

d. "Original Author" means the individual or entity who created the Work.

e. "Work" means the copyrightable work of authorship offered under the terms of this License.

f. "You" means an individual or entity exercising rights under this License who has not previously violated the terms of this License with respect to the Work, or who has received express permission from the Licensor to exercise rights under this License despite a previous violation.

2. Fair Use Rights. Nothing in this license is intended to reduce, limit, or restrict any rights arising from fair use, first sale or other limitations on the exclusive rights of the copyright owner under copyright law or other applicable laws.

3. License Grant. Subject to the terms and conditions of this License, Licensor hereby grants You a worldwide, royalty-free, non-exclusive, perpetual (for the duration of the applicable copyright) license to exercise the rights in the Work as stated below:

a. to reproduce the Work, to incorporate the Work into one or more Collective Works, and to reproduce the Work as incorporated in the Collective Works;

b. to distribute copies or phonorecords of, display publicly, perform publicly, and perform publicly by means of a digital audio transmission the Work including as incorporated in Collective Works;

The above rights may be exercised in all media and formats whether now known or hereafter devised. The above rights include the right to make such modifications as are technically necessary to exercise the rights in other media and formats. All rights not expressly granted by Licensor are hereby reserved.

4. Restrictions. The license granted in Section 3 above is expressly made subject to and limited by the following restrictions:

a. You may distribute, publicly display, publicly perform, or publicly digitally perform the Work only under the terms of this License, and You must include a copy of, or the Uniform Resource Identifier for, this License with every copy or phonorecord of the Work You distribute, publicly display, publicly perform, or publicly digitally perform. You may not offer or impose any terms on the Work that alter or restrict the terms of this License or the recipients' exercise of the rights granted hereunder. You may not sublicense the Work. You must keep intact all notices that refer to this License and to the disclaimer of warranties. You may not distribute, publicly display, publicly perform, or publicly digitally perform the Work with any technological measures that control access or use of the Work in a manner inconsistent with the terms of this License Agreement. The above applies to the Work as incorporated in a Collective Work, but this does not require the Collective Work apart from the Work itself to be made subject to the terms of this License. If You create a Collective Work, upon notice from any Licensor You must, to the extent practicable, remove from the Collective Work any reference to such Licensor or the Original Author, as requested.

b. You may not exercise any of the rights granted to You in Section 3 above in any manner that is primarily intended for or directed toward commercial advantage or private monetary compensation. The exchange of the Work for other copyrighted works by means of digital file-sharing or otherwise shall not be considered to be intended for or directed toward commercial advantage or private monetary compensation, provided there is no payment of any monetary compensation in connection with the exchange of copyrighted works.

c. If you distribute, publicly display, publicly perform, or publicly digitally perform the Work or any Collective Works, You must keep intact all copyright notices for the Work and give the Original Author credit reasonable to the medium or means You are utilizing by conveying the name (or pseudonym if applicable) of the Original Author if supplied; the title of the Work if supplied. Such credit may be implemented in any reasonable manner; provided, however, that in the case of a Collective Work, at a minimum such credit will appear where any other comparable authorship credit appears and in a manner at least as prominent as such other comparable authorship credit.

5. Representations, Warranties and Disclaimer.

a. By offering the Work for public release under this License, Licensor represents and warrants that, to the best of Licensor's knowledge after reasonable inquiry:
   i. Licensor has secured all rights in the Work necessary to grant the license rights hereunder and to permit the lawful exercise of the rights granted hereunder without You having any obligation to pay any royalties, compulsory license fees, residuals or any other payments;
   ii. The Work does not infringe the copyright, trademark, publicity rights, common law rights or any other right of any third party or constitute defamation, invasion of privacy or other tortious injury to any third party.

b. EXCEPT AS EXPRESSLY STATED IN THIS LICENSE OR OTHERWISE AGREED IN WRITING OR REQUIRED BY APPLICABLE LAW, THE WORK IS LICENSED ON AN "AS IS" BASIS, WITHOUT WARRANTIES OF ANY KIND, EITHER EXPRESS OR IMPLIED INCLUDING, WITHOUT LIMITATION, ANY WARRANTIES REGARDING THE CONTENTS OR ACCURACY OF THE WORK.

6. Limitation on Liability. EXCEPT TO THE EXTENT REQUIRED BY APPLICABLE LAW, AND EXCEPT FOR DAMAGES ARISING FROM LIABILITY TO A THIRD PARTY RESULTING FROM BREACH OF THE WARRANTIES IN SECTION 5, IN NO EVENT WILL LICENSOR BE LIABLE TO YOU ON ANY LEGAL THEORY FOR ANY SPECIAL, INCIDENTAL, CONSEQUENTIAL, PUNITIVE OR EXEMPLARY DAMAGES ARISING OUT OF THIS LICENSE OR THE USE OF THE WORK, EVEN IF LICENSOR HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES.

7. Termination.

a. This License and the rights granted hereunder will terminate automatically upon any breach by You of the terms of this License. Individuals or entities who have received Collective Works from You under this License, however, will not have their licenses terminated provided such individuals or entities remain in full compliance with those licenses. Sections 1, 2, 5, 6, 7, and 8 will survive any termination of this License.

b. Subject to the above terms and conditions, the license granted here is perpetual (for the duration of the applicable copyright in the Work). Notwithstanding the above, Licensor reserves the right to release the Work under different license terms or to stop distributing the Work at any time; provided, however that any such election will not serve to withdraw this License (or any other license that has been, or is required to be, granted under the terms of this License), and this License will continue in full force and effect unless terminated as stated above.

8. Miscellaneous.

a. Each time You distribute or publicly digitally perform the Work or a Collective Work, the Licensor offers to the recipient a license to the Work on the same terms and conditions as the license granted to You under this License.

b. If any provision of this License is invalid or unenforceable under applicable law, it shall not affect the validity or enforceability of the remainder of the terms of this License, and without further action by the parties to this agreement, such provision shall be reformed to the minimum extent necessary to make such provision valid and enforceable.

c. No term or provision of this License shall be deemed waived and no breach consented to unless such waiver or consent shall be in writing and signed by the party to be charged with such waiver or consent.

d. This License constitutes the entire agreement between the parties with respect to the Work licensed here. There are no understandings, agreements or representations with respect to the Work not specified here. Licensor shall not be bound by any additional provisions that may appear in any communication from You. This License may not be modified without the mutual written agreement of the Licensor and You.

Bibliography

Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions* (Dover Publications, New York NY, 1972).

Anderson, J. D., *Fundamentals of Aerodynamics* (McGraw–Hill, New York NY, 1991), 2nd edition.

Andrews, L. C., *Special Functions for Engineers and Applied Mathematicians* (Macmillan, New York NY, 1985).

Barenblatt, G. I., *Dimensional Analysis* (Gordon and Breach, New York NY, 1987).

Barenblatt, G. I. and Zel'Dovich, Y. B. (1972). Self–similar solutions as intermediate asymptotics. *Annual Review of Fluid Mechanics*, **4**, 285–312.

Batchelor, G. K., *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, United Kingdom, 1967).

Bejan, A., *Convection Heat Transfer* (John Wiley & Sons, New York NY, 1984).

Bender, C. M. and Orszag, S. A., *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* (Springer–Verlag, New York NY, 1999).

Bergman, T. L., Lavine, A. S., Incropera, F. P., and Dewitt, D. P., *Fundamentals of Heat and Mass Transfer* (John Wiley & Sons, New York NY, 2011), 7th edition.

Berker, R., Intégration des équations du mouvement d'un fluide visqueux incompressible. In S. Flügge, editor, *Handbuch der Physik*, volume VIII/2 (Springer–Verlag, Berlin, 1963).

Beyer, W. H., *CRC Standard Mathematical Tables* (CRC Press, Boca Raton FL, 1984), 27th edition.

Blasius, H. (1908). Grenzschichten in Flüssigkeiten mit kleiner Reibung (translated as "Boundary layers in fluids with little friction" National Advisory Committee for Aeronautics Technical Memorandum #1256, 1950). *Zeitscrift für angewandte Mathematik und Physik*, **56**, 1–37.

Boyce, W. E. and DiPrima, R. C., *Elementary Differential Equations and Boundary Value Problems* (John Wiley & Sons, New York NY, 1977), 3rd edition.

Burmeister, L. C., *Convective Heat Transfer* (John Wiley & Sons, New York NY, 1983).

Carrier, G. F. and Pearson, C. E., *Partial Differential Equations: Theory and Technique* (Academic Press, New York NY, 1976).

Carslaw, H. S. and Jaeger, J. C., *Conduction of Heat in Solids* (Clarendon Press, Oxford, 1959), 2nd edition.

Combellack, W. J., *Introduction to Elementary Functions* (John Wiley & Sons, New York NY, 1962).

Conte, S. D., *Elementary Numerical Analysis* (McGraw–Hill, New York NY, 1965).

Cotta, R. M., *Integral Transforms in Computational Heat and Fluid Flow* (CRC Press, Boca Raton, 1993).

Courant, R., *Differential and Integral Calculus, Volume II* (Interscience, New York NY, 1937).

Courant, R. and John, F., *Introduction to Calculus and Analysis* (John Wiley & Sons, New York NY, 1965).

Currie, I. G., *Fundamentals Mechanics of Fluids* (McGraw–Hill, New York NY, 1993).

Dettman, J. W., *Mathematical Methods in Physics and Engineering* (McGraw–Hill, New York NY, 1962).

Dickson, D. (2000). Mathematicians chase the seven million dollar proofs. *Nature*, **405**, 383.

Doetsch, G. (1936). Integration von Differentialgleichungen vermittels der endlichen Fourier–Transformation. *Mathematische Annalen*, **112**, 52–68.

Eckert, E. R. G. (1981). Pioneering contributions to our knowledge of convective heat transfer. *Journal of Heat Transfer*, **103**, 409–414.

Eckert, M., *The Dawn of Fluid Dynamics* (Wiley–VCH, Weinheim, Federal Republic of Germany, 2006).

Eves, H., *Elementary Matrix Theory* (Allyn and Bacon, Boston, 1966).

Falkner, V. M. and Skan, S. W. (1931). Some approximate solution of the boundary–layer equations. *Philosophical Magazine (Series VII)*, **12**, 865–896.

Feller, W., *An Introduction to Probability Theory and Its Applications*, volume 1 (John Wiley & Sons, New York NY, 1968), 3rd edition.

Fontolliet, P. G., *Telecommunication Systems* (Artech House, Dedham MA, 1986).

Forsythe, G. E., Malcolm, M. A., and Moler, C. B., *Computer Methods for Mathematical Computations* (Prentice Hall, Englewood Cliffs NJ, 1977).

Fry, T. C., Industrial mathematics. In W. L. Schaaf, editor, *Mathematics: Our Great Heritage* (Harper and Brothers, New York NY, 1948).

Goldstein, S. (1969). Fluid mechanics in the first half of this century. *Annual Review of Fluid Mechanics*, **1**, 1–28.

Gradshteyn, I. S. and Ryzhik, I. M., *Tables of Integrals, Series, and Products* (Academic Press, New York NY, 1980), 4th edition.

Gray, A., Mathews, G. B., and MacRobert, T. M., *A Treatise on Bessel Functions and their Applications to Physics* (Macmillan, London, 1952).

Hahn, H., Infinity. In J. R. Newman, editor, *The World of Mathematics* (Simon & Schuster, New York NY, 1956).

Hamming, R. W., *Numerical Methods for Scientists and Engineers* (McGraw–Hill, New York NY, 1962).

Hansen, A. G., Generalized similarity analysis of partial differential equations. In W. F. Ames, editor, *Nonlinear Partial Differential Equations* (Academic Press, New York NY, 1967).

Hellwig, G., *Partial Differential Equations* (Blaisdell Publishing, New York NY, 1964).

Hildebrand, F. B., *Advanced Calculus for Applications* (Prentice Hall, Englewood Cliffs NJ, 1976), 2nd edition.

Holman, J. P., *Heat Transfer* (McGraw–Hill, New York NY, 2010), 10th edition.

Howarth, L. (1938). On the solution of the laminar boundary layer equations. *Proceedings of the Royal Society of London Series A*, **164**, 547–579.

Incropera, F. P. and Dewitt, D. P., *Fundamentals of Heat and Mass Transfer* (John Wiley & Sons, New York NY, 2002), 5th edition.

Joseph, D. D. (1964). Variable viscosity effects on the flow and stability of flow in channels and pipes. *Physics of Fluids*, **7**, 1761–1771.

Joseph, D. D. (1965). Stability of frictionally–heated flow. *Physics of Fluids*, **8**, 2195–2200.

Kays, W. M. and Crawford, M. E., *Convective Heat and Mass Transfer* (McGraw–Hill, New York NY, 1980), 2nd edition.

Kreyszig, E., *Advanced Engineering Mathematics* (John Wiley & Sons, New York NY, 1993), 7th edition.

Mach, E., The economy of science. In J. R. Newman, editor, *The World of Mathematics* (Simon & Schuster, New York NY, 1956).

Martin, W. T. and Reissner, E., *Elementary Differential Equations* (Addison–Wesley Publishing Co., Cambridge MA, 1956).

McAdams, W. H., *Heat Transmission* (McGraw–Hill, New York NY, 1942), 2nd edition.

Mikhailov, M. D. (1968). Generalized finite integral transform. *Inzhenerno–Fizicheskii Zhurnal (Journal of Engineering Physics)*, **14**, 436–438.

Mills, A. F., *Heat Transfer* (Prentice Hall, Upper Saddle River NJ, 1999), 2nd edition.

Moon, P. and Spencer, D. E., *Field Theory Handbook* (Springer–Verlag, Berlin, 1961).

Munson, B. R., Young, D. F., and Okiishi, T. H., *Fundamentals of Fluid Mechanics* (John Wiley & Sons, New York NY, 2006), 5th edition.

Narasimhan, T. N. (1999). Fourier's heat conduction equation: History, influence, and connections. *Reviews of Geophysics*, **37**, 151–172.

Nayfeh, A. H., *Perturbation Methods* (Wiley–VCH, Weinheim, Federal Republic of Germany, 2004).

Nelson, A. L., Folley, K. W., and Coral, M., *Differential Equations* (D. C. Heath & Co., Boston MA, 1960).

Ölçer, N. Y. (1964). On the theory of conductive heat transfer in finite regions. *International Journal of Heat and Mass Transfer*, **7**, 307–314.

Ölçer, N. Y. (1965). On the theory of conductive heat transfer in finite regions with boundary conditions of the second kind. *International Journal of Heat and Mass Transfer*, **8**, 529–556.

Ölçer, N. Y. (1969). General solutions to a class of unsteady heat conduction problems in a rectangular parallelpiped. *International Journal of Heat and Mass Transfer*, **12**, 393–411.

Ong, P. P. (1991). Hidden strengths of dimensional analysis. *The Mathematical Gazette*, **75**, 303–306.

Özişik, M. N., *Heat Conduction* (John Wiley & Sons, New York NY, 1980).

Özişik, M. N., *Heat Transfer: A Basic Approach* (McGraw–Hill, New York NY, 1985).

Panton, R. L., *Incompressible Flow* (John Wiley & Sons, New York NY, 1984).

Piercy, N. A. V. and Preston, J. H. (1936). A simple solution of the flat plate problem of skin friction and heat transfer. *Philosophical Magazine (Series VII)*, **21**, 995–1005.

Pohlhausen, K. (1921). Zur näherungsweise Integration der differential Gleichung der laminaren Reibungschicht. *Zeitscrift für angewandte Mathematik und Mechanik*, **1**, 252–268.

Ross, S. L., *Differential Equations* (Blaisdell Publishing, New York NY, 1965).

Schetz, J. A. and Fuhs, A. E. E., *Fundamentals of Fluid Mechanics* (John Wiley & Sons, New York NY, 1999).

Schlichting, H., *Boundary Layer Theory* (McGraw–Hill, New York NY, 1979), 7th edition.

Sokolnikoff, I. S. and Sokolnikoff, E. S., *Higher Mathematics for Engineers and Physicists* (McGraw–Hill, New York NY, 1941), 2nd edition.

Tani, I. (1977). History of boundary–layer theory. *Annual Review of Fluid Mechanics*, **9**, 87–111.

Tannehill, J. C., Anderson, D. A., and Pletcher, R. H., *Computational Fluid Mechanics and Heat Transfer* (Taylor and Francis, Washington DC, 1997), 2nd edition.

van Dyke, M., *Perturbation Methods in Fluid Mechanics* (Academic Press, New York NY, 1964).

von Arx, W. S., *An Introduction to Physical Oceanography* (Addison–Wesley Publishing Co., Reading MA, 1962).

von Kármán, T. (1921). Über laminare und turbulente Reibung. *Zeitscrift für angewandte Mathematik und Mechanik*, **1**, 233–252.

Wang, C. Y. (1989). Exact solutions of the unsteady Navier–Stokes equations. *Applied Mechanics Reviews*, **42**, S269–S282.

Wang, C. Y. (1990). Exact solutions of the steady–state Navier–Stokes equations — The generalized Beltrami flows — Review and extension. *Acta Mechanica*, **81**, 69–74.

Wang, C. Y. (1991). Exact solutions of the steady–state Navier–Stokes equations. *Annual Review of Fluid Mechanics*, **23**, 159–177.

Watson, E. J. and Preston, J. H. (1951). An approximate solution of two flat plate boundary–layer problems. Technical Report 2537, Aeronautical Research Council, London.

Weast, R. C. and Astle, M. J., *CRC Handbook of Chemistry and Physics* (CRC Press, Boca Raton FL, 1982), 63rd edition.

Wendl, M. C. and Agarwal, R. K. (2000). Couette flow profiles for two nonclassical Taylor–Couette cells. *Journal of Fluids Engineering*, **122**, 435–438.

Wendl, M. C. and Agarwal, R. K. (2002). Viscous dissipation in finite thin–gap Couette devices. *Journal of Heat Transfer*, **124**, 200–203.

White, F. M., *Viscous Fluid Flow* (McGraw–Hill, New York NY, 1974).

Zill, D. G. and Cullen, M. R., *Advanced Engineering Mathematics* (Jones and Bartlett, Sudbury MA, 2000), 2nd edition.

Index